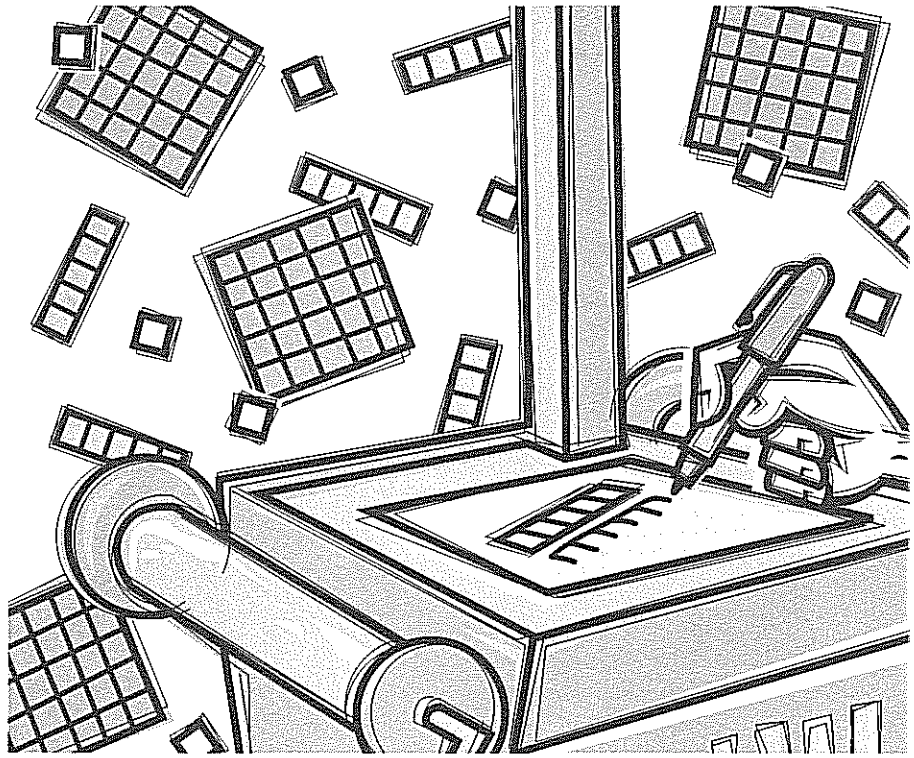
Unit III / Math and the Mind's Eye Activities



Modeling Whole Numbers

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& L. Ted Nelson

Modeling Whole Numbers

1 Grouping and Numeration
Base 5 number pieces are used to examine the role of grouping and place values in recording numbers. The results are extended to other bases.

2 Linear Measure and Dimension
Base 5 number pieces are used to introduce linear measure. The relationship between the dimensions and the area of rectangular regions is discussed.

3 Arithmetic with Number Pieces
Base 5 number pieces are used to perform arithmetical operations. Emphasis is placed on modeling arithmetical operations rather than developing paper-and-pencil processes.

4 Base 10 Numeration
Base 10 number pieces are used to examine the roles of grouping and place value in a base 10 numeration system.

5 Base 10 Addition and Subtraction
Base 10 number pieces are used to portray methods for adding and subtracting multidigit numbers.

6 Number Piece Rectangles
Base 10 number pieces are used to find the area and dimensions of rectangles as a preliminary to developing models for multiplication and division.

7 Base 10 Multiplication
Base 10 number pieces and base 10 grid paper are used to portray methods of multiplying whole numbers.

8 Base 10 Division
Base 10 number pieces and base 10 grid paper are used to portray methods of dividing whole numbers.

Math and the Mind's Eye materials are intended for use in grades 4-9. They are written so teachers can adapt them to fit student backgrounds and grade levels. A single activity can be extended over several days or used in part.

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Math and the Mind's Eye

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Grouping and Numeration

O V E R V I E W

Number pieces are used to examine the role of grouping and place value in recording numbers.

Prerequisite Activity

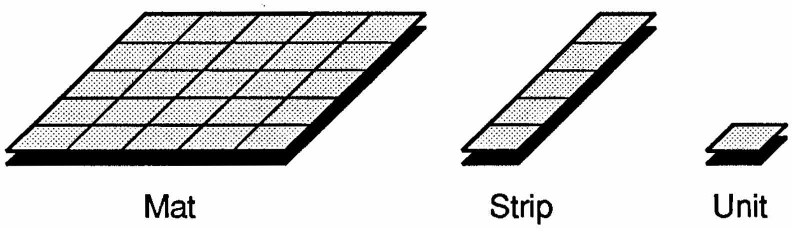
None.

Materials

Base 5 number pieces (see Comment 1), grid paper.

Actions

1. Distribute base 5 mats, strips and units to each student. Tell the students the names of the pieces. Ask them to determine the number of units in each piece and ask them to describe the relationships among the pieces.



Base 5 Pieces

2. Ask each student to form a collection of 4 mats, 3 strips and 7 units. Have the students determine (a) the number of pieces and (b) the total number of units in this collection.

3. Have the students find different collections of number pieces which total 79 units. Make a chart on the chalkboard or overhead listing the various collections the students find. Include a column for the number of pieces in the collection.

Comments

1. Base 5 number pieces can be made by copying page 7 of this activity on tagboard and cutting along the indicated lines. You may wish to have your students do the cutting.

Each student should have a supply of about 5 mats, 10 strips and 15 units. The large oblong pieces will be used later in this activity and should be distributed then.

A small square is a *unit square* or simply a *unit*. A group of 5 units arranged in a row is a *strip* and a group of 5 strips arranged in a square is a *mat*. Each mat contains 25 units.

2. Each mat, strip and unit is counted as a single piece. Hence this collection contains 4 + 3 + 7 or 14 pieces. The total number of units is 4×25 + 3×5 + 7 or 122.

3. Some possible collections are listed below. The asterisk marks the collection with the fewest number of pieces.

M	S	U	No. of Pieces
2	3	14	19
2	5	4	11
1	9	9	19
0	0	79	79
3	0	4	7*
2	2	19	23

Actions

4. Discuss the *base 5 representation* of 79.

5. Write the following chart on the chalkboard or overhead:

Total Units	M	S	U
113	___	___	___
___	2	1	4
95	___	___	___
21	___	___	___
___	1	0	3
50	___	___	___
___	4	4	4

Tell the students that, from now on, all collections are to contain the fewest number of pieces. Work with the students to complete the first two lines of the chart. Write numerical statements for these two lines. Then ask each student to complete the chart and write numerical statements for the remaining lines.

Comments

4. The collection which totals 79 units and contains *the least number of pieces* is the collection which contains 3 mats, 0 strips and 4 units. It can be described by the notation 304_5. This is called the base 5 representation of 79. Thus $304_5 = 79$.

5. The completed chart is:

Total Units	M	S	U
113	4	2	3
59	2	1	4
95	3	4	0
21	0	4	1
28	1	0	3
50	2	0	0
124	4	4	4

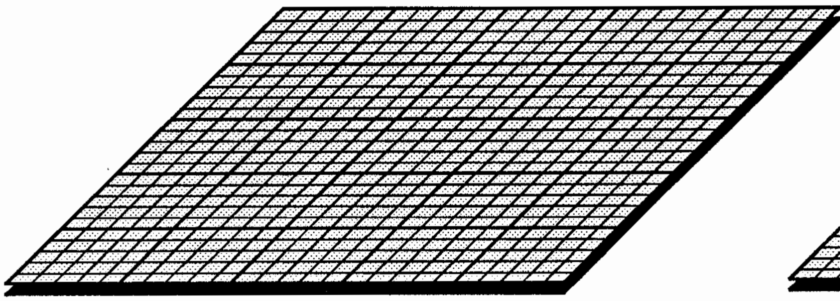
The corresponding numerical statements are:

$$113 = 423_5$$
$$59 = 214_5$$
$$95 = 340_5$$
$$21 = 41_5$$
$$28 = 103_5$$
$$50 = 200_5$$
$$124 = 444_5$$

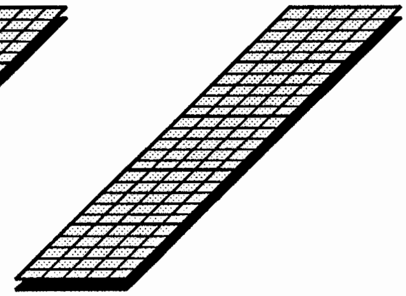
Students may write 041_5 instead of 41_5. This is correct, however zeroes on the left are usually not recorded since no information is lost if they are omitted.

Actions

6. Provide each student with one of the large oblong number pieces. Discuss with the students what larger base 5 number pieces might look like. Provide names for the new pieces introduced.



Base 5 Mat-Mat



Base 5 Strip-Mat

7. Ask students to find the base 5 representations of 200 and 2000.

8. Discuss why 0, 1, 2, 3 and 4 are the only digits which occur in base 5 representations.

Comments

6. Each large oblong piece is a group of 5 mats arranged in a row. Thus, it is a strip of mats or *strip-mat*. It contains a total of 125 units. The next larger base 5 piece is a group of 5 strip-mats. These are arranged to form a square of 25 mats. Hence, this piece is a mat of mats or *mat-mat*. It contains 625 units.

This process of forming base 5 pieces can be continued indefinitely. Thus, 5 mat-mats are grouped to form a strip of mat-mats or *strip-mat-mat* (3125 units). Five strip-mat-mats are grouped to form a *mat-mat-mat* (15,625 units), etc.

Note that base 5 number pieces are successive groups of five.

7. The collection of 200 units which contains the fewest number of base 5 pieces consists of 1 strip-mat, 3 mats, 0 strips and 0 units. Thus, $200 = 1300_5$. The collection for 2000 with the fewest pieces contains 3 mat-mats, 1 strip-mat, 0 mats, 0 strips and 0 units. Hence, $2000 = 31000_5$.

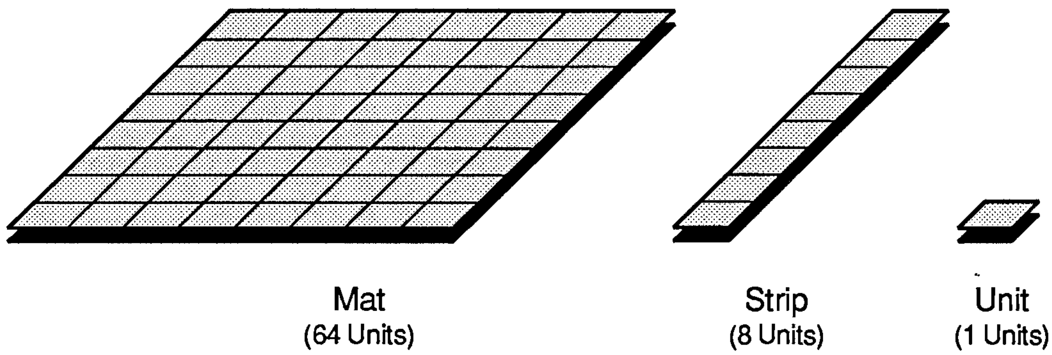
8. The collection which contains the fewest number of pieces will never contain 5 of the same kind of piece . If it did, the 5 pieces could be exchanged for the next larger piece and the number of pieces in the collection reduced.

Actions

9. Ask the students to imagine base 8 number pieces. Ask for volunteers to describe what individual pieces look like and the number of units they contain.

Comments

9. The first three base 8 pieces are illustrated. A base 8 strip-mat contains 512 units and a mat-mat contains 4096 units.



Base 8 Pieces

10. Ask the students to find the base 8 representations of 180 and 1000.

10. The collection of 180 units which contains the fewest number of base 8 pieces consists of 2 mats, 6 strips and 4 units. Thus, 180 = 264_8. The collection with the fewest pieces for 1000 consists of 1 strip-mat, 7 mats, 5 strips and 0 units. Thus, 1000 = 1750_8.

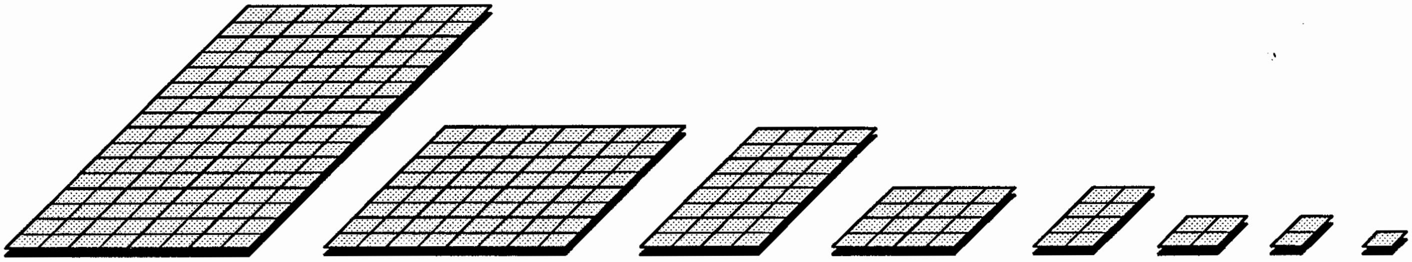
11. Have the students imagine base 10 number pieces. Ask them to describe the collection with the fewest pieces that contains 1275 units.

11. A base 10 strip is a group of 10 units, a mat groups 10 strips and totals 100 units, a strip-mat groups 10 mats and totals 1000 units.

The collection of 1275 units which contains the fewest number of base 10 pieces consists of 1 strip-mat, 2 mats, 7 strips and 5 units. This collection can be denoted as 1275_{10}. However, if the base is 10, it is customary to omit the subscript indicating the base. Conversely, if no base is indicated, it is assumed to be 10.

Actions

12. (Optional.) Have the students cut out base 2 number pieces.



Base 2 Pieces

13. (Optional.) Ask the students to find the base 2, or *binary*, representations of 9, 23, and 100. Point out the "on-off" nature of binary representations.

Comments

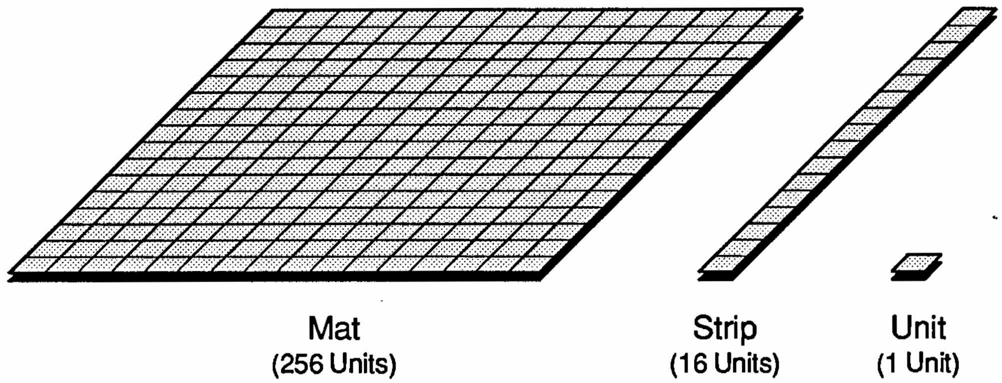
12. The first 8 base 2 pieces may be cut from a 16x16 grid. Page 8 of this activity is a master for centimeter grid paper.

13. The collection of 9 units which contains the fewest number of base 2 pieces consists of 1 strip-mat, 0 mats, 0 strips and 1 unit. Thus $9 = 1001_2$. Also, $23 = 10111_2$ and $100 = 1100100_2$.

The two digits, 0 and 1, that occur in base 2 representations can be interpreted as the two positions of an electrical switch, say, 1 is "on" and 0 is "off". Using the binary representation of a whole number allows it to be represented as a sequence of switches in on or off positions. This is analagous to the way computers store numerical information.

Actions

14. (Optional.) Have the students visualize base 16 pieces. Ask them to find the collection with the fewest number of pieces that contains (a) 100 units, (b) 500 units. Discuss the base 16 representations of 100 and 500.



Base 16 Pieces

Comments

14. A base 16 mat contains 16^2 or 256 units, a strip-mat contains 16^3 or 4096 units.

The collection of 100 units which contains the fewest number of base 16 pieces consists of 6 strips and 4 units; that for 500 units consists of 1 mat, 15 strips and 4 units.

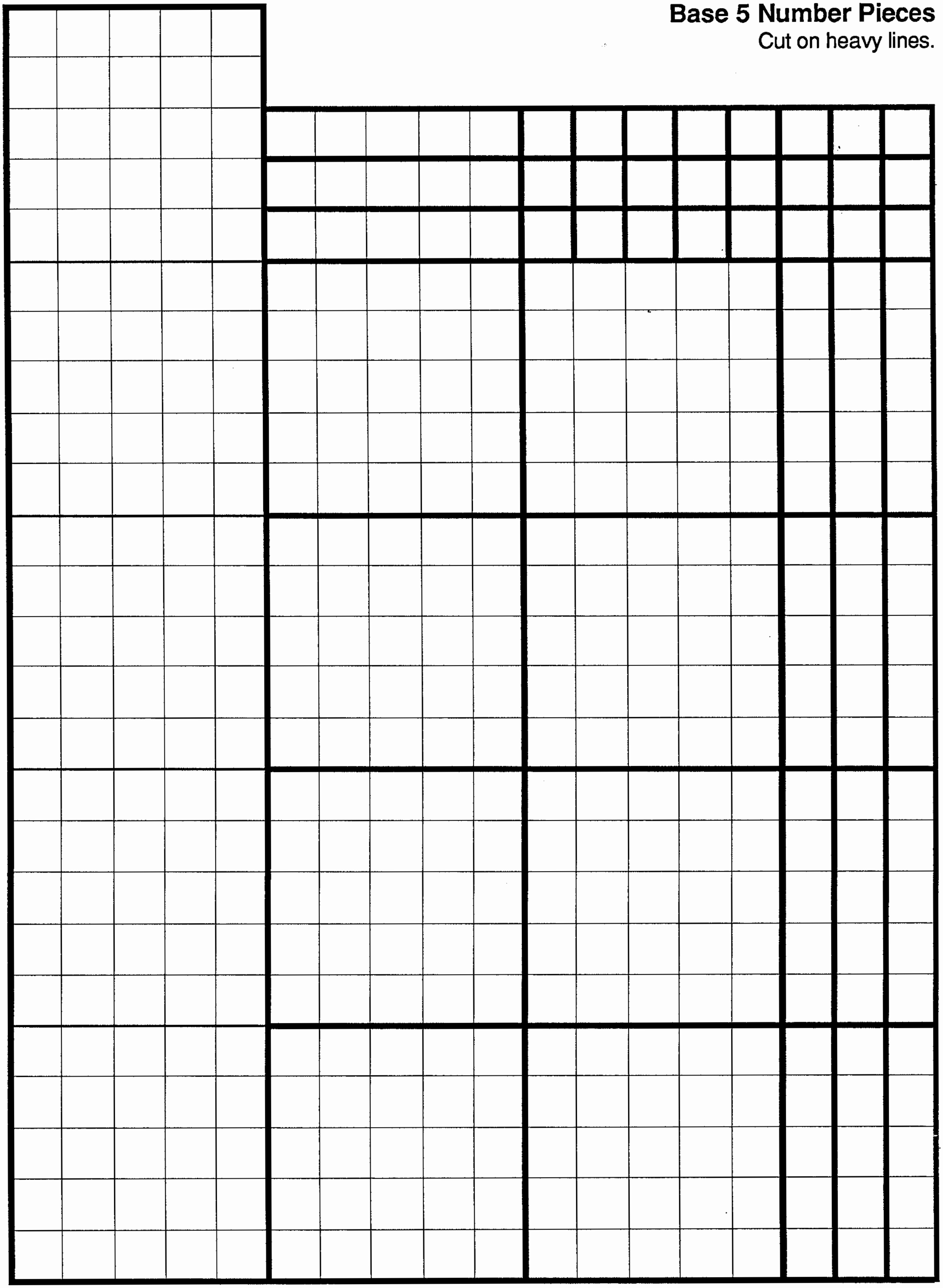
Base 16 representations require 16 digits. New digits representing 10, 11, 12, 13, 14 and 15 must be added to the standard collection of digits, 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Students may wish to invent their own symbols for these additional digits.

In machine language compter programming, which uses base 16, or *hexadecimal*, representation, it is customary to use the symbola A, B, C, D, E and F to represent 10 through 15, respectively. Thus $D6B_{16}$ represents a collection of base 16 pieces consisting of 13 mats, 6 strips and 11 units for a total of $13 \times 256 + 6 \times 16 + 11$ or 3435 units.

The base 16 representations of 100 and 500 are 64_{16} and $1E4_{16}$, respectively.

Base 5 Number Pieces

Cut on heavy lines.



Centimeter Grid Paper

Unit III • Activity 2

Linear Measure and Dimension

O V E R V I E W

Base 5 number pieces are used to introduce linear measure. Distinction is made between the dimensions and the area of regions formed with number pieces.

Prerequisite Activity

Grouping and Numeration, Unit III, Activity 1

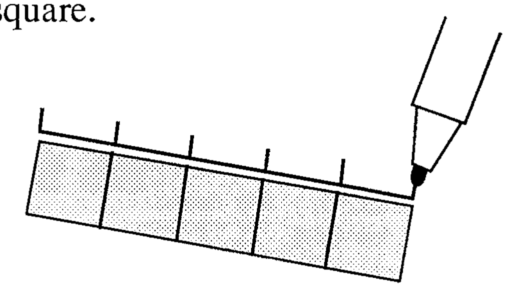
Materials

Base 5 number pieces, Activity Sheets A, B and C for each student. Base 5 rulers are optional (see Comment 6). Transparencies of base 5 number pieces for the instructor's use. A base 5 measuring tape is optional (see Comment 5).

Actions

1. Distribute base 5 number pieces to each student.

2. Place a base 5 strip on the overhead and, using the strip as a straightedge, draw a line segment equal to the length of one side of the strip. Subdivide this line segment into 5 equal parts, so that each subdivision is the length of a side of a unit square.



Introduce the terms *chain* and *unit length.* Draw several line segments on the overhead and give their lengths in terms of chains and unit lengths.

3 units

2 chains

3 chains, 2 units

Comments

1. Each student, or group of students, should have about 10 units, 15 strips, 8 mats and 1 strip-mat. (See Unit III/Activity 1 for a description of these pieces.)

2. Transparencies of base 5 number pieces can be made by copying page 7 of Unit III, Activity 1, *Grouping and Numeration,* on transparency film and cutting as indicated.

