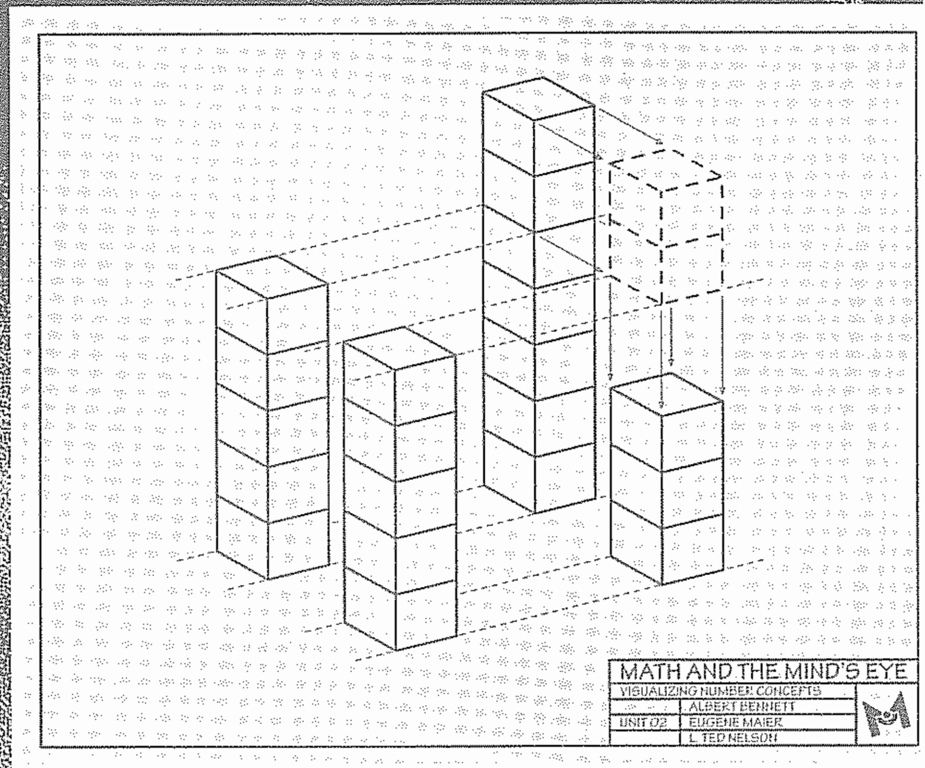
Unit II / Math and the Mind's Eye Activities



Visualizing Number Concepts

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Visualizing Number Concepts

1 Basic Operations
Students use tile to express their perceptions of the four basic arithmetic operations. Models for these operations used in subsequent activities are introduced.

2 Odd and Even Numbers
A tile pattern is used to visually represent the concepts of evenness and oddness of numbers. Sums and differences of odd and even numbers are investigated.

3 Factors and Primes
Counting numbers are represented by rectangular arrays of tile. These arrays are used to determine the factors of a number and introduce the concepts of prime and composite numbers.

4 Averaging
The average of two or more numbers is related to the process of evening off columns of cubes. Methods for arriving at the average of set of numbers are developed.

5 Greatest Common Divisors
The greatest common divisor of two numbers is obtained by finding the largest square that tiles a rectangle which has those numbers as dimensions.

6 Least Common Multiples
The least common multiple of two numbers is obtained by finding the smallest square composed of rectangles which has those numbers as dimensions.

Math and the Mind's Eye materials are intended for use in grades 4-9. They are written so teachers can adapt them to fit student backgrounds and grade levels. A single activity can be extended over several days or used in part.

A catalog of Math and the Mind's Eye materials and teaching supplies is available from The Math Learning Center, PO Box 3226, Salem, OR 97302, 1 800 575–8130 or (503) 370–8130. Fax: (503) 370–7961.

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Math and the Mind's Eye

These materials were prepared with the support of National Science Foundation Grant MDR-840371.

ISBN 1-886131-14-7

Basic Operations

O V E R V I E W	**Prerequisite Activity**
Students use tile to express their perceptions of the four basic arithmetic operations. Models for these operations which are used in subsequent activities are introduced. Part I deals with multiplication and division and Part II discusses addition and subtraction.	None **Materials** At least 15 tile per student.

Actions

Comments

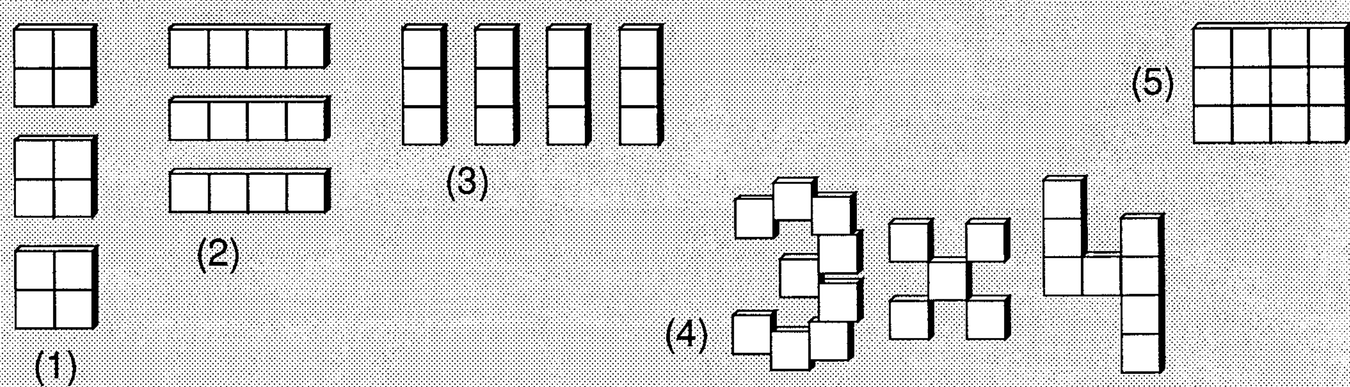
Part I **Multiplication and Division**

1. Distribute 15 to 20 tile to each student. Write "3 × 4" on the blackboard. Ask each student to use tile to represent the expression on the board. Emphasize that there is no *right* way to do this and that you expect a variety of representations.