A *chain* is the length of the long side of a strip. A *unit length* is the length of a side of a unit square. Thus, one chain equals 5 unit lengths.

one unit length

one chain

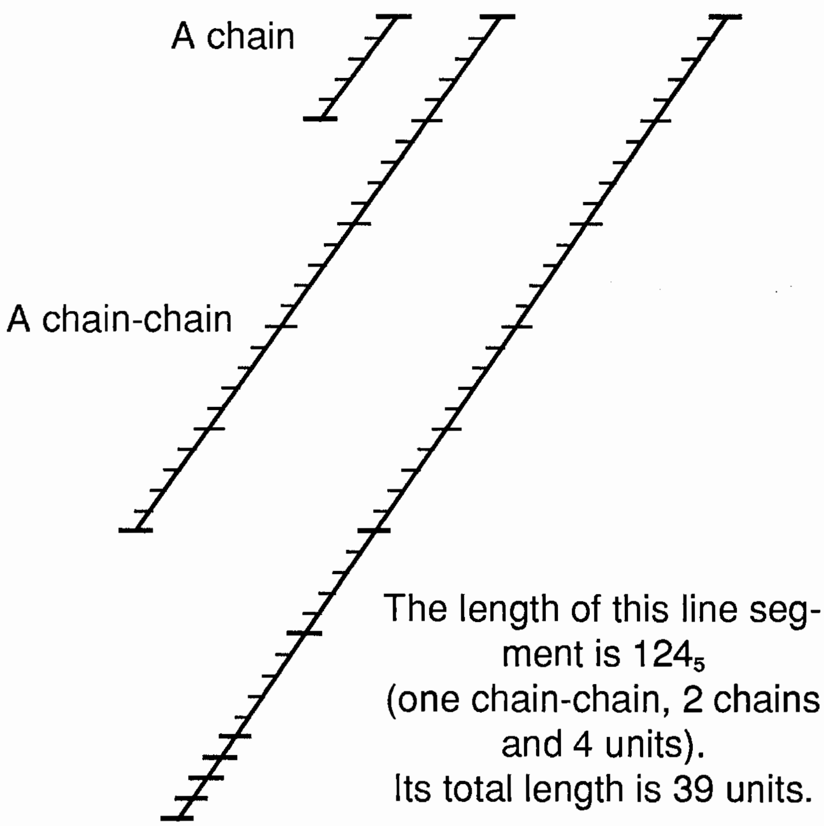
If there is no ambiguity, a unit length may be referred to simply as a unit. However, in some contexts in this activity, the word "unit" may refer to a unit square. Hence, when the word "unit" is used, it is important to understand whether the word is referring to a unit square or a unit length.

The side of a transparency of a strip-mat is useful for drawing line segments of varying lengths.

Actions

3. Distribute a copy of Activity Sheet A to each student. Ask the students to complete the table. When most students have finished, place a transparency of the activity on the overhead and ask for volunteers to fill the blanks in the table. Discuss any questions the students have.

4. Discuss with the students a system of measuring lengths based on groups of five, and how base 5 notation may be used to represent lengths in this system. Illustrate by drawing several line segments on the chalkboard or overhead and recording their lengths in base 5 notation. Ask the students to find the number of unit lengths in each segment drawn.



Comments

3. Masters of the activity sheets are attached.

You may wish to work through the first line of the table with the whole class. If this is done on the overhead, making a transparency of a strip-mat and using an edge of it as a measure will allow students to see chain and unit subdivisions.

Following is the completed table:

Segment	LENGTH Chains	Units	Total units of lengths
A	2	2	12
B	0	4	4
C	3	0	15
D	2	3	13
E	3	1	16
F	2	0	10

4. A chain is the length of a group of 5 unit lengths placed end-to-end. In a system based on groups of five, the next largest measure would be the length of a group of 5 chains placed end-to-end (this is the length of the longest side of a strip-mat). This length is referred to as a *chain of chains* or, simply, a *chain-chain.* Notice a chain-chain is 25 units long.

The next larger measure is the length of a group of five chain-chains placed end-to-end. This length is a *chain-chain-chain.* It equals 125 unit lengths. This process of grouping by fives can be continued indefinitely.

The length of the line segment shown, written in base 5 notation, is 124_5, indicating the segment is 1 chain-chain, 2 chains and 4 units long. This equals 1x25 + 2x5 + 4 or 39 unit lengths. Similarly, a segment of length 302_5 is 3 chain-chains, 0 chains and 2 units long, which totals 3x25 + 0x5 + 2, or 77, unit lengths. A segment of length 2000_5 is 2 chain-chain-chains long which equals 2x125, or 250, unit lengths.

Longer segments can be drawn on the chalkboard. You may wish to construct a base 5 measuring tape (see Action 7) to measure them. They can also be measured by marking off chain-chains using the side of a strip-mat.

Actions

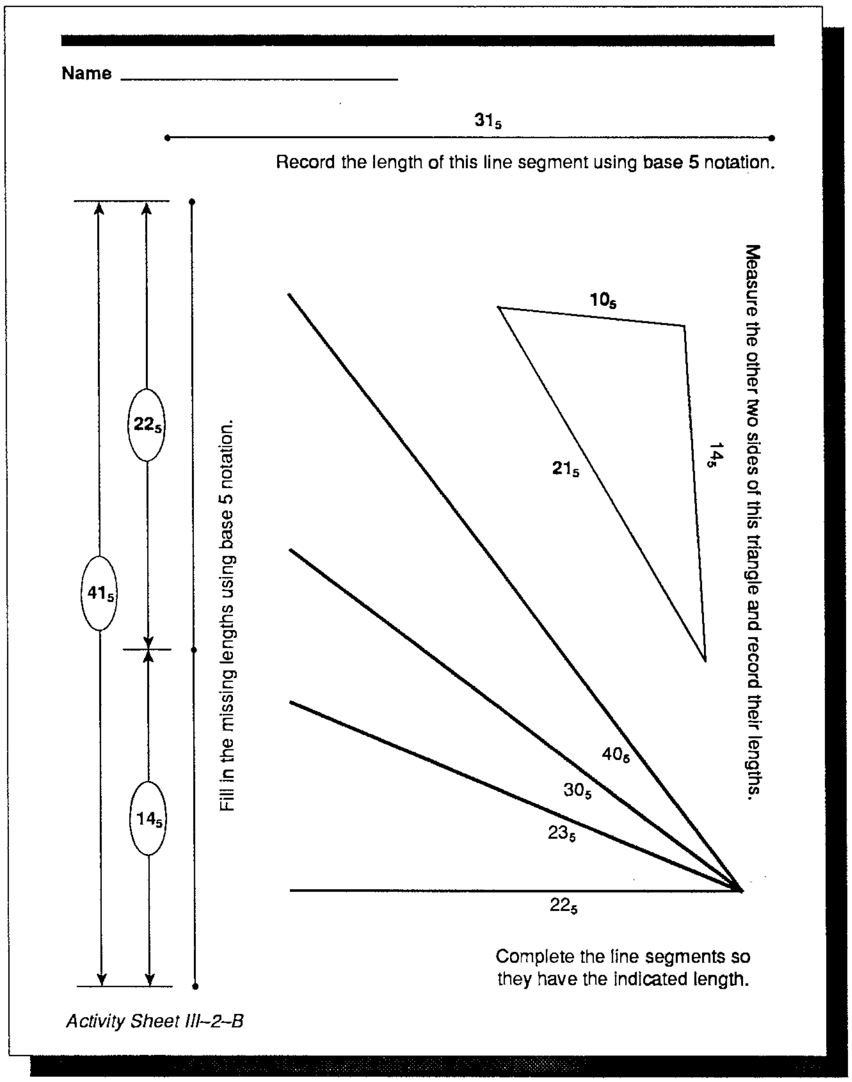
5. Ask the students to record the lengths of the line segments on Activity Sheet A in base 5 notation.

Comments

5. The length of segment A is 2 chains and 2 units or 22_5. The lengths of line segments B through F, respectively, are 4_5, 30_5, 23_5, 31_5 and 20_5.

Actions

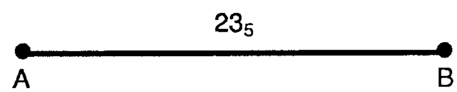
6. Discuss with the students ways of indicating the length of a line segment. Then distribute copies of Activity Sheet B and ask the students to complete the activities on the sheet.



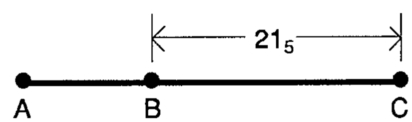
Comments

6. The students can measure lengths with an edge of a strip-mat. However, you may want to provide them with base 5 rulers. The rulers can be prepared by copying the attached master on cardstock and cutting on the heavy lines. Using base 5 rulers to measure length, rather than an edge of a strip-mat, helps students distinguish between measures of length (chain, chain-chain, etc.) and measures of area (strip, mats, strip-mats, etc.).

Sometimes the length of a line segment is simply written alongside the segment.



If there is danger of confusion, the length can be written between two arrows showing the extent of the segment.



In this situation, arrows are used to indicate that 21_5 is the length of segment BC, and not the length of the entire segment AC.

The completed sheet should resemble the one at left. Note that if the line segments are completed correctly, their endpoints lie on a line.

Actions

7. (Optional.) Ask students to construct a base 5 measuring tape and use it to measure and record the lengths of various items in the classroom.

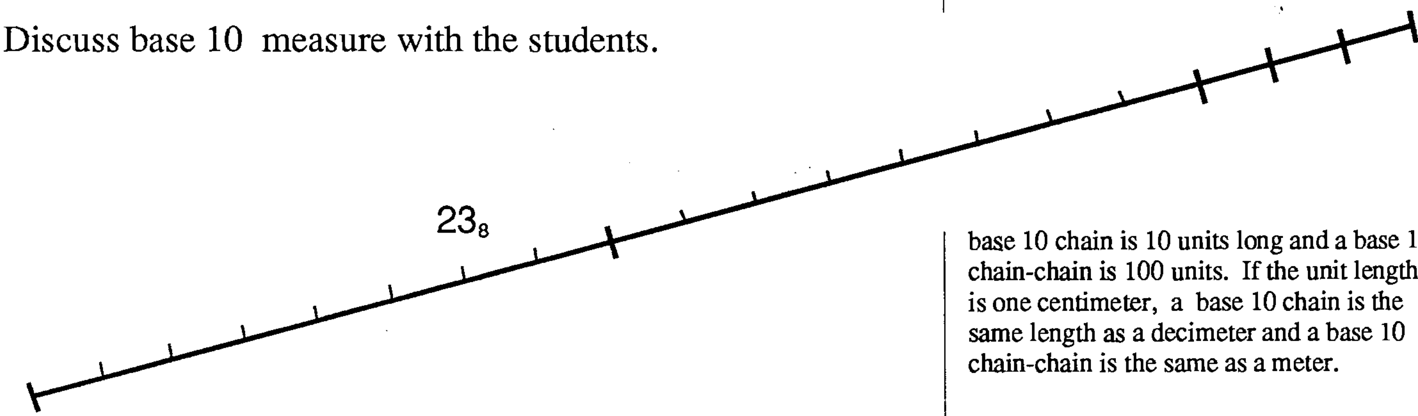
Comments

7. Included with this activity is a pattern for constructing a base 5 measuring tape. To construct a tape, one needs a copy of this pattern, scissors and scotch tape. The length of the tape is one chain-chain-chain (1000_5), which is 125 units.

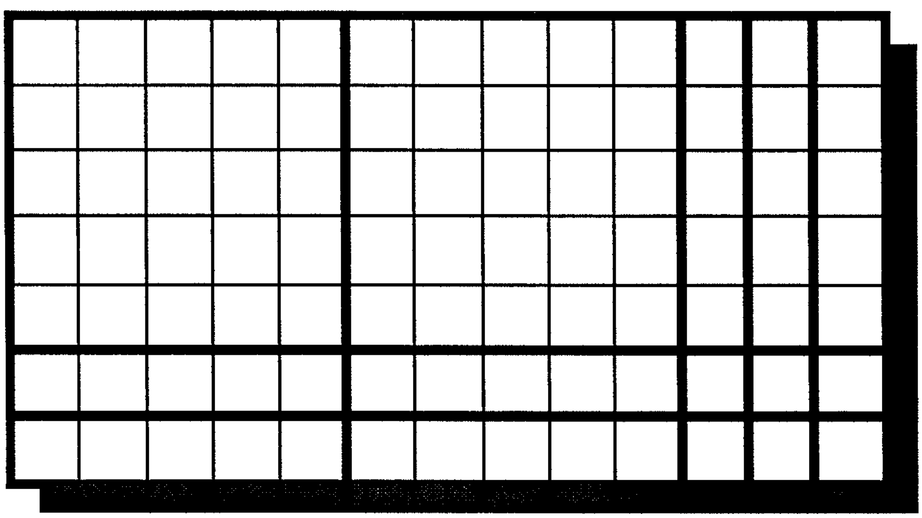
Actions

8. Ask the students to determine the number of unit lengths in a base 8 chain and in a base 8 chain-chain. Have them draw a line segment whose length is 23_8 and determine the number of unit lengths in the segment.

Discuss base 10 measure with the students.



9. Place the following rectangle of base 5 pieces on the overhead. Ask the students how the area and dimensions of this rectangle would be recorded in base 5 notation.



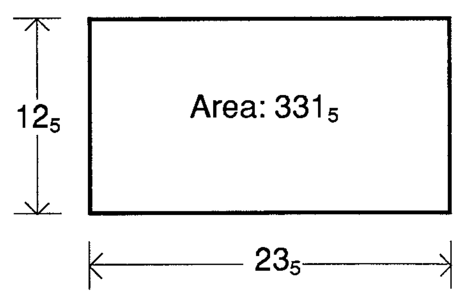
Comments

8. A base 8 chain contains 8 unit lengths. A base 8 chain-chain contains 8 chains which totals 64 unit lengths. A segment of length 23_8 is 2 (base 8) chains and 3 units long which totals 2x8 + 3 or 19 units. A base 10 chain is 10 units long and a base 10 chain-chain is 100 units. If the unit length is one centimeter, a base 10 chain is the same length as a decimeter and a base 10 chain-chain is the same as a meter.

9. The area of the rectangle is the number of unit squares it contains. Its dimensions are the lengths of its different sides.

The rectangle is comprised of 2 mats, 7 strips and 6 units. A collection which totals the same number of units in the fewest pieces contains 3 mats, 3 strips and 1 unit (5 of the units can be traded for 1 strip and 5 of the strips can be traded for 1 mat). Hence the area of the rectangle is 331_5 unit squares. The length of the longest dimension is 2 chains and 3 units or 23_5 unit lengths. The other dimension is 1 chain and 2 units or 12_5 unit lengths.

This information may be shown in a sketch as follows:

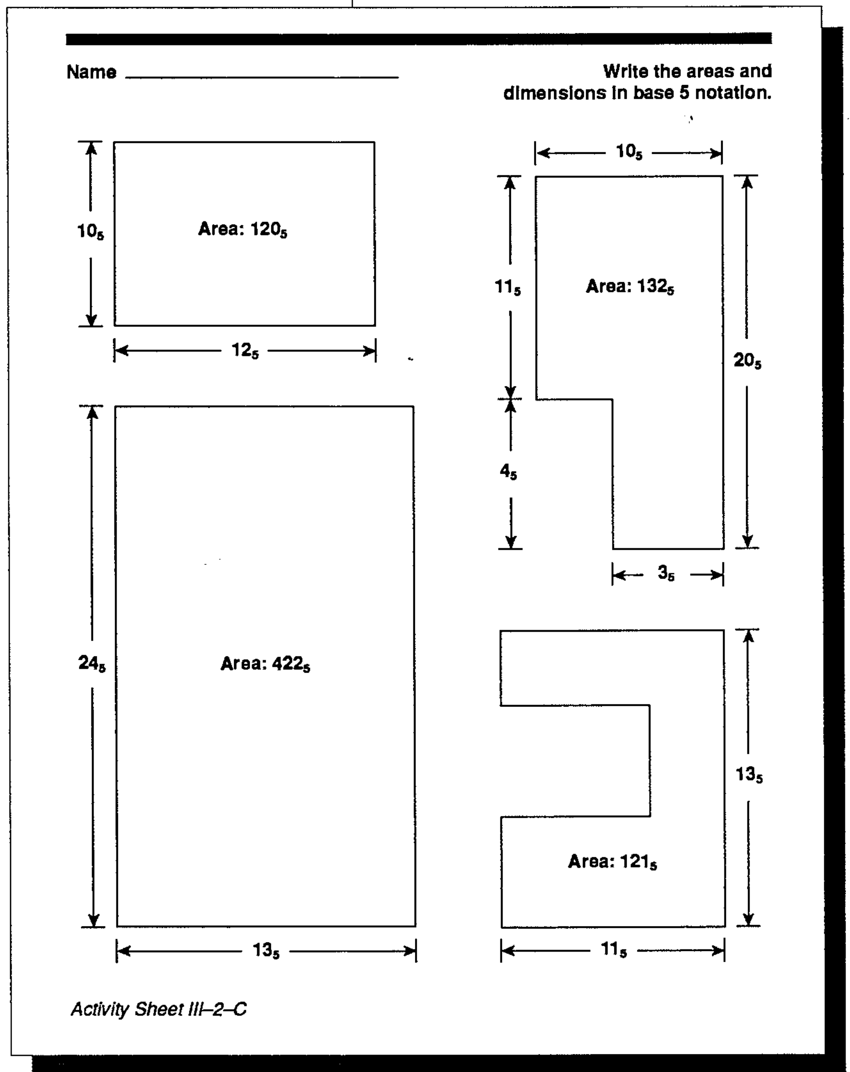


Actions

10. Distribute a copy of Activity Sheet C to each student and ask them to fill in the dimensions and areas as indicated. Discuss with the students how they arrived at their answers.

Comments

10. Following is a completed sheet:



Students may use various methods to arrive at their answers. Some may fill in the areas with base 5 pieces and then make exchanges to arrive at minimal collections. Others may mark off regions of the figures which are equivalent to different number pieces.

Students can be asked to demonstrate their methods on the overhead using a transparency of the activity sheet and transparencies of base 5 pieces.

Actions

11. Ask the students to construct the following rectangles with base 5 pieces and provide the information requested using base 5 notation.

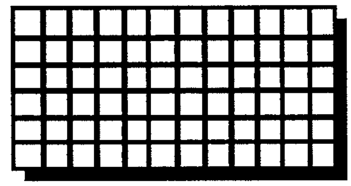
(a) A rectangle whose area is 242_5. Record its dimensions.
(b) A square whose side has length 13_5. Record its area.
(c) A rectangle with dimensions 32_5 and 13_5. Record its area.
(d) A rectangle with area 134_5 and one dimension 4_5. Record its other dimension.
(e) A rectangle with area 1341_5 and one dimension 23_5. Record its other dimension.

Discuss with the students how they arrived at their answers.

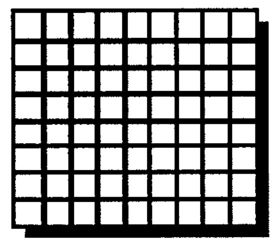
Comments

11. Students may demonstrate their construction of the rectangles on the overhead using transparencies of base 5 pieces.

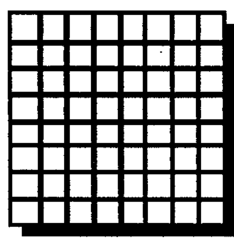
(a) Here is one rectangle. Its dimensions are 22_5 and 11_5.



Other rectangles of this area are also possible. By trading one mat for five strips and two strips for ten units, the following rectangle can be formed. Its dimensions are 13_5 and 14_5.

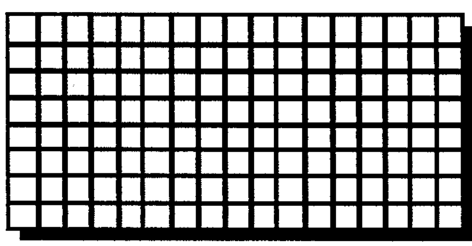


(b) Here is a square whose side has length 13_5.

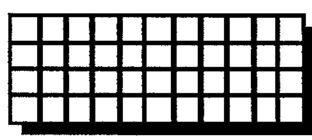


The pieces forming this square are equivalent to 2 mats, 2 strips and 4 units. Hence its area is 224_5.

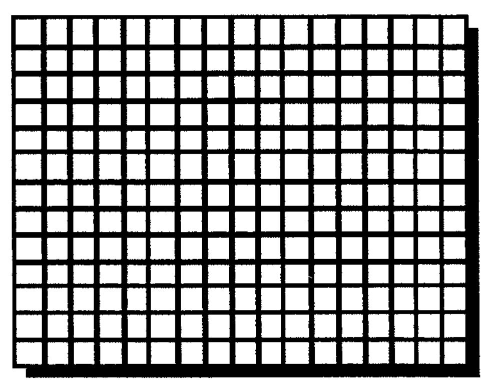
(c) This rectangle has area 1021_5.



(d) The other dimension is 21_5. In order to construct this rectangle, one must trade the mat for 5 strips.



(e) One must make a number of trades to construct this rectangle. Its other dimension is 32_5.

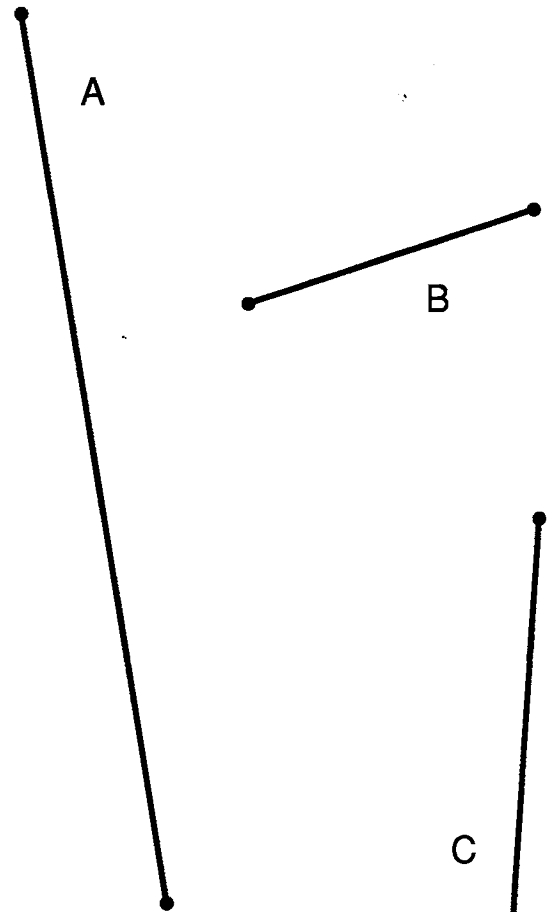


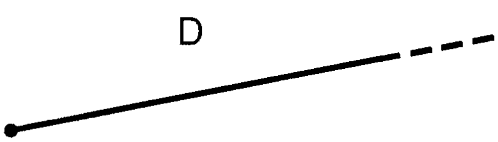
Name ______________________________

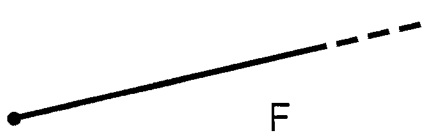
1. Complete the table.

2. Finish drawing segments D, E and F so their length are those given in the table.

Line Segment	LENGTH Chains	Units	Total Units of Length
A	____	____	____
B	____	____	____
C	____	____	____
D	2	3	____
E	____	1	16
F	2	____	10

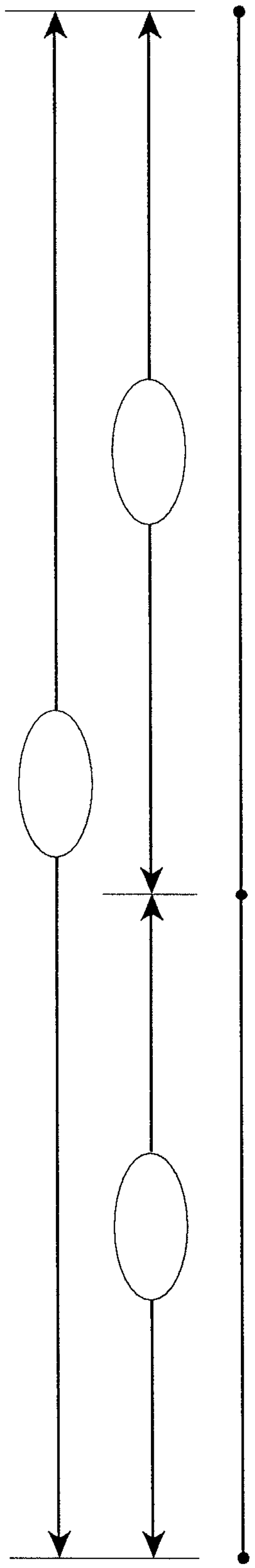






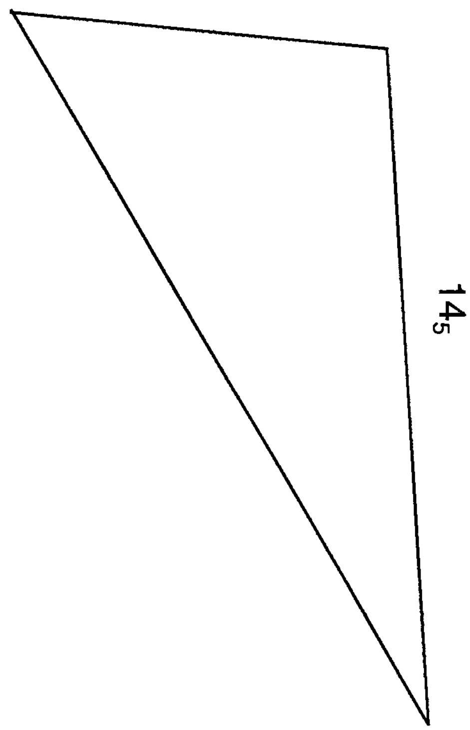
Name ______________________________

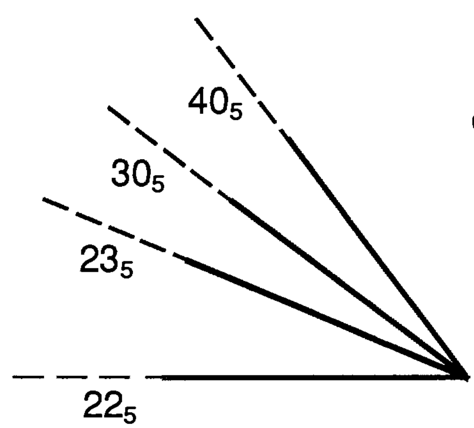
Record the length of this line segment using base 5 notation.



Fill in the missing lengths using base 5 notation.

Measure the other two sides of this triangle and record their lengths.

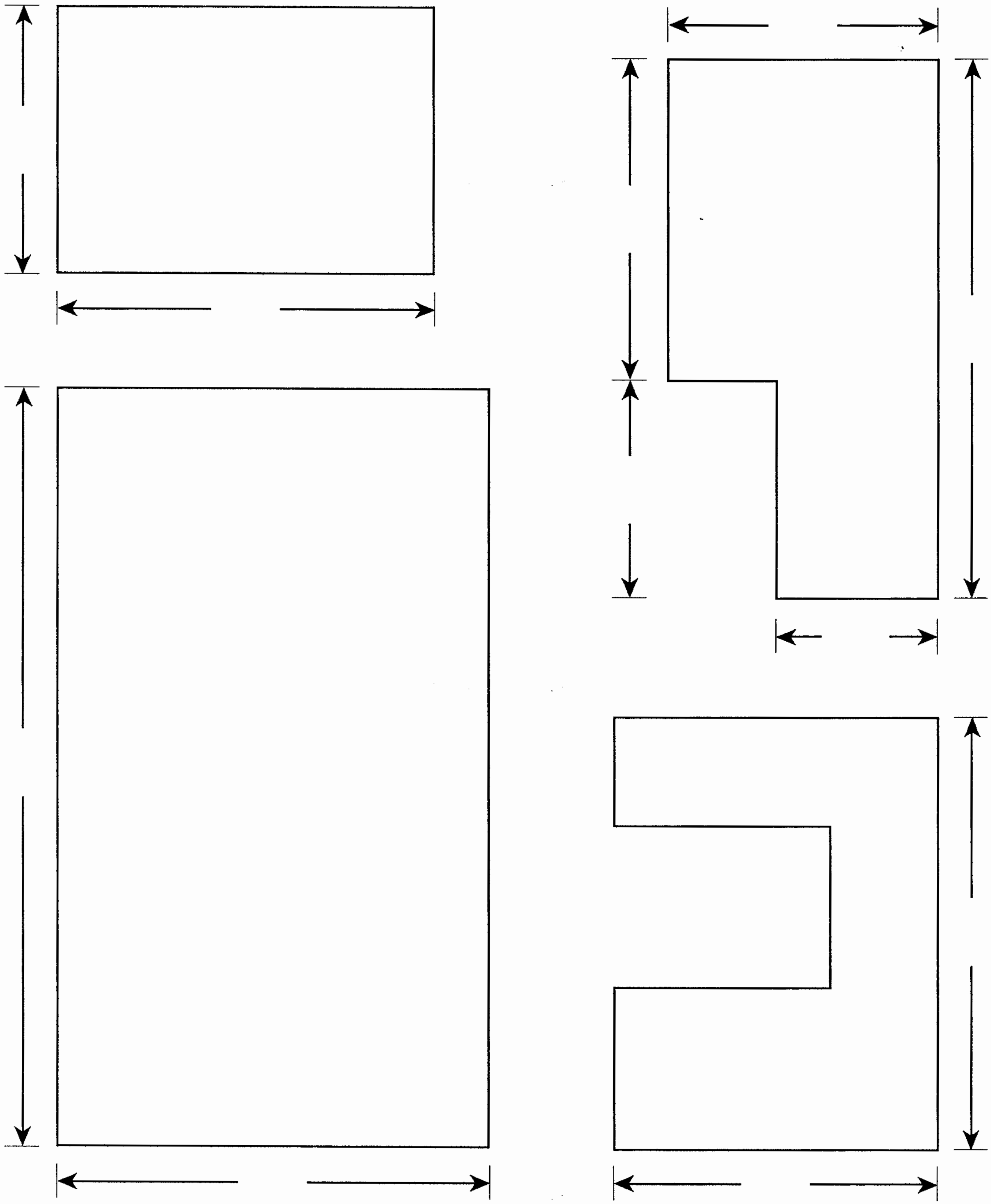




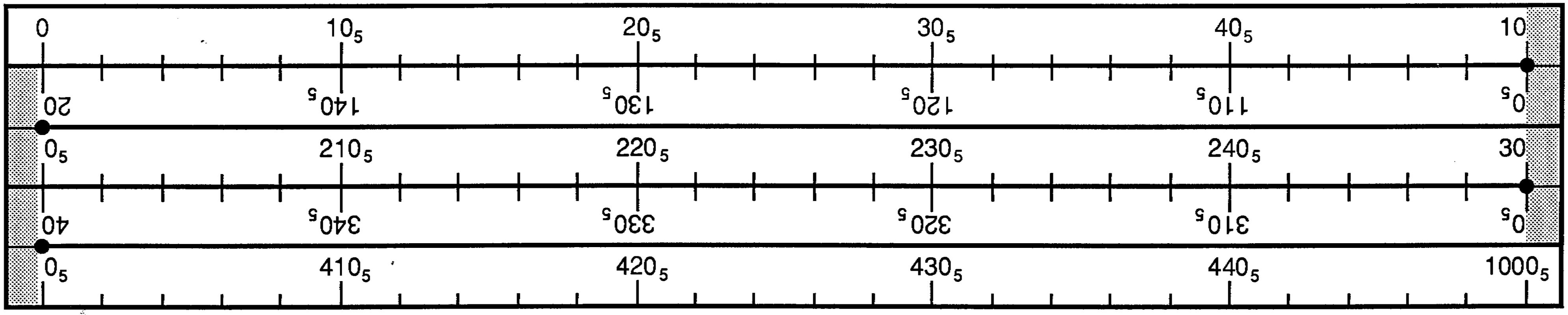
Complete the line segments so they have the indicated length.

Name ______________________

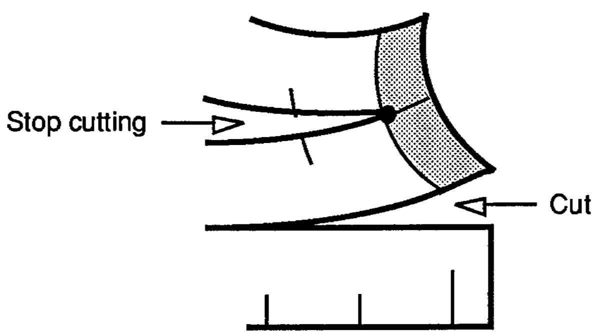
Write the areas and dimensions in base 5 notation.



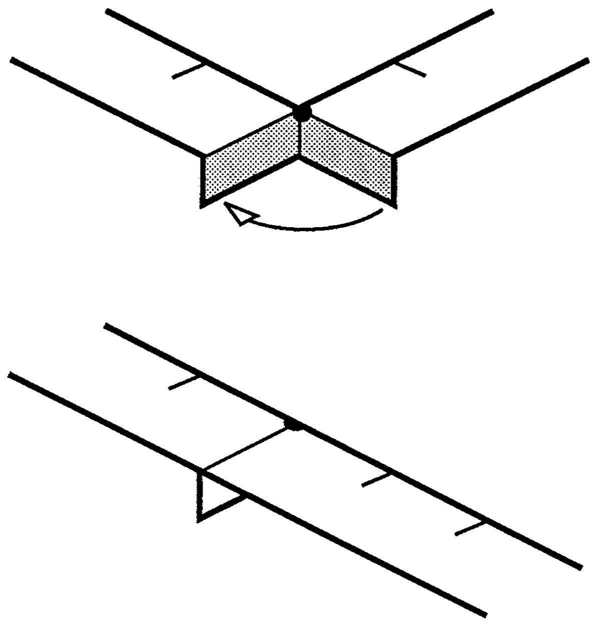
Pattern for Base 5 Measuring Tape



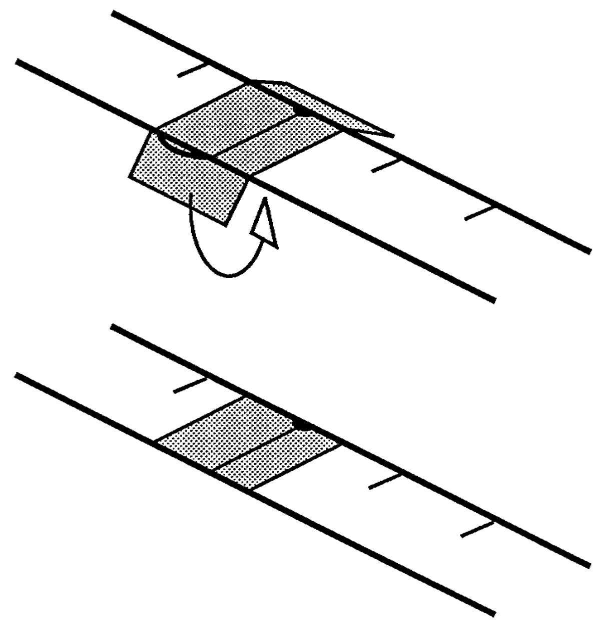
1. Cut along all heavy lines.



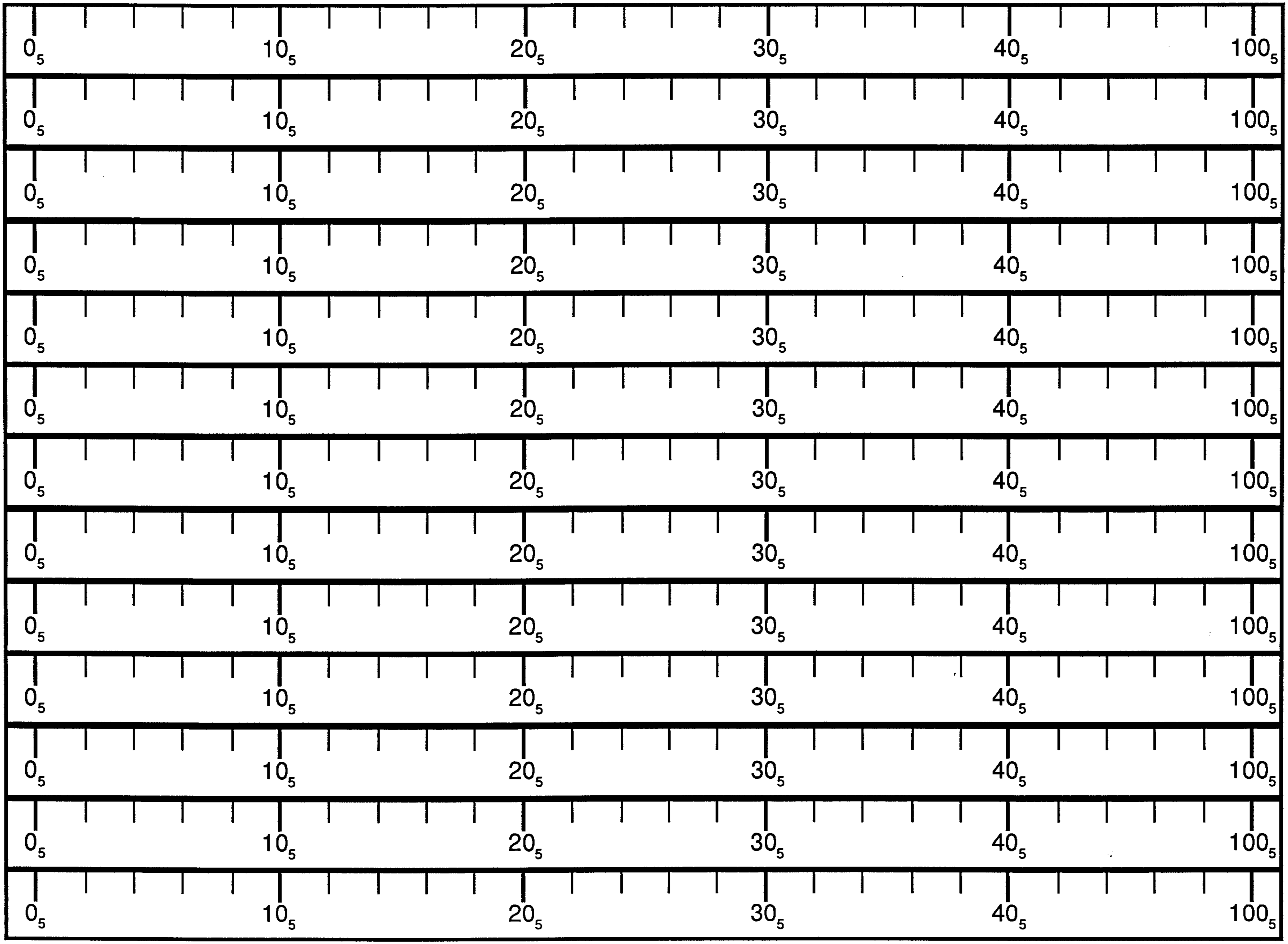
2. Fold in shaded areas:



3. Flatten tab and wrap connection with scotch tape:



Master for Base 5 Rulers



Arithmetic with Number Pieces

O V E R V I E W

Number pieces are used to perform arithmetical computations. The emphasis is placed on ways of viewing the arithmetical operations rather than the development of paper-and-pencil algorithms.

Prerequisite Activity

Unit II, Activity 1, *Basic Operations;* Unit III, Activity 1, *Grouping and Numeration;* Unit III, Activity 2, *Linear Measure and Dimensions*

Materials

Base 5 number pieces and base 5 grids, transparencies of base 5 pieces for use on the overhead (see Comment 1, Unit 3, Activity 1, *Grouping and Numeration*)

Actions

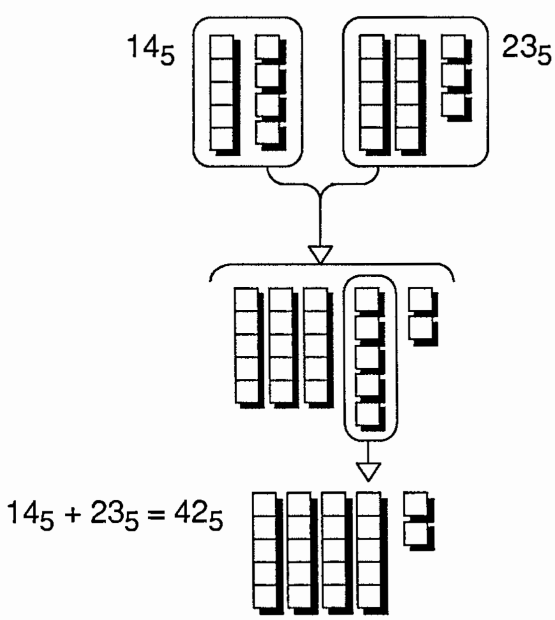
1. Distribute base 5 number pieces to each student or group of students.

2. Write the following arithmetical expressions on the chalkboard:

(a) $14_5 + 23_5$ (b) $224_5 + 343_5$

(c) $301_5 - 134_5$ (d) $1124_5 - 321_5$

Have the students devise ways of performing the indicated computations by manipulating base 5 number pieces, using paper and pencil to only record answers. Discuss.



Comments

1. Each student or group of students should have a supply of 8 mats, 15 strips and 20 units.

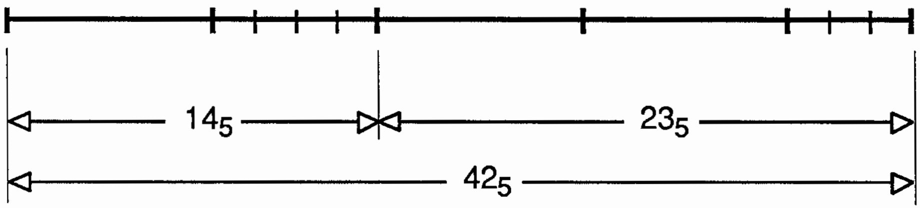
2. Circulate among the students, offering hints as appropriate (see Unit II, Activity 1, *Basic Operations* for ways of viewing arithmetical operations). Encourage the students to discuss with each other ways of performing the computations.

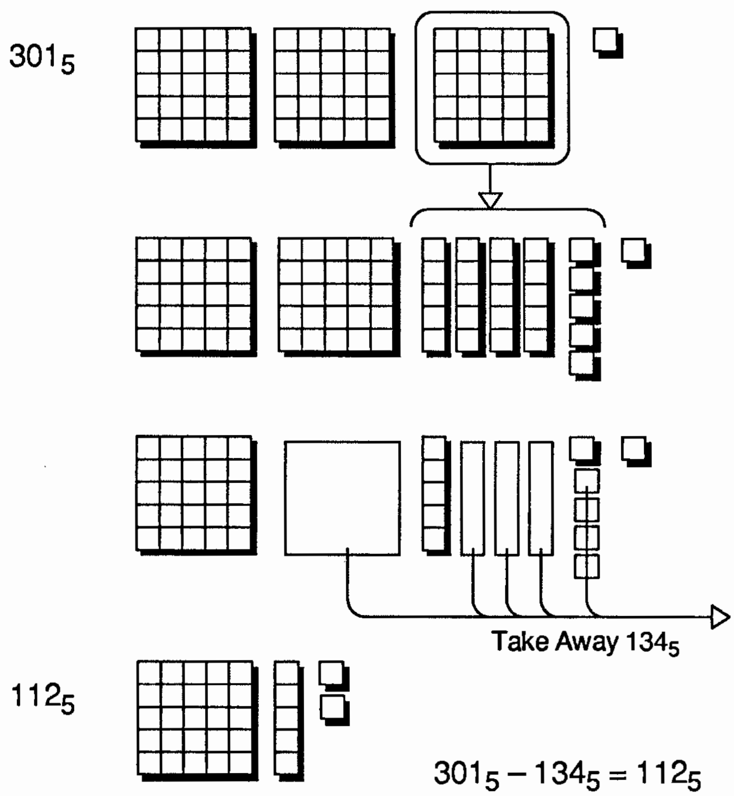
You may wish to ask volunteers to show how they did the computations. This can be done on the overhead using transparencies of base 5 pieces.

(a) An addition may be performed by combining collections of base 5 pieces, and then converting this combined collection to an equivalent collection containing a minimum number of pieces. See the diagram.

Continued next page.

Actions





Comments

2. (a) *Continued.* The addition in part (a) can also be performed using lengths. Successive lengths of 14_5 and 23_5 are marked off on a line and their total length measured to find the sum (measurements can be made with the edge of a strip-mat).

(b) Combining collections for 224_5 and 343_5, and then making exchanges, results in a collection of 1 strip-mat, 1 mat, 2 strips and 2 units. Hence $224_5 + 343_5 = 1122_5$.

(c) Here are two ways of doing this subtraction with number pieces:

In the "take-away" method, after making exchanges, a collection for 134_5 is taken away from a collection for 301_5.

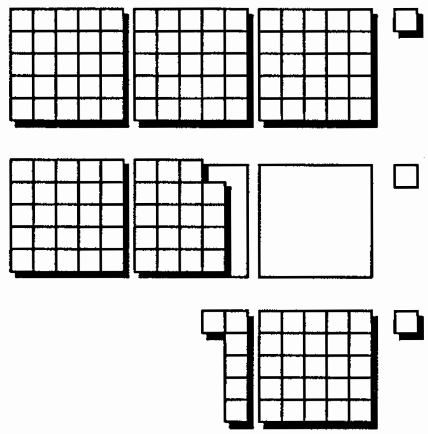
Continued next page.

Actions

301_5

134_5

112_5



3. Repeat Action 2 for the following computations:

(a) $3_5 \times 142_5$ (b) $24_5 \times 13_5$

(c) $311_5 \div 3_5$ (d) $314_5 \div 24_5$

Comments

2 (c). *Continued.*

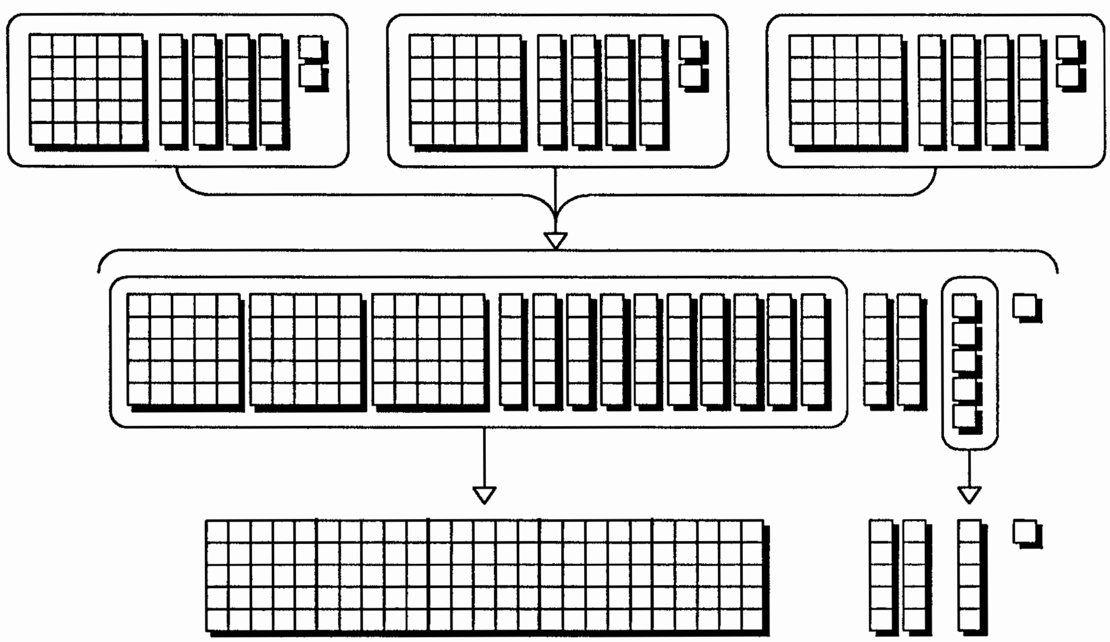
In the "difference" method, a collection is found that will make up the difference between collections for 301_5 and 134_5.

(d) The methods of part (c) may be used to obtain $1314_5 - 421_5 = 343_5$.

3. (a) This product may be found by "repeated addition", i.e., combining 3 collections for 142_5 and then making exchanges. See the diagram below.

3 groups of 142_5

1021_5



$3_5 \times 142_5 = 1031_5$

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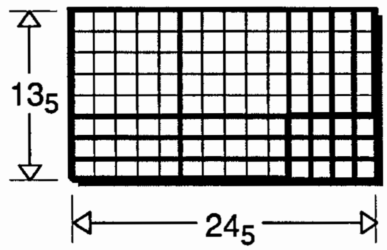
Actions

Comments



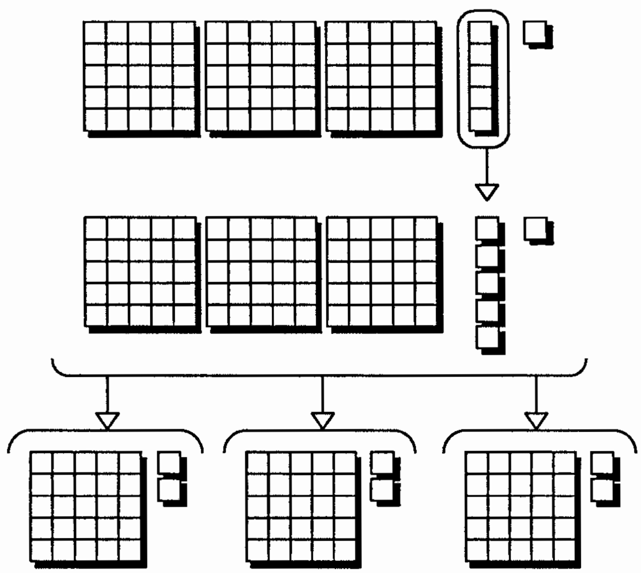
$24_5 \times 13_5 = 422_5$

(b) One way to compute this product is to form a rectangle whose dimensions are the numbers being multiplied. The area of this rectangle is the desired product.

Pieces in the rectangle can be exchanged to obtain 4 mats, 2 strips and 2 units.

Base 5 rulers (see Comment 6, Unit III, Activity 2, *Linear Measure and Dimension*) may help student determine the dimensions of the rectangles they form.

311_5

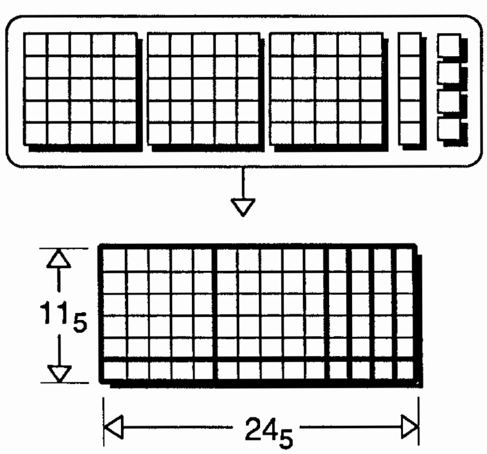


3 groups of 102_5

$311_5 \div 3_5 = 102_5$

(c) To find this quotient, a collection for 311_5 may be divided into 3 groups, making exchanges as necessary.

314_5



(d) This quotient can be found by taking a collection for 314_5 and arranging its pieces into a rectangle with one dimension 24_5 (to obtain the rectangle shown, one mat must be exchanged for 5 strips). The other dimension is the desired quotient.

Notice that this divides 311_5 into 24_5 groups, each group being a column of the rectangle. The number of objects in each group is the number of rows.

This method may be adapted to any division.

Actions

4. Ask the students to perform additional computations, as necessary, to become familiar with ways of computing with base 5 pieces.

5. Distribute base 5 grid paper to each student. Ask the students to compute $12_5 \times 23_5$ by sketching a rectangle whose dimensions are 12_5 and 23_5 and finding its area.

Comments

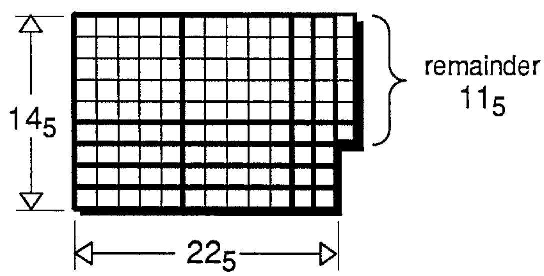
4. Computations may be selected from the following:

(a) $1021_5 + 324_5$ (d) $20_5 \times 41_5$
(b) $1314_5 - 421_5$ (e) $1232_5 \div 22_5$
(c) $33_5 \times 22_5$ (f) $424_5 \div 14_5$

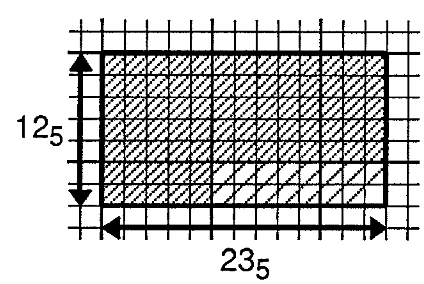
Answers:

(a) 1400_5 (d) 1320_5
(b) 343_5 (e) 31_5
(c) 1331_5 (f) 22_5 w/ rem. 11_5

Notice the remainder in (f). If a collection of pieces equivalent to 4 mats, 2 strips and 4 units is arranged into a rectangular array in which one dimension is 14_5, the other dimension is 22_5 with 1 strip and 1 unit left over. (In forming the rectangle below, 2 mats were exchanged for 10 strips and one strip was exchanged for 5 units.)



5. A master for base 5 grid paper is included with this activity. Pencil sketches show up better on dittoed copies than on blackline copies.



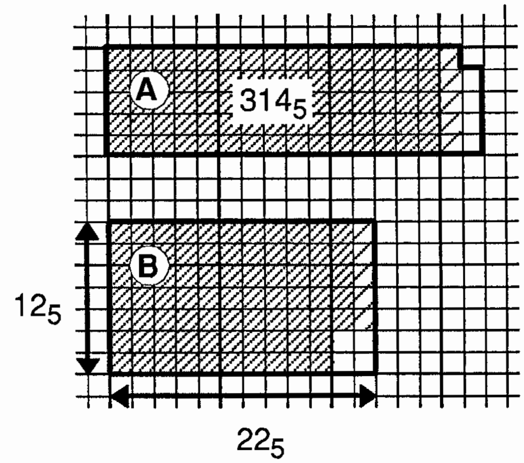
The darker shaded region is equivalent to 3 mats, the lighter shaded region is equivalent to 3 strips and the unshaded region is 1 unit. Hence the area is 331_5. Since the area of the rectangle is the product of its dimensions, $12_5 \times 23_5 = 331_5$.

Actions

6. Ask each student to enclose a region, on base 5 grid paper, whose area is 314_5. Then ask them to sketch a rectangle which has the same area and has one dimension equal to 12_5. Have the students use their completed sketches to compute $314_5 \div 12_5$.

Comments

6. A region A of area 314_5 can be obtained by enclosing 3 mats (the darker shaded portion), 1 strip (the lighter shaded portion and 4 units (the unshaded portion). This amount of area can be redistributed into a rectangle B with one dimension 12_5 as shown in the following sketches. The other dimension of the rectangle is 22_5. Hence $314_5 \div 12_5 = 22_5$.



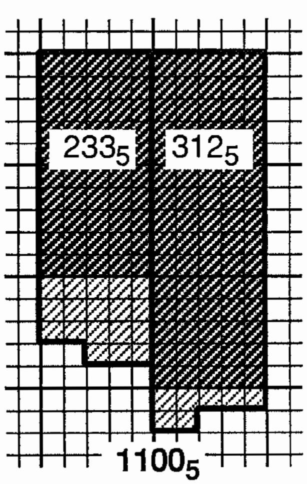
7. Ask the students to make sketches on base 5 grid paper to help them do the following computations:

(a) $233_5 + 312_5$
(b) $321_5 - 143_5$
(c) $32_5 \times 24_5$
(d) $404_5 \div 13_5$

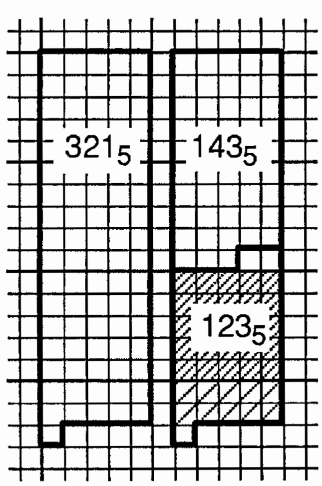
7. In the following sketches, each dark-shaded region is equivalent to a strip mat, each medium-shaded region can be converted into one or more mats and each light-shaded region can be converted into strips.

(a) $233_5 + 312_5 = 1100_5$
(b) $321_5 - 143_5 = 123_5$
(c) $32_5 \times 24_5 = 1423_5$
(d) $404_5 \div 13_5 = 23_5$

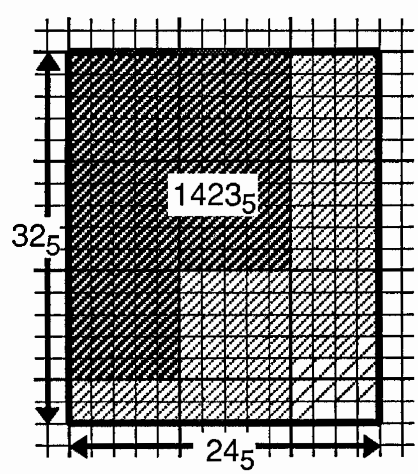
A



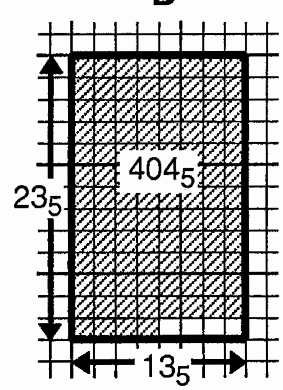
B



C



D



Base 5 Grid Paper

Base 10 Numeration

O V E R V I E W

Base 10 number pieces are used to provide a model for place value.

Prerequisite Activity

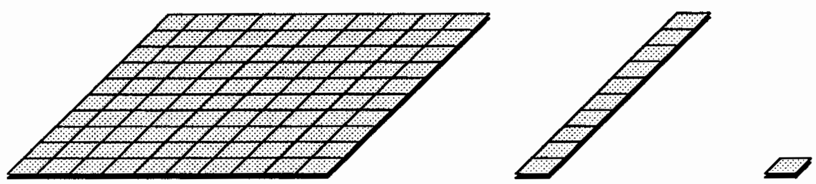
None is required but Unit III /Activity 1, *Grouping and Numeration*, is helpful

Materials

Base 10 number pieces and transparencies of base 10 number pieces for use on the overhead (see Comment 1)

Actions

1. Distribute the base 10 number pieces to each student. Discuss the relationship among the pieces and the value of each piece.



Mat Strip Unit

Comments

1. Base 10 number pieces can be made by copying the last page of this activity on tagboard and cutting along the indicated lines. Copies can be made on transparency film and the individual pieces cut out for use on the overhead.

Each student should have at least 13 units, 8 strips and 4 mats. Then, if the students work in small groups, each group will have enough number pieces for each activity. You may wish to have some extra units available for Action 2 (see comment 2).

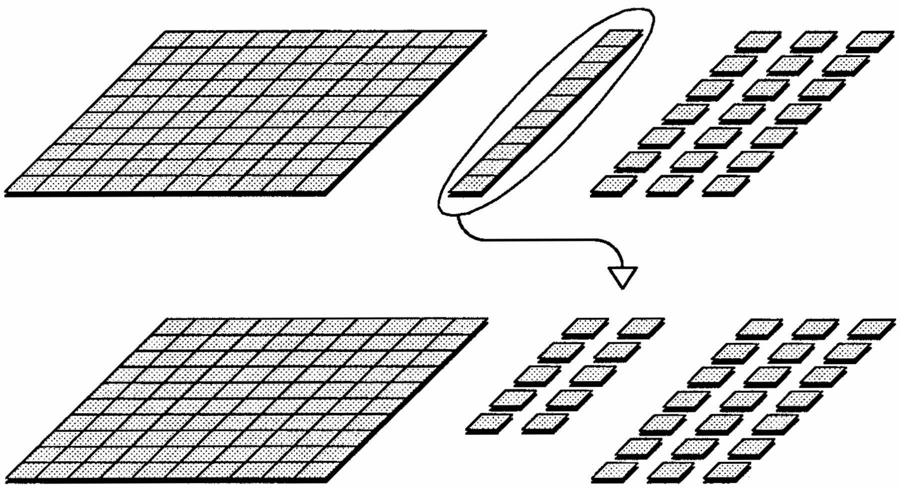
Ten *unit squares*, or simply *units*, arranged in a row form a *strip*. Ten strips side-by-side form a *mat*. Each mat contains 100 units.

In this activity the assumption is made that students know how to count but may not understand the place value nature of our numeration system. For example, they may know how to count one hundred twenty-four objects and even be able to write the symbol 124. They may not, however, view 124 as 1 group of one hundred, 2 groups of ten and 4 units.

Actions

2. Place the following collection of base 10 number pieces on the overhead: 1 mat, 1 strip and 21 units. Point out that altogether this collection contains 23 base 10 number pieces. Make a chart, like the one below, on the chalkboard or overhead and record the information about this collection on the first line of the chart.

Trade the 1 strip for 10 units and record the resulting collection on the second line of the chart.



Ask the students to copy the chart and to add to their chart by making more equal exchanges and recording each result.

mats	strips	unit	Total Number of Pieces
1	1	21	23
1	0	31	32

Have the students help you make a master list of different collections.

Comments

2. For this activity you may want the students to work in groups. Many of the exchanges require a large number of units. Therefore, once students are familiar with the notion of making exchanges, they may want to imagine the exchanges taking place rather than physically carrying them out.

There are 18 different collections that can be listed in the chart. The asterisk marks the collection with the fewest number of base 10 pieces.

Mats	Strips	Units	Total Number of Pieces
1	1	21	23
1	0	31	32
1	2	11	14
1	3	1	5*
0	13	1	14
0	12	11	23
0	11	21	32
0	10	31	41
0	9	41	50
0	8	51	59
0	7	61	68
0	6	71	77
0	5	81	86
0	4	91	93
0	3	101	104
0	2	111	113
0	1	121	122
0	0	131	131

Actions

3. Discuss the completed chart. Ask the students for their observations.

Point out that all 18 equivalent collections in the chart represent the number 131, and that the collection which uses the fewest number of base 10 pieces is called the *minimal collection* for the number 131.

Comments

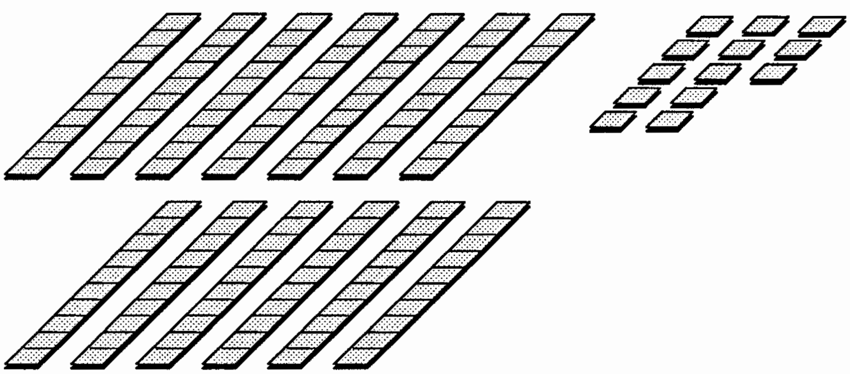
3. Discussion can be prompted by asking questions like: What do the different collections in the chart have in common? Would any one of the collections of pieces cover more area than another?

It is important to see that each collection has the same value. This can be expressed in different ways: if all of the number pieces in each collection were exchanged for units, all collections would contain the same number of units — in this case 131; each collection of pieces covers the same area; or, while making exchanges, the number of base 10 pieces changes but the amount of material remains the same.

Note that the minimal collection for a number will never have an entry larger than 9 in any column of the chart because every group of 10 number pieces, of the same kind, can be traded for the next larger size.

Actions

4. Have each student, or group of students, form a collection of 13 strips and 13 units. Ask them to make the minimal collection for the number represented by this set of number pieces.



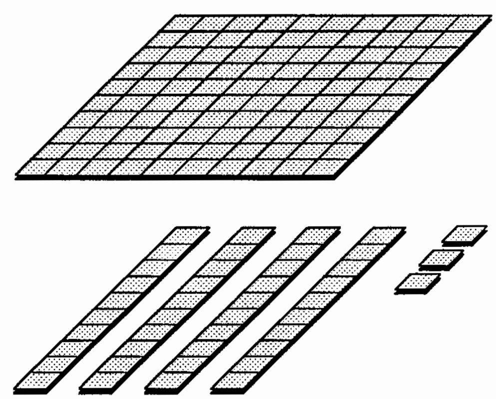
Have the students form minimal collections for the the numbers represented by the following sets of number pieces.

a) 1 mat, 12 strips
b) 14 strips, 15 units
c) 22 units
d) 9 mats, 9 strips, 9 units
e) 1 mat, 10 strips, 2 units
f) 100 units

Comments

4. The minimal collection for the number represented by 13 strips and 13 units is 1 mat, 4 strips and 3 units.

The minimal collections are:



	mat	strips	units
a)	2	2	0
b)	1	5	5
c)	0	2	2
d)	9	9	9
e)	2	0	2
f)	1	0	0

Actions

5. Put the collection consisting of 13 strips and 13 units on the overhead again. Ask the students to determine the total number of units represented by these pieces if all pieces are exchanged for units. Compare this result with the minimal set obtained in Action 4. Discuss the results.

Repeat this action with other collections of pieces from Action 4.

6. Hold up a large handful of unit squares and tell the students that you have two hundred thirty-seven altogether. Ask them to imagine what the minimal collection for this number of units would be and to represent this minimal collection with their base 10 number pieces. Discuss.

Repeat with some other numbers.

7. (Optional) Display the set of pieces consisting of 9 mats, 13 strips and 11 units and ask the students for the minimal collection for the number represented by this set.

Discuss with students what larger base 10 number pieces might look like. Provide names for these pieces, build or sketch diagrams of them, and determine their values.

Comments

5. With exchanges, the 13 strips and 13 units total of 143 units. The minimal set consists of 1 mat, 4 strips and 3 units. Because a mat is a group of 100 and a strip is a group of ten, the symbol 143 that arose from counting units, can also be viewed as 1 group of 100, 4 groups of 10 and 3 units.

6. This Action is intended to help students think of numbers (and represent them) in terms of place value.

After the students have represented a few numbers with their number pieces, you may want to ask them to draw diagrams of the number pieces which represent selected numbers.

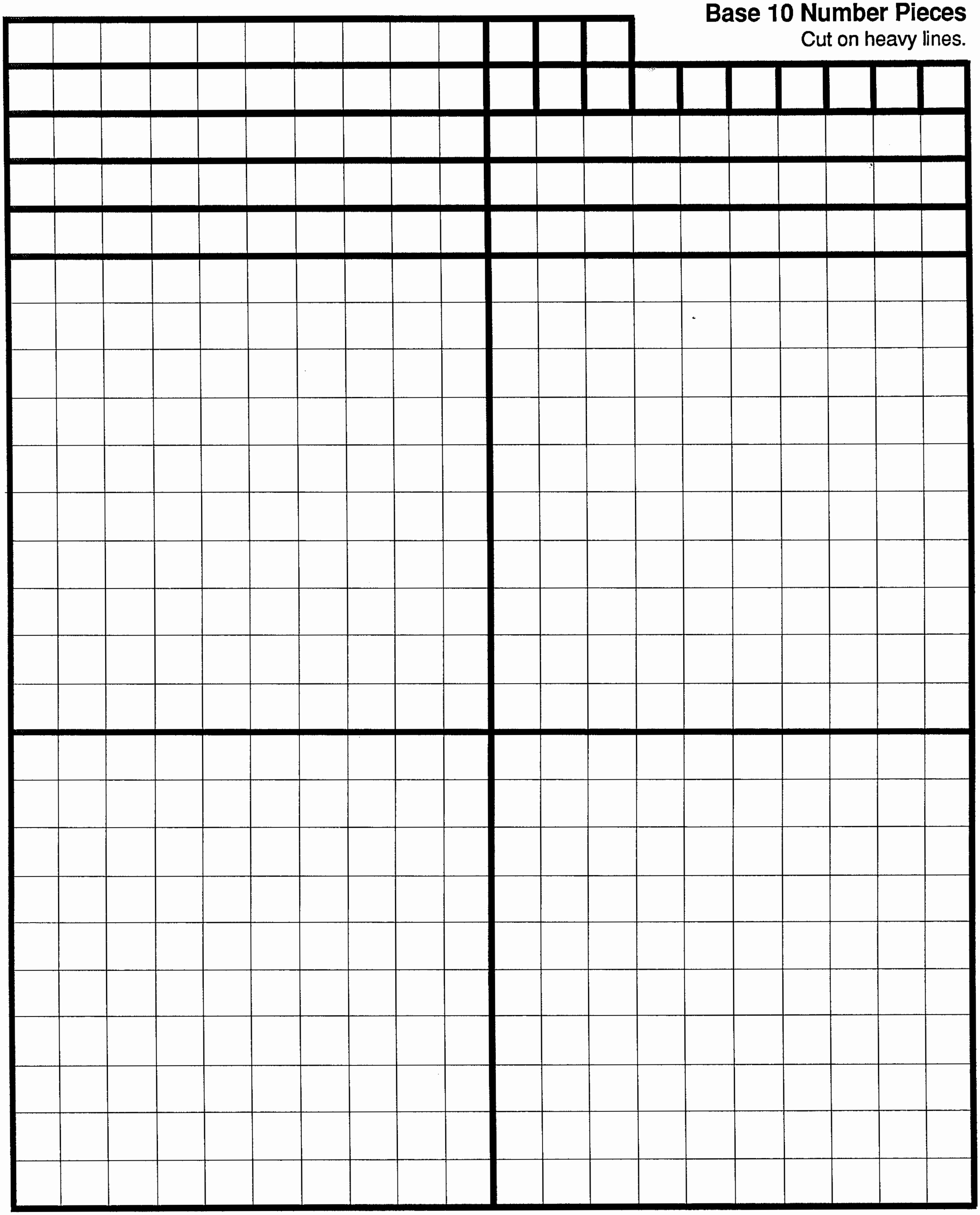
You may wish to extend this action by asking students to imagine what a collection of 237 units would look like if they exchanged as many units as possible for strips. It will be valuable for student to be able to visualize 237 as 2 mats, 3 strips and 7 units or 23 strips and 7 units or 237 units.

7. The minimal collection might appear to be 10 mats, 4 strips and 1 unit. This seems unsatisfactory because it requires more than 9 pieces of the same kind. Creating a larger base 10 number piece solves this problem.

The base 10 number piece model extends to higher powers of 10. An oblong piece consisting of 10 mats in a row is called a *strip-mat* (it represents 1000). A square formed from 10 strip-mats represents 10,000 and is a mat of mats or, simply, *mat-mat*. This process of creating base ten pieces can be continued indefinitely. Ten mat-mats can be grouped to form a *strip-mat-mat* (100,000), ten strip-mat-mats grouped to form a *mat-mat-mat* (1,000,000), etc.

Base 10 Number Pieces

Cut on heavy lines.



Base 10 Addition and Subtraction

O V E R V I E W

Base 10 number pieces are used to develop an understanding of the operations of addition and subtraction of multidigit numbers.

Prerequisite Activity

Unit II, Activity1, *Basic Operations*; Unit III, Activity 4, *Base 10 Numeration*.

Materials

Base 10 number pieces, transparencies of base 10 pieces for use on the overhead (see Comment 1 in Unit III, Activity 4, *Base 10 Numeration*)

Actions

1. Distribute base 10 pieces to each student or group of students.

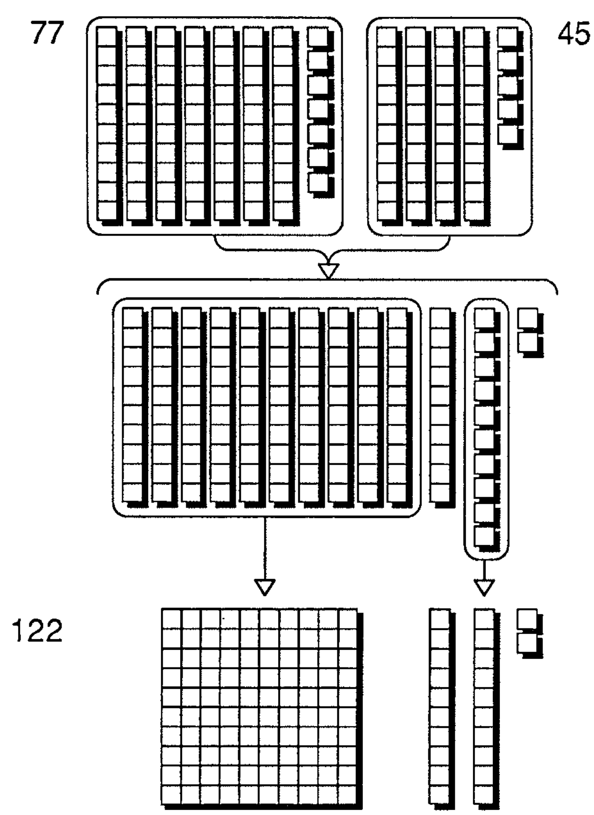
2. Ask the students to use base 10 number pieces to find the sum of 77 and 45. Discuss.

Comments

1. Each student or group of students should have about 5 mats, 15 strips and 15 units.

2. You may want to ask volunteers to demonstrate their methods on the overhead.

One way to find the sum is to combine number piece collections for 77 and 45 and then make exchanges:



77 + 45 = 122

Actions

3. Ask the students to use their base 10 pieces to find these sums:

133 + 18 156 + 229 46 + 27 + 88

Observe the students' methods and discuss.

Comments

3. Some students may find a sum without exchanging pieces (e.g. a student may think of 10 units as 1 strip without physically making the exchange). Also, students may vary the order in which they make exchanges. There is no particular order in which exchanges ought to be made.

4. Show the students how the process of finding sums by combining collections of number pieces and making exchanges can be recorded with the use of a place value table. Discuss options for doing this.

4. There is no standard procedure for using a place value table to record the steps in finding a sum using number pieces. Some possibilities for the sum 264 + 378 are given below. Students may wish to create there own ways of recording.

If 264 + 378 is determined by first combining minimal collections for 264 and 378 and then making exchanges, beginning with the units, the process might be recorded thus:

M	S	U	
2	6	4	(minimal collection for 264)
+ 3	7	8	(minimal collection for 378)
5	13	12	(combined collection)
5	14	2	(combined collection after exchanging 10 units for 1 mat)
6	4	2	(combined collection after exchanging 10 strips for 1 strip)

If exchanges are made as individual pieces are combined, starting with the units, the recording might look as follows:

M	S	U	
2	6	4	(minimal collection for 264)
+ 3	7	8	(minimal collection for 378)
	1	2	(result of combining units and exchanging 10 for 1 strip)
1	4		(result of combining strips and exchanging 10 for 1 mat)
6			(result of combining mats)

If, on the other hand, exchanges are made as individual pieces are combined, starting with the mats, the recording might be:

M	S	U	
2	6	4	(minimal collection for 264)
+ 3	7	8	(minimal collection for 378)
5			(result of combining mats)
1	3		(result of combining strips and exchanging 10 for 1 mat)
	1	2	(result of combining units and exchanging 10 for 1 strip)
6	4	2	(minimal collection for sum)

It is usually more efficient to begin combining pieces at the right (with the units) and proceed to the left, than to begin at the left and proceed to the right.

Actions

5. (Optional) Explain how paper and pencil procedures for addition can be described in terms of combining number pieces and making exchanges.

6. Write the arithmetical expression 82 – 47 on the overhead or chalkboard. Ask the students to use their base 10 pieces to perform this subtraction. Discuss the methods they use.

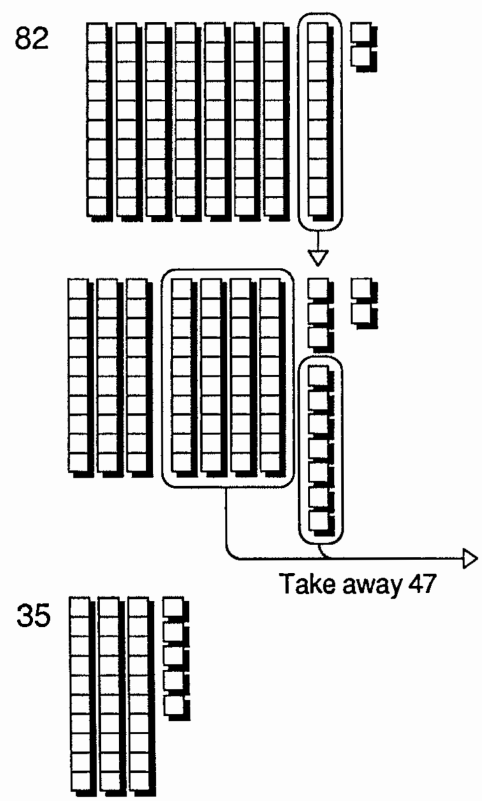
Comments

5. One standard procedure for adding 156 and 279 results in the following recording:

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 156 \\ +\ 279 \\ \hline 435 \end{array}$$

This can be described as follows: The 15 units are combined, 10 of these are exchanged for 1 strip (recorded above the existing strips) and the remaining 5 units are recorded. Next, the 13 strips are combined, 10 of these are exchanged for 1 mat (recorded above the existing mats) and the remaining 3 strips are recorded. Finally, the mats are combined and their number recorded.

6. Many students will subtract using the *take-away* method. In this case, they will take 4 strips and 7 units from a collection of 8 strips and 2 units, after exchanging one of the 8 strips for 10 units:



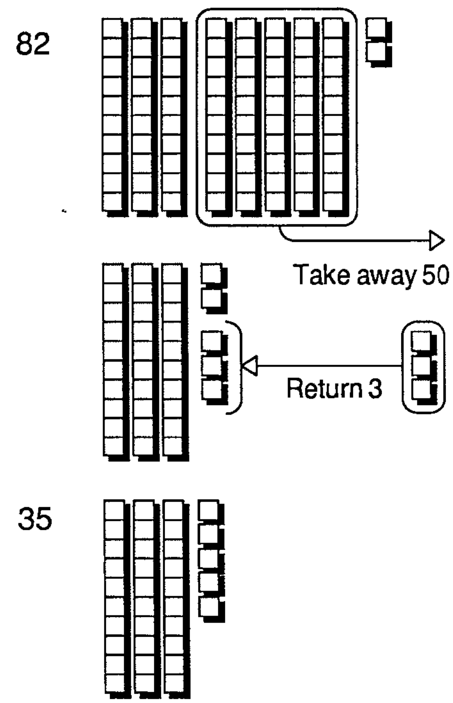
82 – 47 = 35

Continued on next page.

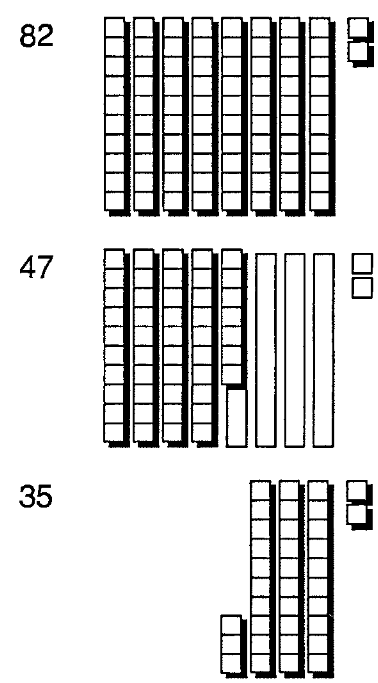
Actions

Comments

6. *Continued.*
Another way to take 47 units from a collection of 8 strips and 2 units is to take 5 strips from the collection and return 3 units *change.*



The subtraction can also be done by the *difference* method. In this method, a collection is found that makes up the difference between collections for 82 and 47:



Actions

7. Ask the students to use their base 10 pieces to do these subtractions:

183 – 65   245 – 78   412 – 267

8. (Optional) Describe how paper and pencil procedures for subtraction can be described in terms of number piece manipulations.

Comments

7. You may want to have the students try various methods and discuss which they prefer.

8. Subtracting 385 from 647 by the procedure generally taught results in the following recording.

$$\begin{array}{rrr} \overset{5}{\cancel{6}} & {}^{1}4 & 7 \\ -\ 3 & 8 & 5 \\ \hline 2 & 6 & 2 \end{array}$$

This can be described as follows: First, 5 units are taken from 7 units leaving 2 units. Next, 8 strips are to be taken from the initial collection. In order to do this, 1 mat is exchanged for 10 strips so that there are now 5 mats and 14 strips (recorded above the initial collection). Taking 8 strips from 14 leaves 6 strips. Finally, taking 3 mats from 5 leaves 2 mats.

The above procedure is based on the first subtraction method described in *Comment 6*. Your students may wish to devise recording procedures suggested by other methods used there.

Another procedure, once popular, results in the following recording:

$$\begin{array}{rrr} 6 & {}^{1}4 & 7 \\ -\ {}^{4}\cancel{3} & 8 & 5 \\ \hline 2 & 6 & 2 \end{array}$$

As before, taking 5 units from 7 units leaves 2 units. Now the same amount is added to both collections: 10 strips to the top collection and 1 mat to the bottom collection. Then taking 8 strips from 14 leaves 6 strips and taking 4 mats from 6 mats leaves 2 mats.

This procedure depends on the observation that the difference between two collections of number pieces is not changed if the same amount is added to both collections.

Number Piece Rectangles

O V E R V I E W

Number pieces are used to find the area and dimensions of a rectangle as a preliminary to multiplying and dividing with base 10 number pieces, which is discussed in subsequent units.

Prerequisite Activity

Unit III/Activity 4, *Base 10 Numeration*

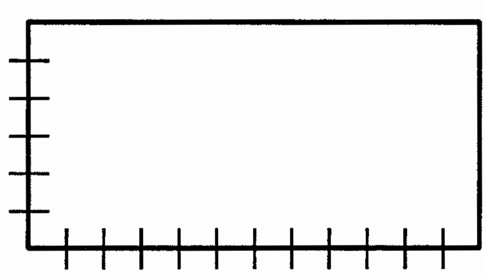
Materials

Base 10 number pieces and copies of Activity Sheet III-6 for each student

Actions

1. Distribute base 10 number pieces and a copy of Activity Sheet III-6 to each student or group of students. Discuss with the students how number pieces can be used to find the area of the rectangle at the top of this sheet, without doing any arithmetic other than counting.

2. Ask the students to suggest ways number pieces may be used to find the dimensions of the rectangle.

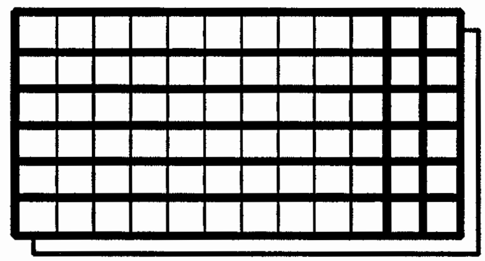


6 x 12 rectangle

Comments

1. Each student, or group of students, should have a supply of 5 mats, 15 strips and 20 units. A master of Activity Sheet III-6 is attached.

One way to find the area of the rectangle is to fill it with number pieces:



It takes 6 strips and 12 units to fill the rectangle. This is equivalent to 7 strips and 2 units. Hence the area of the rectangle is 72 square units.

This can be demonstrated on the overhead using a transparency of the activity sheet and transparencies of base 10 pieces.

2. The *dimensions* of a rectangle are the lengths of adjacent sides. These lengths may be determined by using a base 10 strip to mark off unit lengths along the sides (see figure at left). Counting subdivisions, one finds that one dimension is 6 unit lengths and the other is 12 unit lengths.

A rectangle is often described by giving its dimensions. For example, this rectangle is a "6 x 12 rectangle". In this context, the symbol "x" is usually read "by". Thus, "6 x 12 rectangle" is read "6 by 12 rectangle".

Actions

Comments

3. Discuss the processes of measuring area and measuring length.

3. The *area* of a surface is generally measured by the number of unit squares required to cover it. (A unit of area doesn't have to be a square — it could be an equilateral triangle, a regular hexagon, a rectangle the size of a particular book cover, or some other shape.)

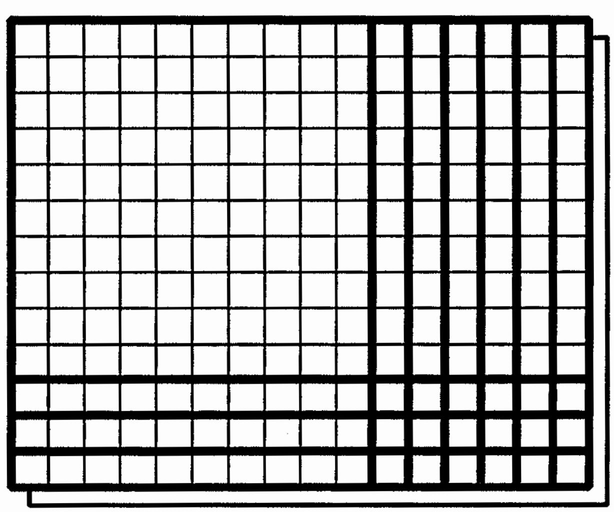
The *length* of a line segment is the number of unit lengths that can be marked off along the segment.

In this activity, a unit square is a base 10 number piece unit. A unit length is the length of an edge of a number piece unit.

Changing the unit square and unit length will result in different numerical values for the area and dimensions of a rectangle. Thus, in reporting a numerical value for an area or dimension, it is important that the unit of measure is known.

It is generally clear from the context whether the unit of measure is the unit square or the unit length. For example, if it is stated that the area of a rectangle is 325, it is understood that the unit of measure is the unit square.

4. Ask the students to use number pieces to find, without doing any arithmetic other than counting, the area and dimensions of the rectangle at the bottom of the Activity Sheet.



4. The rectangle can be filled with 1 mat, 9 strips and 18 units.

Exchanges can be made to convert this collection of number pieces to an equivalent collection containing a minimal number of pieces. Doing this, results in a collection of 2 mats and 8 units. Hence the area of the rectangle is 208.

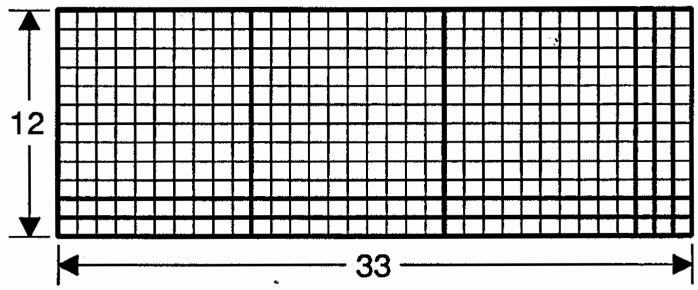
Since 1 strip and 3 units fit along one edge and 1 strip and 6 units fit along an adjacent edge, its dimensions are 13 and 16.

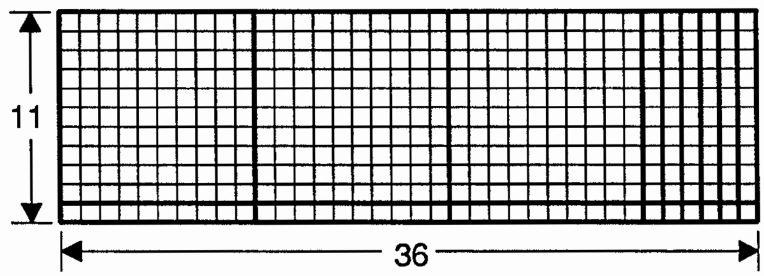
Actions

5. Ask each student, or group of students, to form a collection of 3 mats, 9 strips, and 6 units. Have them form a rectangle using these pieces and then find its area and dimensions. After they have found one rectangle, ask them to find other rectangles of the same area that can be formed with number pieces.

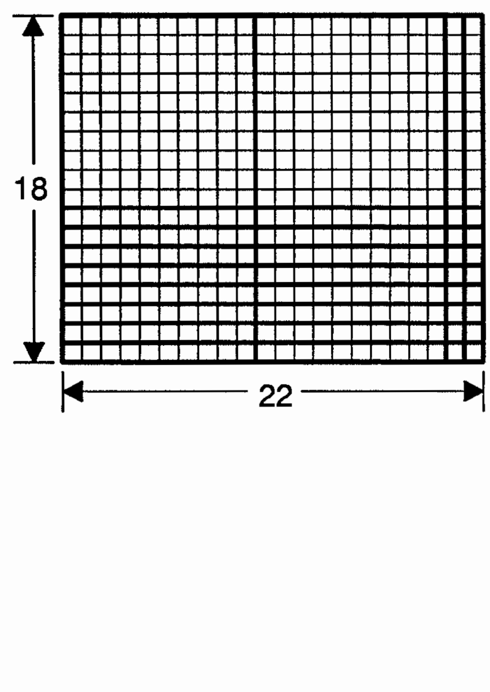
Comments

5. This collection of number pieces can be arranged to form either a 12 x 33 rectangle or an 11 x 36 rectangle.





There are other number piece rectangles which have the same area. If, for example, a mat is exchanged for 10 strips and one strip is exchanged for 10 units, the resulting collection can be formed into an 18 x 22 rectangle:



Actions

6. For each of the following collections of number pieces, ask the students to find the area and dimensions of a rectangle which can be formed with the given collection or an equivalent collection.

(a) 2 mats, 8 strips, 6 units
(b) 4 mats, 8 strips, 3 units
(c) 3 mats, 8 units
(d) 5 mats, 5 strips, 2 units

7. Ask the students to construct the following rectangles using number pieces, and provide the requested information.

(a) A rectangle with dimensions 13 x 34. Record its area.
(b) A square whose area is 529. Record its dimensions.
(c) A rectangle with area 288 and one dimension 12. Record the other dimension.
(d) A rectangle with area 598 and one dimension 23. Record the other dimension.

Discuss with the students how they arrived at their answers.

Comments

6. (a) Without making exchanges, this collection can be formed into either a 13 x 22 or 11 x 26 rectangle.

(b) This collection can be made into a 21 x 23 rectangle.

(c) This collection of number pieces cannot be made into a rectangle. However, exchanging 1 mat for 10 strips, results in an equivalent collection of 2 mats, 10 strips and 8 units which can be made into either a 14 x 22 or an 11 x 28 rectangle.

(d) Exchanging 1 mat for 10 strips and 1 strip for 10 units gives an equivalent collection which can be formed into either a 23 x 24 or a 12 x 46 rectangle.

7. Some students may be reluctant to manipulate number pieces and undertake to obtain the requested information using paper and pencil arithmetic If this happens, urge them to do this activity without using any arithmetic other than counting.

(a) This rectangle can be constructed with 3 mats, 13 strips and 12 units. An equivalent minimal collection contains 4 mats, 4 strips and 2 units. Hence the area of the rectangle is 442.

(b) A collection of 5 mats, 2 strips and 9 units cannot be formed into a square. However, exchanging 1 mat for 10 strips results in a collection of 4 mats, 12 strips and 9 units. This collection can be arranged in a 23 x 23 square.

(c) Without making exchanges, a collection of 2 mats, 8 strips and 8 units can be formed into a rectangle with 12 unit lengths along one edge. The other dimension is 24.

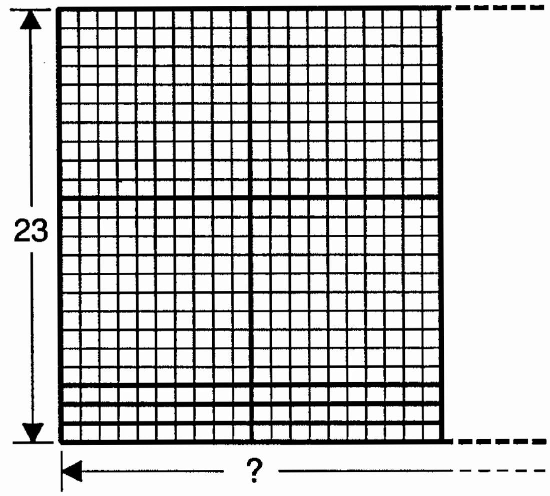
Continued next page.

Actions

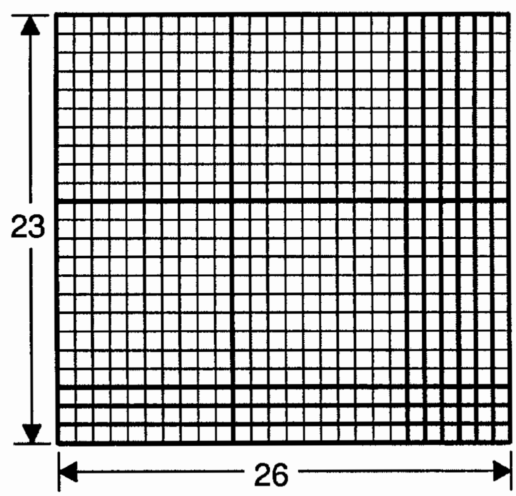
Comments

7. *Continued.*

(d) One way to form the desired rectangle is to start with a collection of 5 mats, 9 strips and 8 units and begin to arrange it into a rectangle with 23 unit lengths along one edge. After arranging 4 mats and 6 strips as



shown, the remaining mat is exchanged for 10 strips. Then additional columns of 2 strips and 3 units each are added to the rectangle until — with the exchange of 1 strip for 10 units — the supply of number pieces is exhausted. The result is a rectangle whose other dimension is 26.



Base 10 Multiplication

OVERVIEW

Base 10 number pieces and grid paper are used to find products of whole numbers.

Prerequisite Activity

Unit II, Activity I, *Basic Operations;*
Unit III, Activity 4, *Base 10 Numeration*;
Unit IV, Activity 6, *Base 10 Rectangles*

Materials

Base 10 number pieces and base 10 grid paper for each student

Actions

1. Distribute base 10 number pieces to each student or group of students.

2. Ask the students to devise ways to find the following products using their number pieces and no arithmetical procedures other than counting.

(a) 3 x 34   (b) 13 x 24

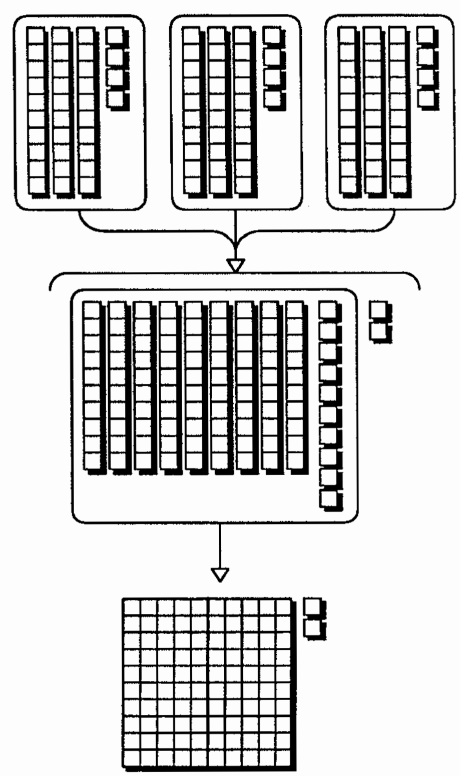
Observe the processes the students use and discuss their methods with them.

Comments

1. Each student or group of students should have a supply of 5 mats, 18 strips and 15 units.

2. Some students may set out to find these products by "repeated addition", i. e. they will find 3 x 34 by forming 3 collections for 34, combining them, and then making exchanges:

3 Groups of 34



$3 \times 34 = 102$

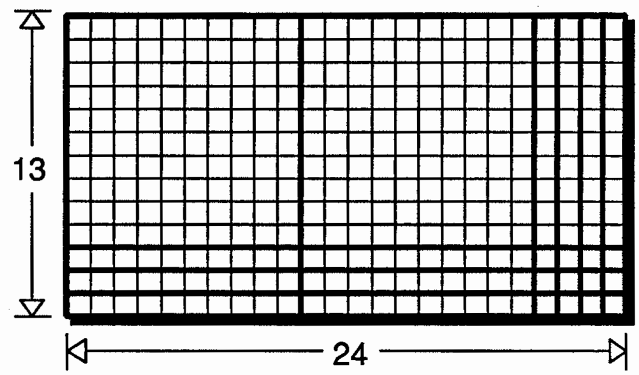
Continued next page.

Actions

Comments

2. *Continued.* Finding 13 x 24 by the above method is not as convenient. For one thing, the students may not have enough strips and units to form 13 collections for 24. An alternative is to form a rectangle whose dimensions are 13 by 24. The area of this rectangle is the desired product. Pieces in the rectangle can be exchanged to obtain 3 mats, 1 strip and 2 units. Hence its area is 312.

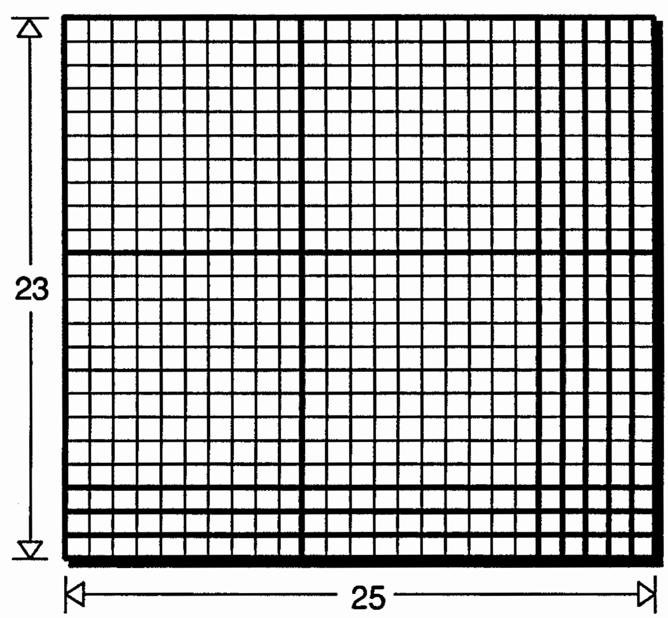
You may want to point out that the 13 rows of the rectangle can be viewed as 13 collections of 24 units each.



$13 \times 24 = 312$

3. Ask the students to find the product 23 x 25 by forming a rectangular array of number pieces.

3. There are 4 mats, 16 strips and 15 units in the rectangle shown. Pieces can be exchanged to obtain a minimal collection of 5 mats, 7 strips and 5 units. Hence, 23 x 25 = 575.



Actions

4. Distribute base 10 grid paper to each student. Ask the students to draw a sketch on their grid paper that will enable them to find the product 12 x 24 without using any arithmetical procedures other than counting. Discuss.

5. Ask the students to find the following products by drawing sketches on base 10 paper.

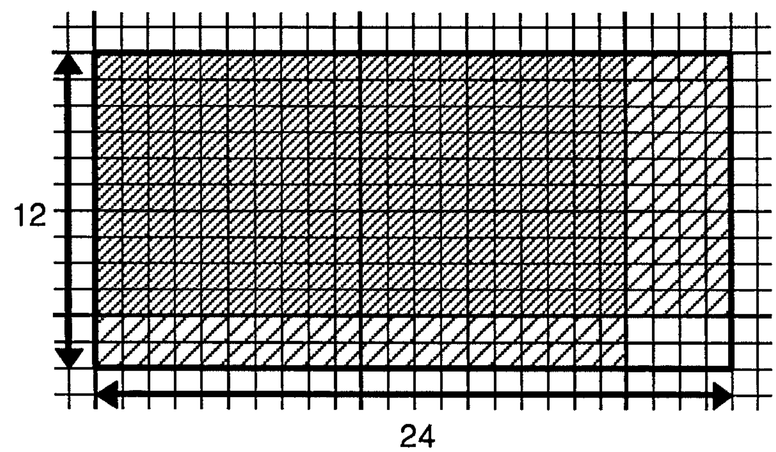
(a) 16 x 23 (b) 22 x 27 (c) 31 x 46

Discuss.

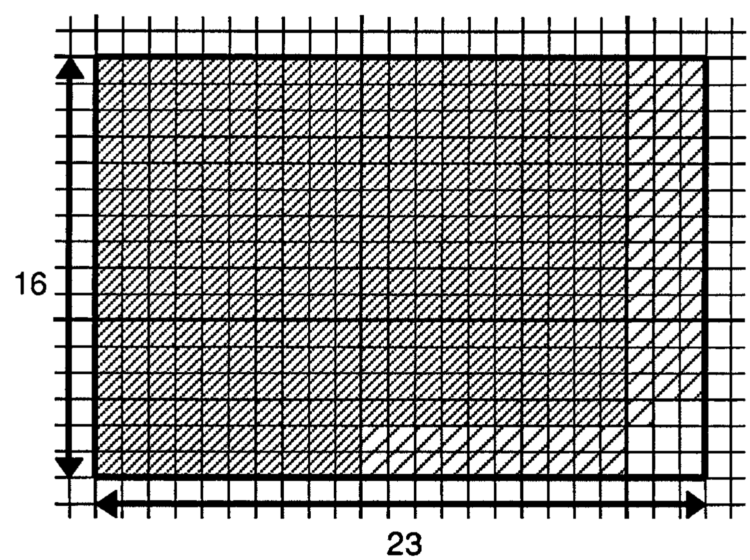
Comments

4. A master for base 10 grid paper is included in this activity. Each student will need at least 2 sheets. Pencil sketches show up better on dittoed copies than on blackline copies.

One way to find the product is to sketch a 12 by 24 rectangle. In the sketch shown, the area of the darker shaded region is 2 mats, the area of the lighter shaded region is 8 strips and the unshaded area is 8 units. Hence the area of the rectangle is 288. Since the area of a rectangle is the product of its dimensions, 12 x 24 = 288.



5. (a) The area of a 16 x 23 rectangle is 368 since it is equivalent to 3 mats (the darker shaded region), 6 strips (the lighter shaded region) and 8 units. Students may use various ways to determine this.

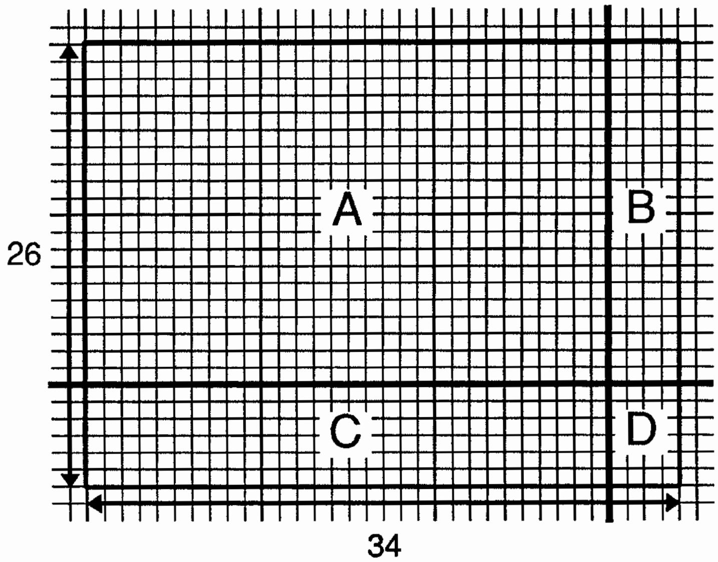


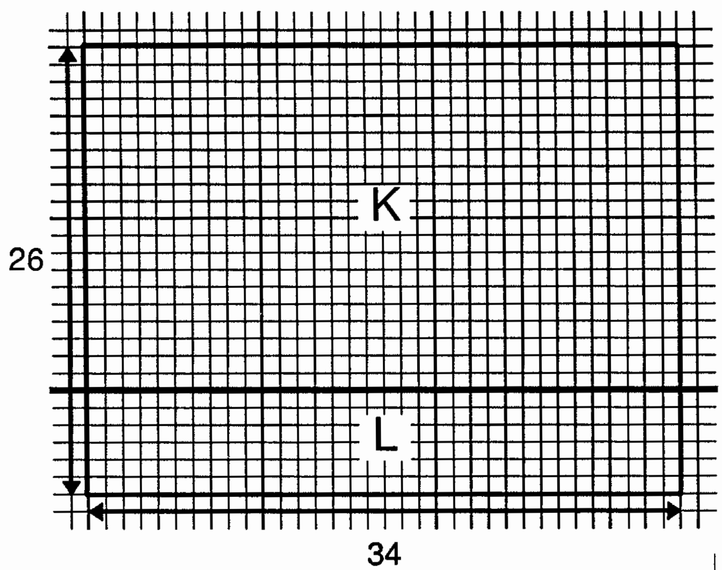
(b) 22 x 27 = 594.

(c) The area in a 31 x 46 rectangle is equivalent to 1 strip-mat, 4 mats, 2 strips and 6 units. Hence, 31 x 46 = 1426.

Actions

6. (Optional.) Explore with the students procedures for finding the area of a rectangle by dividing it into smaller rectangles.





Comments

6. This leads into Action 7 which deals with paper and pencil procedures for finding products.

A 26 x 34 rectangle can be divided into 4 rectangular subregions as shown here. Rectangle A is composed of mats, rectangles B and C are composed of strips and D is a rectangle of units. The area of the original rectangle is obtained by adding the areas of rectangles A, B, C and D:

24	Area of D (24 units)
180	Area of C (18 strips)
80	Area of B (8 strips)
600	Area of A (6 mats)
884	Total area

Alternatively, the rectangle could be divided into 2 regions, K and L.

204	Area of L (18 strips, 24 units)
680	Area of K (6 mats, 8 strips)
884	Total area

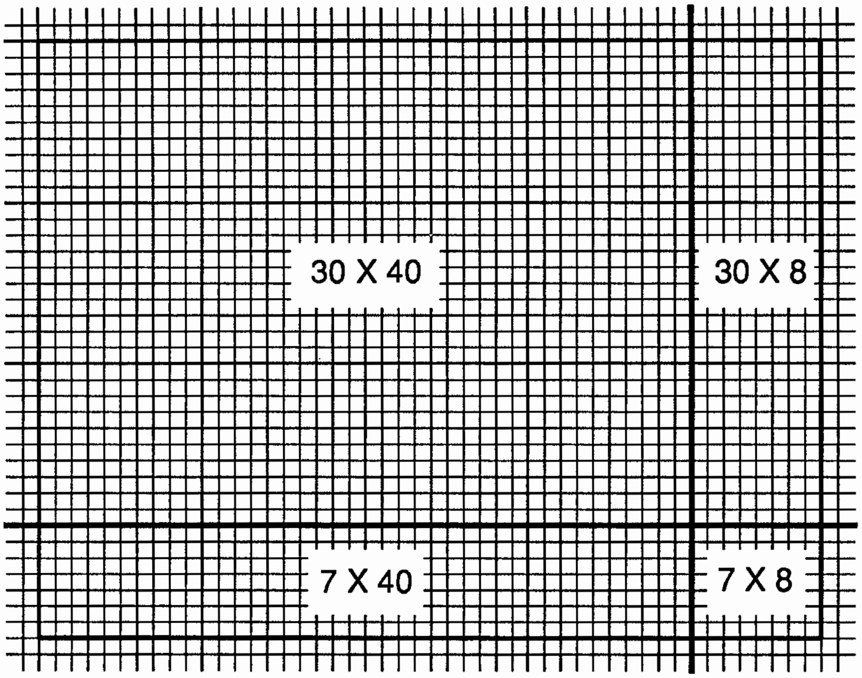
You may want to ask the students to find and record the areas of rectangles of other dimensions, using one or both of the above procedures. Here are some possible dimensions: 21 x 27, 32 x 19, 24 x 43.

The students may devise other ways of subdividing a rectangle.

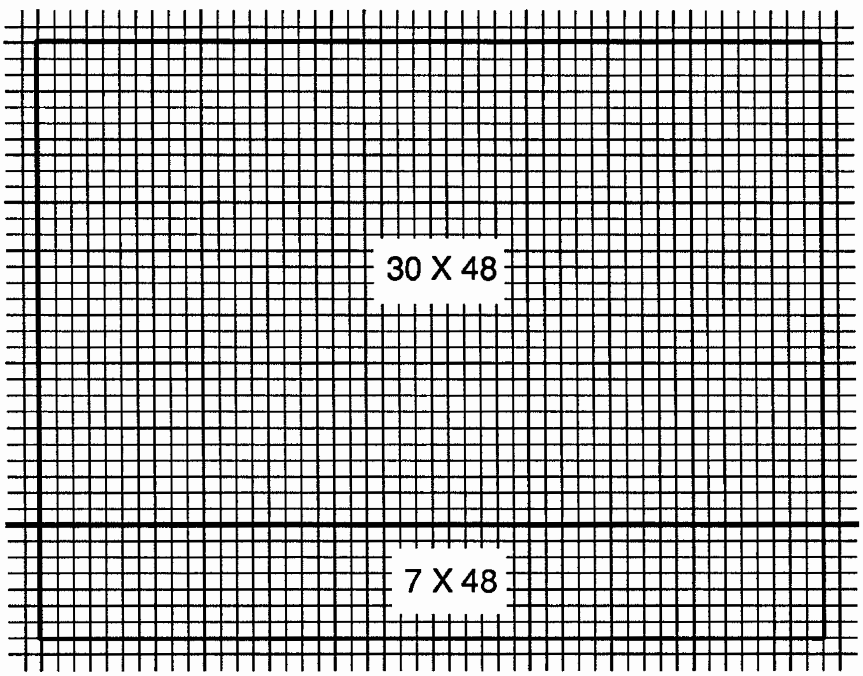
Actions

7. (Optional.) Discuss paper and pencil procedures for finding the product of two numbers.

37 × 48:



37 × 48:



Comments

7. Paper and pencil schemes for finding products can be based on the procedures discussed in Action 6.

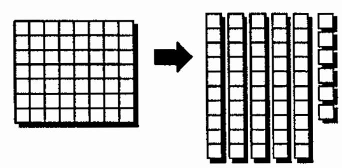
For example, 37 x 48 may be determined by finding the sum of 4 partial products. This corresponds to finding the area of 37 x 48 rectangle by dividing it into 4 rectangular subregions and summing their areas.

$$
\begin{array}{rl}
48 & \\
\underline{\times\ \ 37} & \\
56 & (7 \times 8) \\
280 & (7 \times 40) \\
240 & (30 \times 8) \\
\underline{1200} & (30 \times 40) \\
1776 &
\end{array}
$$

Alternatively, 37 x 48 can be thought of as the sum of 2 partial products. This corresponds to finding the area of a 37 x 48 rectangle by breaking it into 2 rectangles.

$$
\begin{array}{rl}
48 & \\
\underline{\times\ \ 37} & \\
336 & (7 \times 48) \\
\underline{1440} & (30 \times 48) \\
1776 &
\end{array}
$$

In using paper and pencil procedures to multiply, it is necessary to know the product of single-digit numbers. For example, in the above illustration, one must know that 7 x 8 = 56. This is equivalent to the observation that a 7 x 8 rectangle made of number pieces can be converted into 5 strips and 6 units. A table of single-digit products can be constructed by



building number piece rectangles, or drawing sketches of them, and finding their areas in terms of strips and units.

You may want to discuss with your students how the product of two 3-digit numbers may be treated as a sum of partial products. In general, the product of two numbers can be found by summing partial products. However, products of multi-digit numbers are most efficiently obtained with the use of a calculator.

Base 10 Division

O V E R V I E W

Base 10 number pieces and grid paper are used to find quotients of whole numbers.

Prerequisite Activity

Unit II, Activity I, *Basic Operations;* Unit III, Activity 4, *Base 10 Numeration.*; Unit IV, Activity 6, *Base 10 Rectangles*

Materials

Base 10 number pieces and base 10 grid paper for each student.

Actions

1. Distribute base 10 number pieces to each student or group of students.

2. Ask the students to devise ways to find the following quotients using their number pieces, and no arithmetical procedures other than counting.

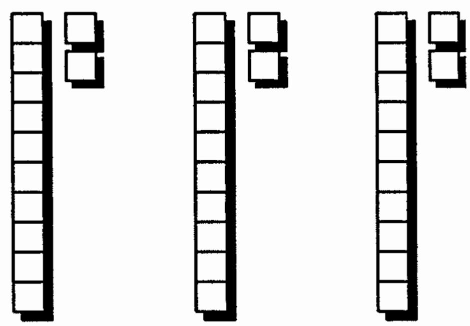
(a) $36 \div 3$ (b) $156 \div 12$

Observe the processes the students use and discuss their methods with them.

Comments

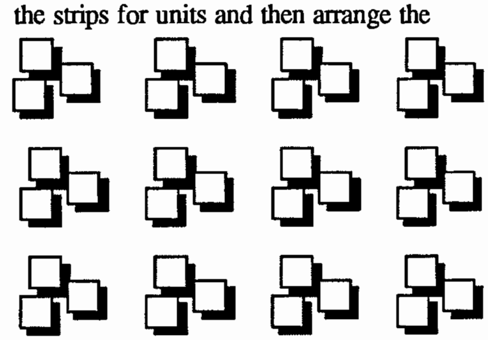
1. Each student or group of students will need 7 mats, 15 strips and 15 units. Some students may request more units. See Comment 2 (a).

2. (a) One way to determine the quotient is to separate a collection of 3 strips and 6 units into 3 groups of the same size. This



is the *sharing method* of divison. The quotient is the size of each group.

Some students may want to exchange all of the strips for units and then arrange the



units in groups of 3. This is the *grouping method* of division. The quotient is the number of groups. You may want to have a collection of 36 units available to demonstrate this method on the overhead.

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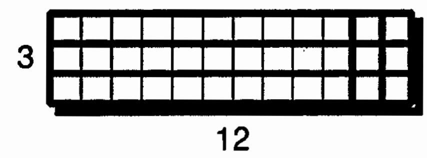
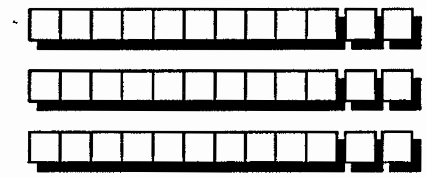
Actions

Comments

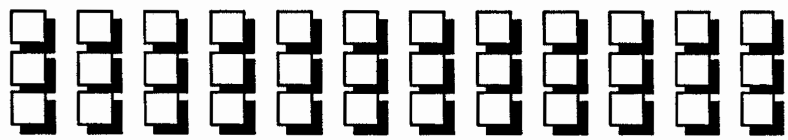
2. *Continued.* The quotient can also be determined by forming a rectangle with area 36 and one dimension equal to 3. The quotient is the other dimension.



Note that in the sharing method, the 3 groups can be arranged in rows:

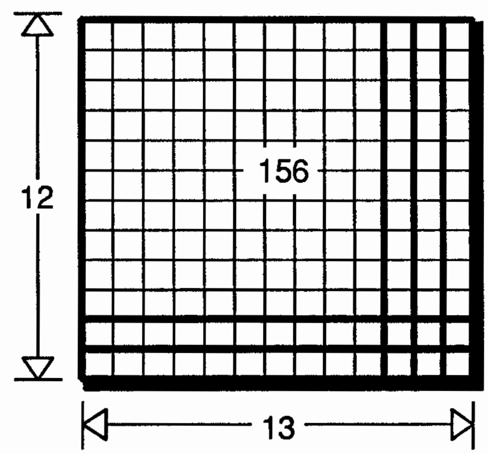


In the grouping method, the groups of 3 can be arranged in columns:



In both cases, pushing the groups together results in a 3 x 12 rectangle.

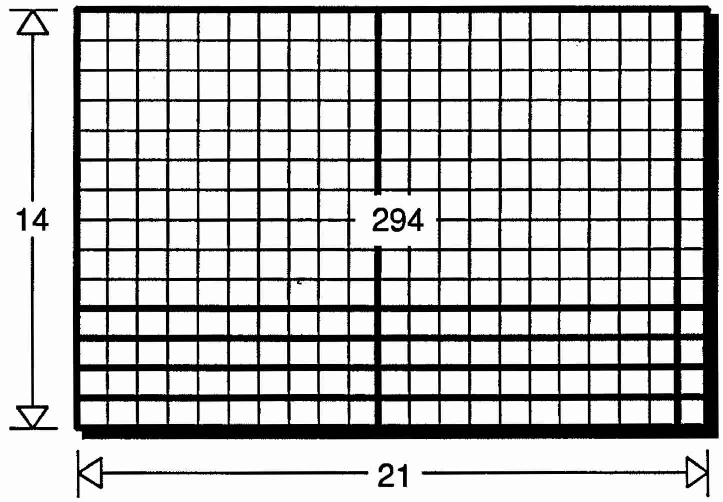
(b) This quotient can be found by arranging 1 mat, 5 strips and 6 units in a rectangle which has 12 as one of its dimensions. The other dimension is 13. Thus 156 + 12 = 13.

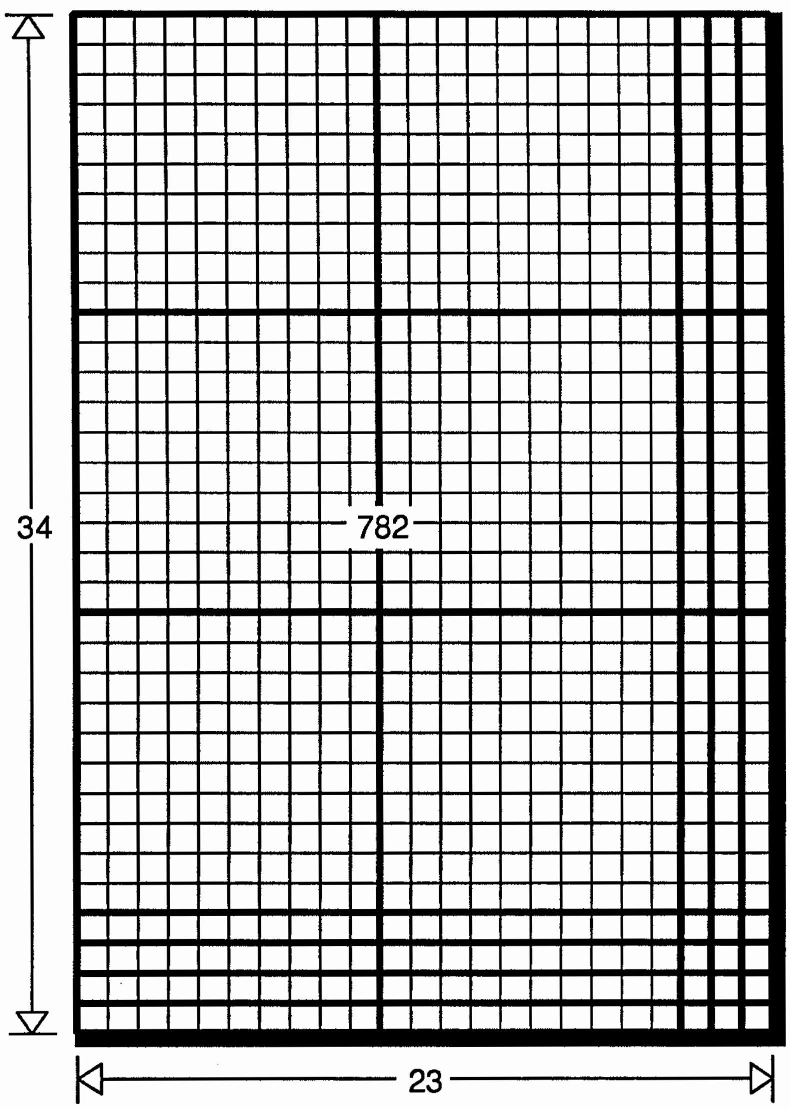


Actions

3. Ask the students to use number pieces, and no arithmetic other than counting, to find the following quotients:

(a) $294 \div 14$ (b) $782 \div 34$ (c) $290 \div 13$





Comments

3. (a) If 2 mats, 9 strips and 4 units are arranged in a rectangle so that one dimension is 14, the other dimension is 21.

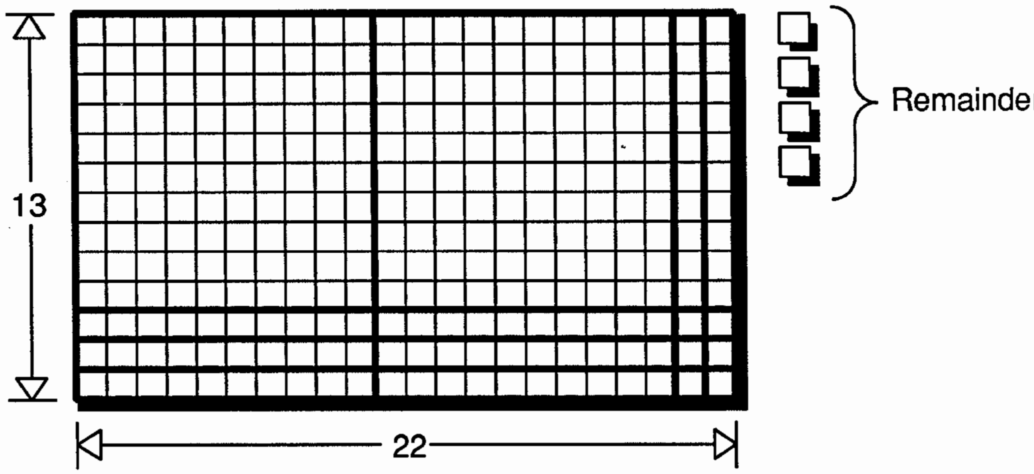
(b) Starting with 7 mats, 8 strips and 2 units, a rectangle with a dimension of 34 can be formed if one mat is exchanged for 10 strips and 1 strip is exchanged for 10 units. The other dimension is 23.

Continued next page.

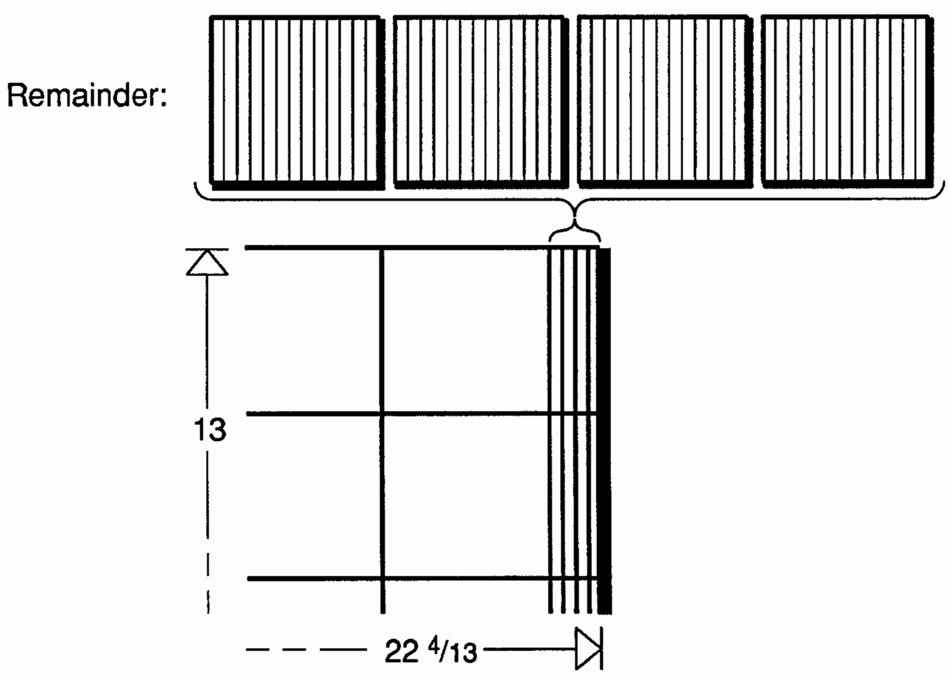
Actions

Comments

3. *Continued.* (c) A collection of number pieces equivalent to 2 mats and 9 strips cannot be arranged in a rectangle which has 13 as one of its dimension. However, if 1 strips is exchanged for 10 units, the resulting collection can be arranged in a 13 x 22 recatangle with 4 units left over. Hence 290 ÷ 13 = 22 with a remainder of 4.



If the students are familiar with fractions, you may want to point out that the remainder may be written in fractional form. Note that if each of the 4 left over units were sliced into 13 equal parts and these were distributed among the rows of the rectangle, each row would get 4 of these parts and the result would be a 13 x 22 $^4/_{13}$ rectangle. Thus 290 ÷ 13 = 22 $^4/_{13}$.



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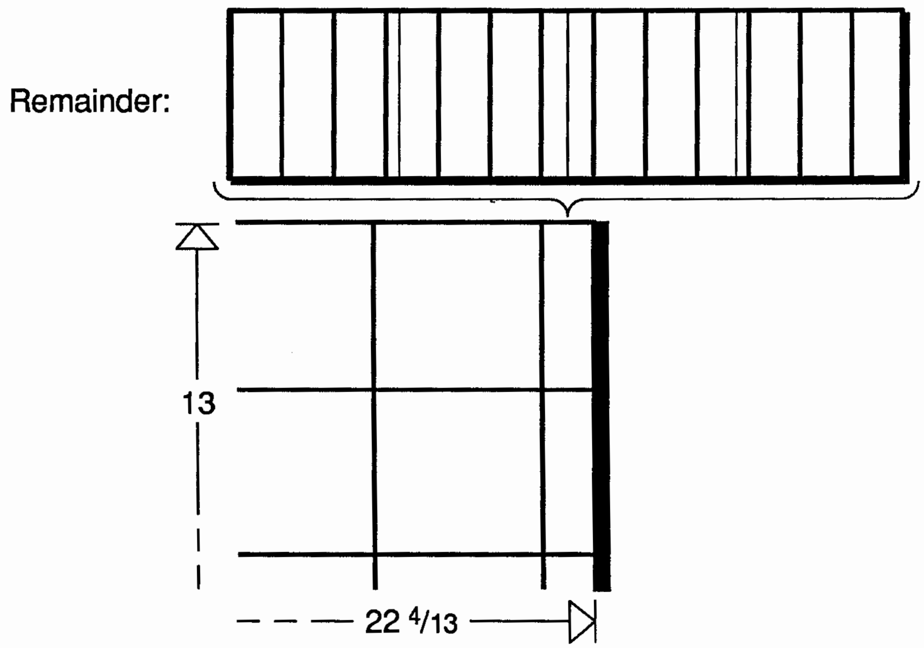
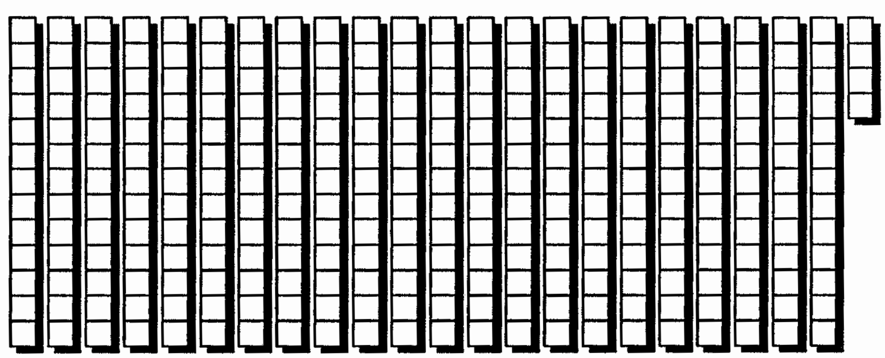
Actions

Comments

3. (c) *Continued.* Alternatively, instead of dividing each of the 4 remaining units into 13 parts, they could be placed in a row. This row could then be divided into 13 equal parts and these parts distributed, one to each row of the rectangle.



The remainder can also be considered in the context of the grouping method of division. In this method, 220 ÷ 13 is the number of groups of 13 units that can be formed from 220 units. In this case, there are 22 such groups plus a partial group of 4 units. The partial group contains 4 parts of the 13 needed for another group, i. e. it is $^4/_{13}$ of a group. So the number of groups of 13 is 22 $^4/_{13}$.



22 $^4/_{13}$ groups of 13

Actions

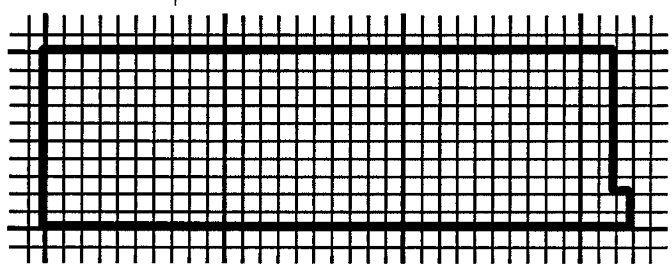
4. Distribute base 10 grid paper to each student. Ask the students to draw a sketch of a rectangle that will enable them to find the quotient $322 \div 14$, without using any arithmetical procedures other than counting. Discuss the methods the students use.

Comments

4. A master for base 10 grid paper is attached to Unit III, Activity 7, *Base 10 Multiplication.* Each student will need at least 2 sheets. Pencil sketches show up better on dittoed copies than on blackline masters.

The quotient may be found by sketching a rectangle which has an area of 322 and a dimension of 14. The other dimension will be the quotient.

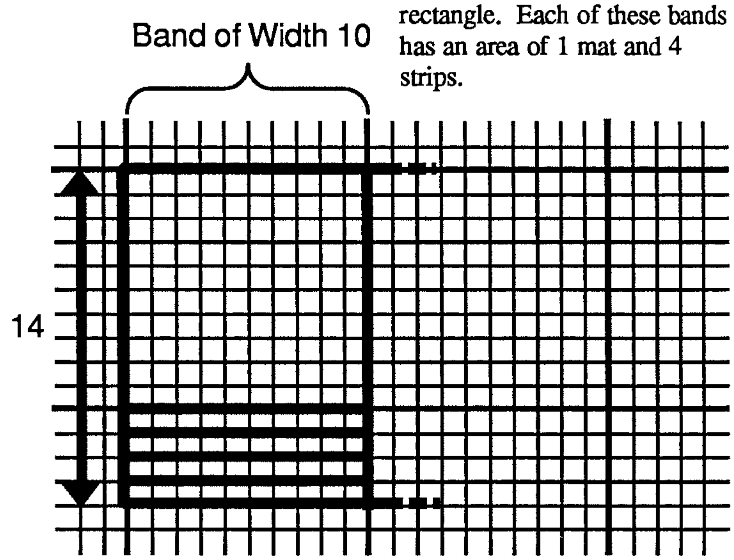
Some students may have difficulty sketching an appropriate rectangle. It may help to have them enclose a region on their grid paper whose area is 3 mats, 2 strips and 2 units:



Then discuss with them ways to construct a rectangle which encloses the same area and has a side of length 14.

There are a number of ways to construct a rectangle of area 322 with one dimension of 14. The following method is used in Comment 6 to describe the long division algorithm.

First, determine how many bands of width 10 and edge 14 can be incorporated into the rectangle. Each of these bands has an area of 1 mat and 4 strips.



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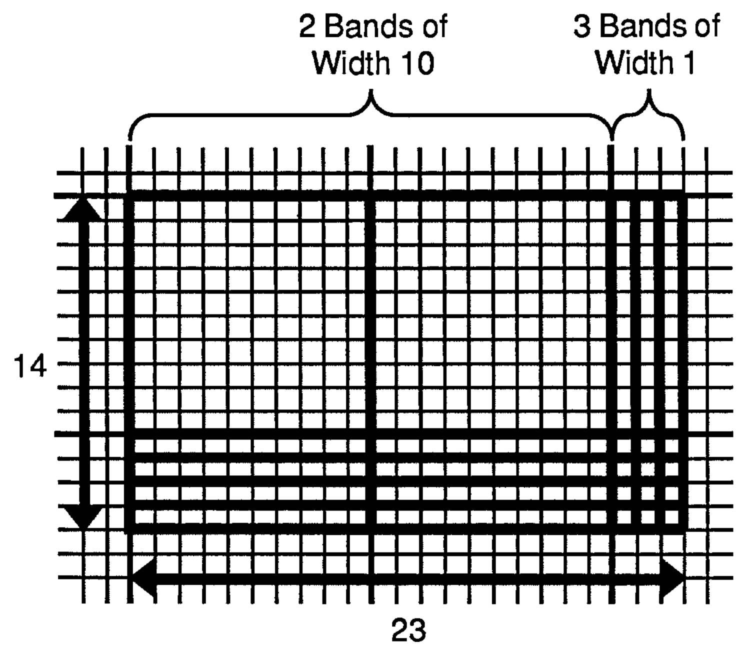
Actions

Comments

4. *Continued.* Two bands of width 10 provide an area of 2 mats and 8 strips. Adding another band of width 10 creates too large an area, so bands of width 1 and edge 14 are added until an area equivalent to 3 mats, 2 strips and 2 units is obtained.



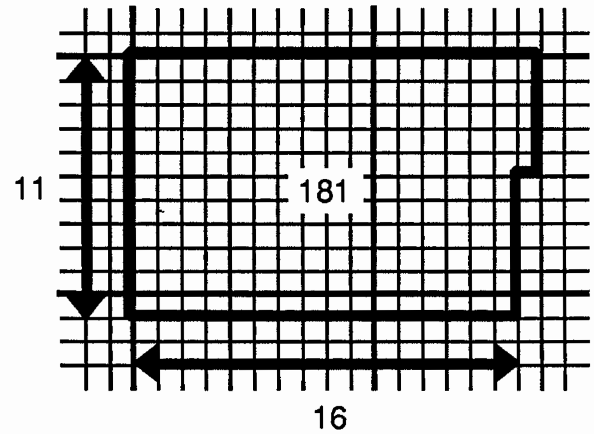
Actions

5. Ask the students to find the following quotients by drawing sketches on base 10 paper.

(a) $182 \div 13$ (b) $315 \div 21$ (c) $181 \div 11$

Comments

5. (c) Attempting to construct a rectangle of 181 square units so that one dimension of the rectangle is 11, results in a 11 x 16 rectangle with 5 unit squares remaining. Hence $181 \div 11 =$ 16 with a remainder of 5.



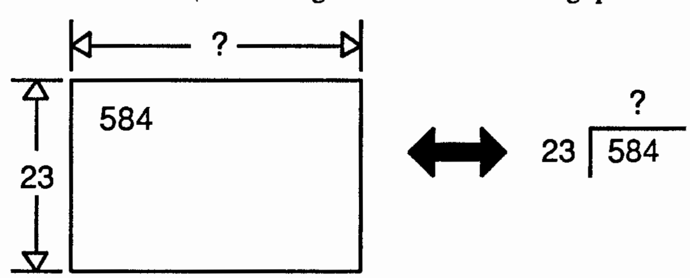
The remainder can be written in fractional form. Similar to the example in Comment 3c, each of the 5 units in the remainder could be divided into 11 parts and distributed among the rows of the rectangle. The result would be an 11 x 16 $^5/_{11}$ rectangle. Thus $181 \div 11 = 16\ ^5/_{11}$.

Actions

6. (Optional) Discuss the paper and pencil procedure for finding quotients that is commonly called "long division".

Comments

6. The long division procedure can be related to the method of sketching rectangles discussed in Comment 4. For example, finding $584 \div 23$ by the long division method can be related to sketching a rectangle with area 584 and 23 as one dimension. You may want to point out the similarity between a sketch of a rectangle with a missing dimension and the notation for a long division with a missing quotient.



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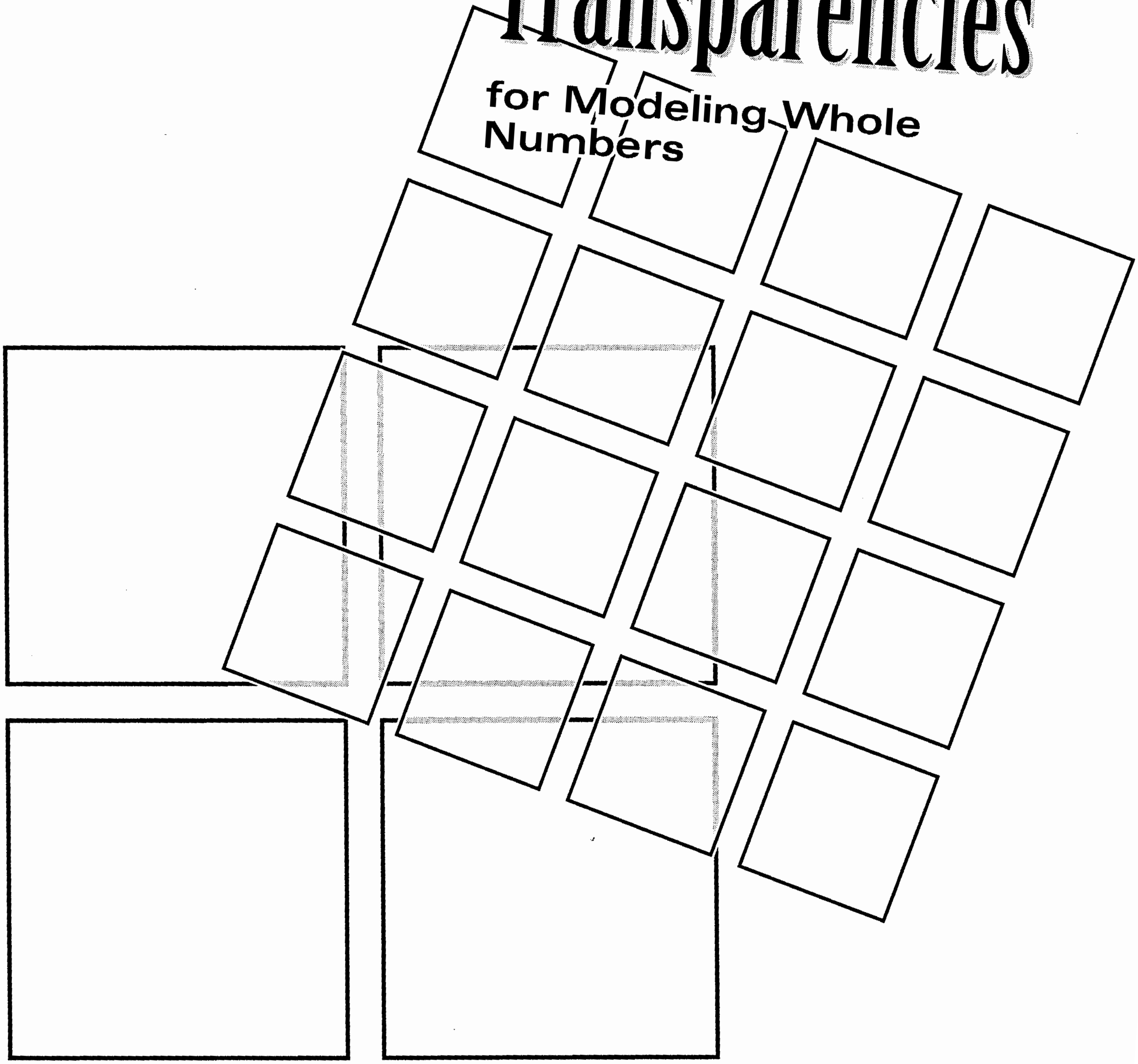
Unit III / Math and the Mind's Eye

Transparencies

for Modeling Whole Numbers



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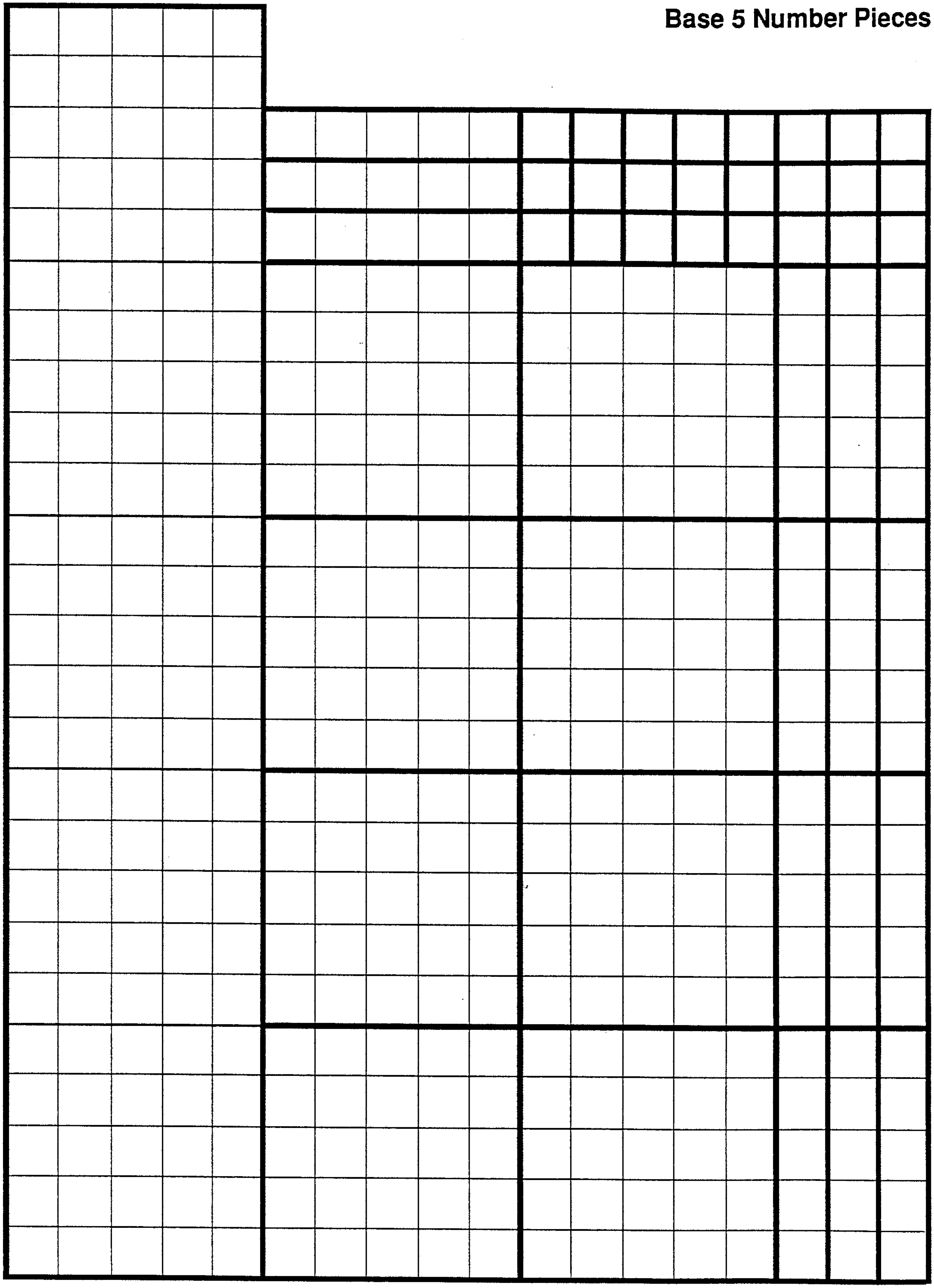
Catalog #MET3

Centimeter Grid Paper

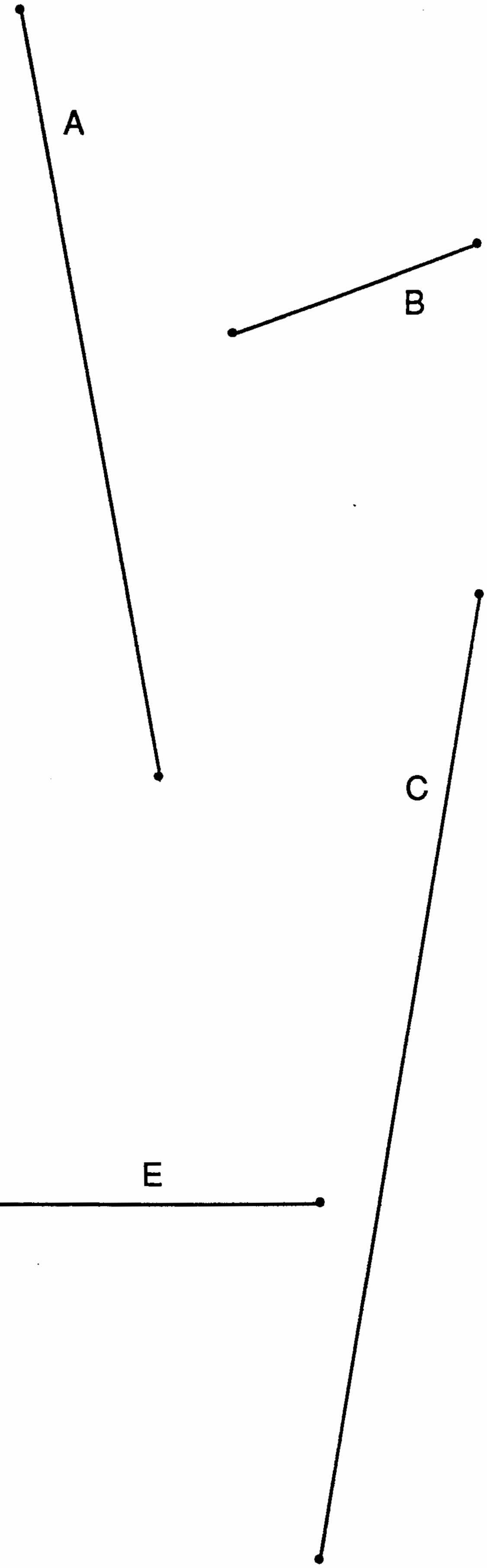
Base 5 Number Pieces

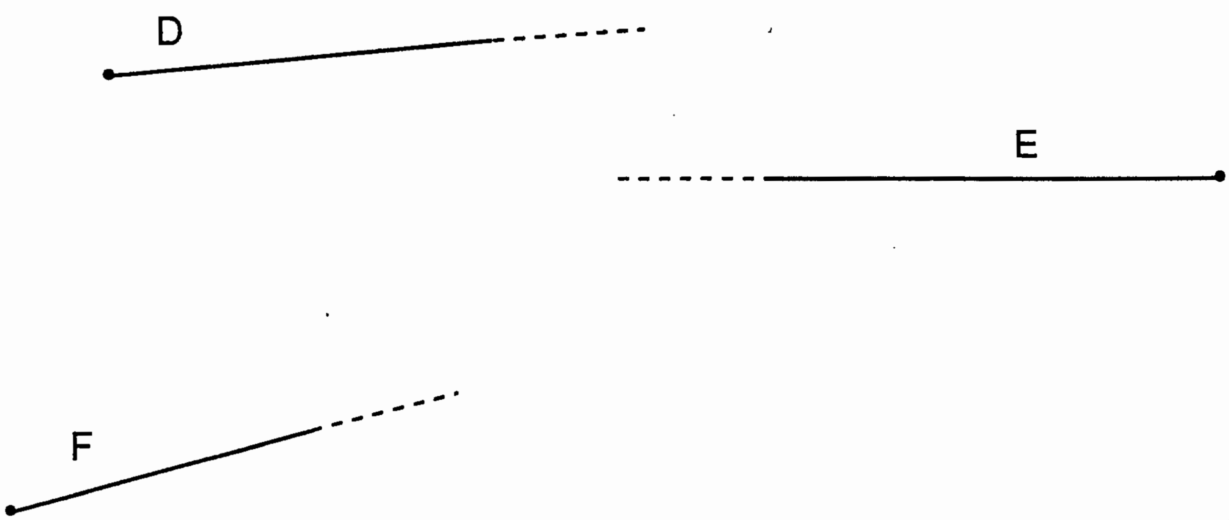


1. **Complete the table.**

2. **Finish drawing segments D, E and F so their length are those given in the table.**

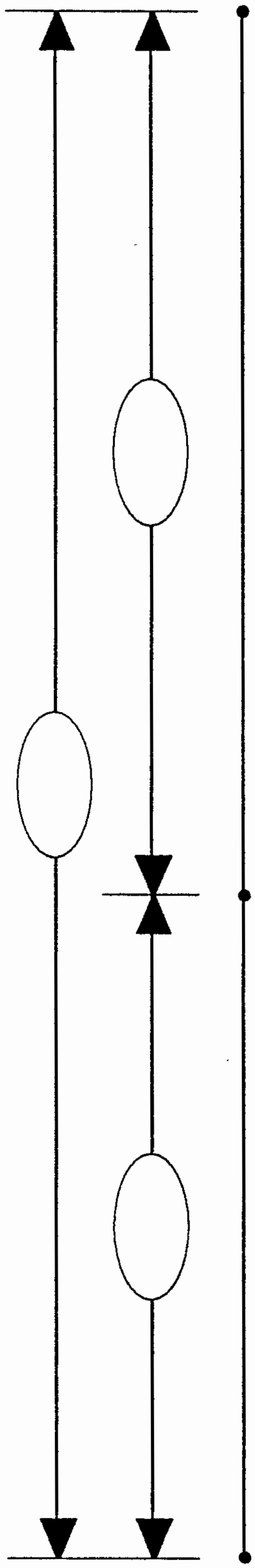
Line Segment	LENGTH Chains	Units	Total Units of Length
A	____	____	____
B	____	____	____
C	____	____	____
D	2	3	____
E	____	1	16
F	2	____	10



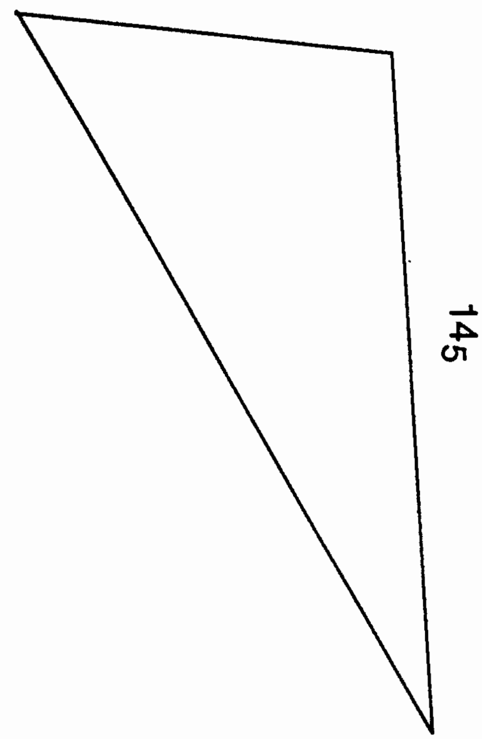


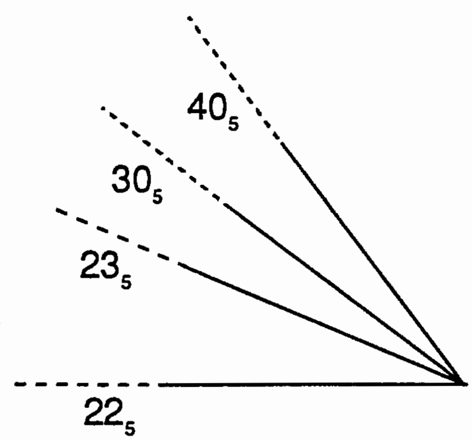
Activity Sheet III-2-A

Record the length of this line segment using base 5 notation.



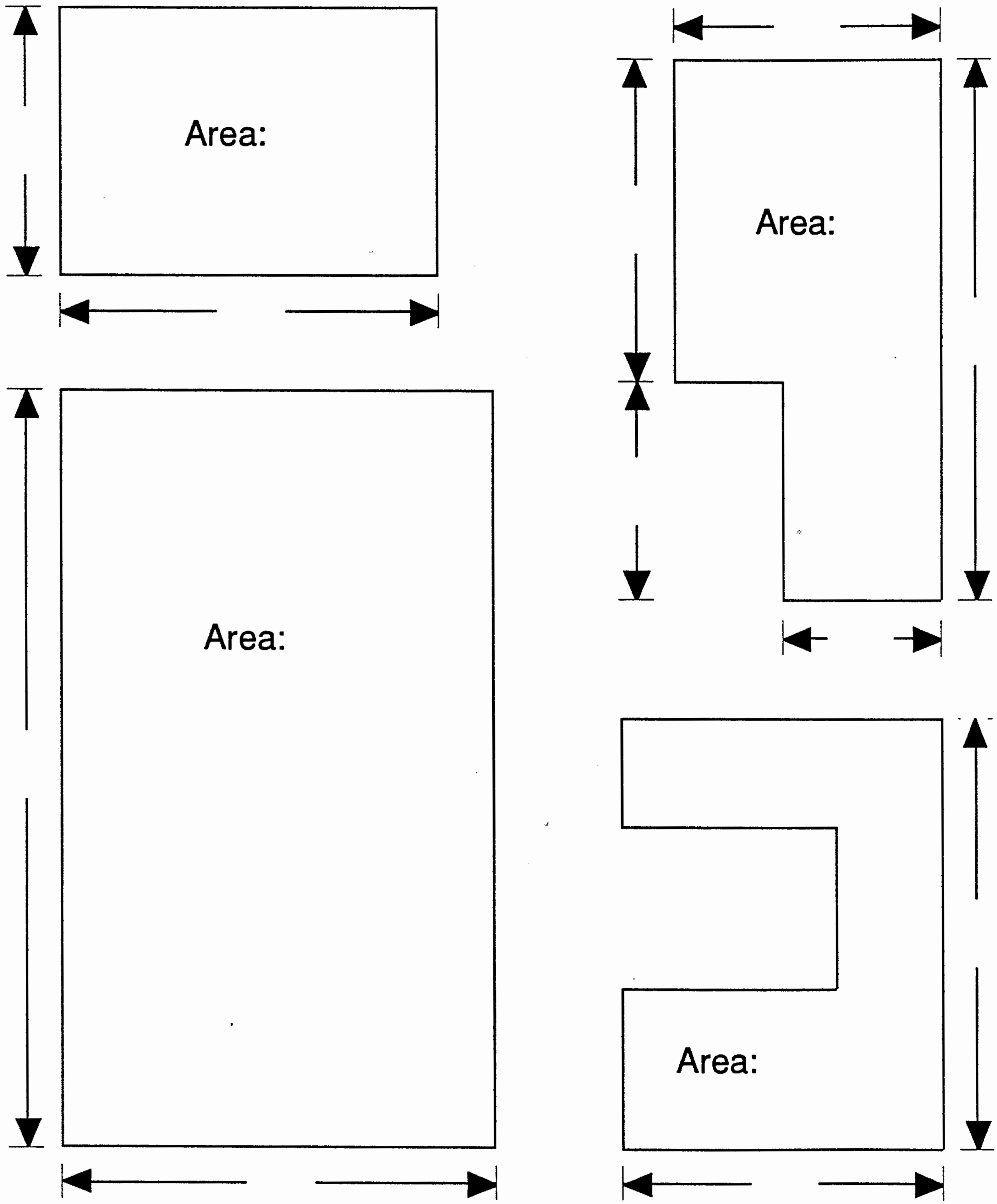
Fill in the missing lengths using base 5 notation.





Complete the line segments so they have the indicated length.

Write the areas and dimensions in base 5 notation.

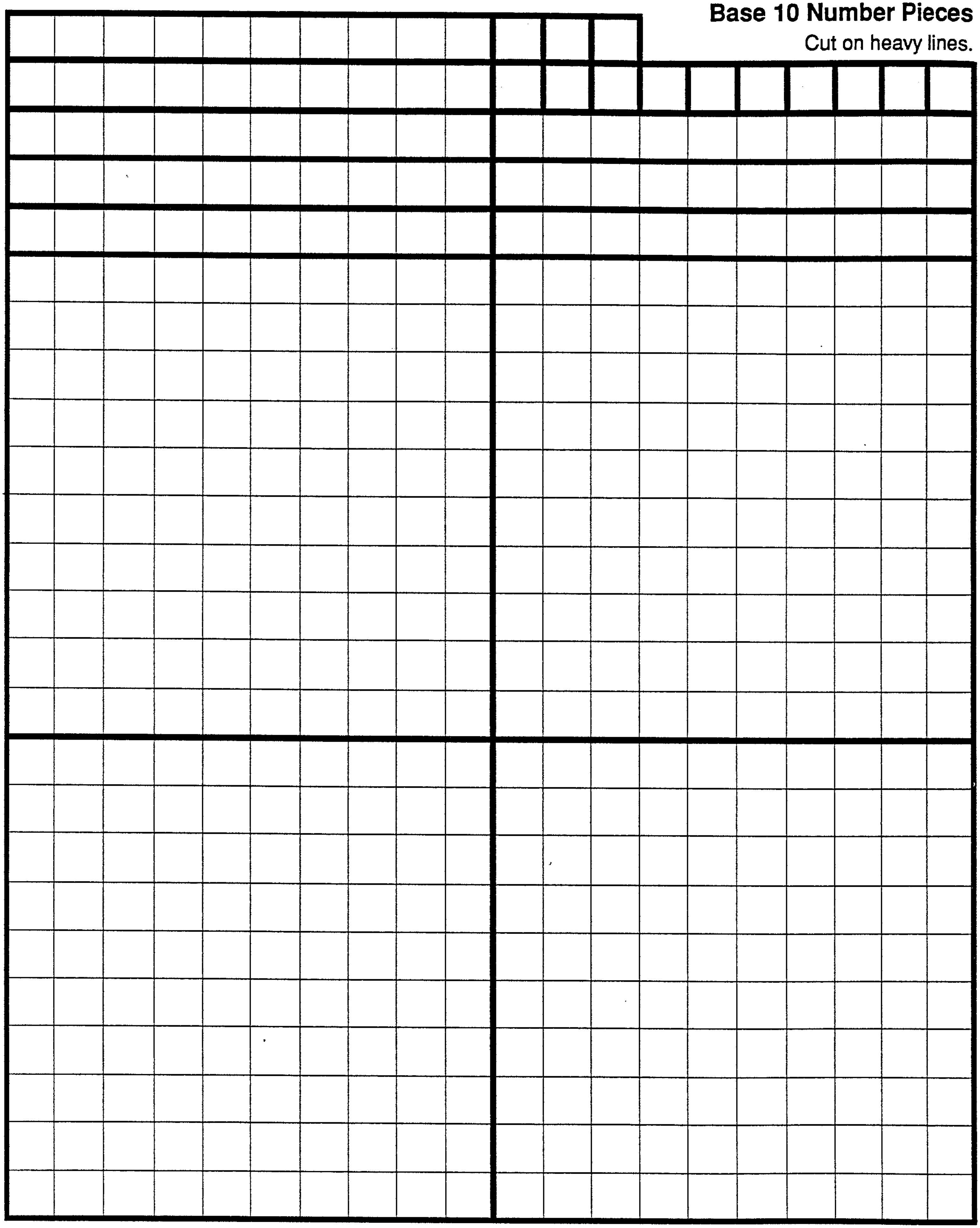


Activity Sheet III-2-C

Base 5 Grid Paper

Base 10 Number Pieces

Cut on heavy lines.



Base 10 Grid Paper