1. Writing "3 × 4" on the blackboard rather than saying "three times four", "the product of three and four", "three multiplied by four", or some other phrase, forces the students to focus on the symbols. The intention is that students determine the meaning of the symbols for themselves.

2. (a) Call attention to the various representations. Ask students to talk about theirs. Accept all without judgment.

2. The representations will give you some insight about the ways in which your students view multiplication. Some possible representations are shown below.



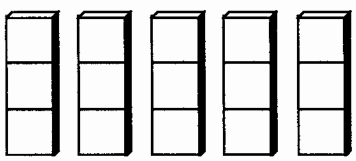
(b) Draw attention to the rectangular array model (5). If no one makes this model, make it yourself and add it to the collection of representations of 3 × 4.

Some students may think of 3 × 4 as 3 sets of 4 as in sketches (1) and (2). Others will think of it as 4 sets of 3 as in (3). Those who form symbols, as in (4), may think of arithmetic predominantly as symbols and symbol manipulation without other meaning. The rectangular array (5) is a particularly useful model of multiplication. Note that models (2) and (3) are easily converted to rectangles by pushing the rows or columns together. *(Continued)*

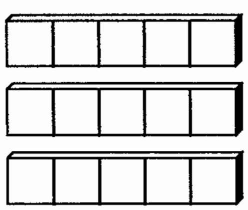
Actions

3. Write “15 ÷ 3” on the board. Ask your students to use tile to represent this expression. Call attention to the various representations and discuss them with your students. Again, accept all representations without judgment.

4. Discuss the *grouping* and *sharing* methods of division. Give examples, or ask your students to give examples of both uses of division.

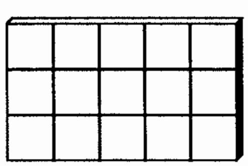


15 ÷ 3 : grouping method



15 ÷ 3 : sharing method

(Note that both tile arrangements become the same rectangular array when rows or columns are pushed together:

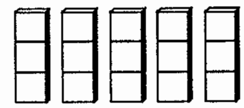


The array contains 15 tiles in 3 rows and 5 columns.)

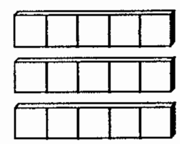
Comments

2. *(Continued)* Thus, a 3 × 4 rectangle may be viewed as either 3 sets of 4 or as 4 sets of 3. This illustrates the commutative nature of multiplication.

3. Both symbolic and non-symbolic representations may occur. Two common non-symbolic models are five groups of three,



or three groups of five.



4. Representing 15 ÷ 3 by 5 groups of 3 is an example of the *grouping* model of division. In this case, 15 ÷ 3 is thought of as the number of groups of 3 into which 15 can be divided, as in the question: “If each student is to receive 3 pencils, how many students will 15 pencils supply?”

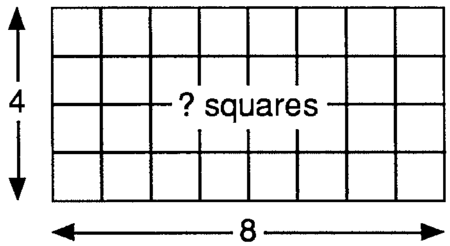
Representing 15 ÷ 3 by 3 sets of 5 is an example of the *sharing* method of division. Here, 15 ÷ 3 is thought of as the number of objects in each of 3 sets when 15 objects are shared equally among them, as in the question: “If 15 pencils are divided equally among 3 students, how many will each student get?”

The grouping method of division is also called the *subtractive* or *measurement* method since groups are subtracted away or measured off. The sharing method is also called the *dealing* or *partitive* method since objects are dealt or partitioned into sets.

Actions

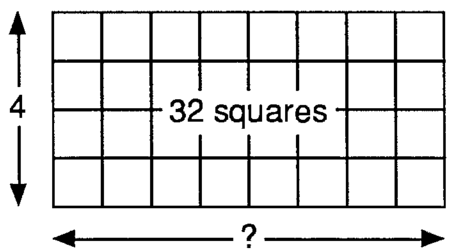
5. Show how multiplication and division are related in the rectangular array model.

Figure A



4 × 8 is the number of squares

Figure B



32 ÷ 4 is the missing dimension

Comments

5. The process of multiplication may be thought of as determining the number of tile in a rectangular array when the number of tile along each edge is known (or, equivalently, finding the area of a rectangle when the dimensions are given. See Figure A).

The process of division may be thought of as determining the number of tile on one side of the array when the number of tile on the other side and the total number in the array are known (or, finding one dimension of a rectangle given the other dimension and the area. See Figure B.).

Part II **Addition and Subtraction**

6. Ask your students to form two groups of tile, one containing 9 tile and the other 5. Write "9 + 5" on the blackboard and ask your students to use their tile to represent the expression on the board. Discuss what happens.

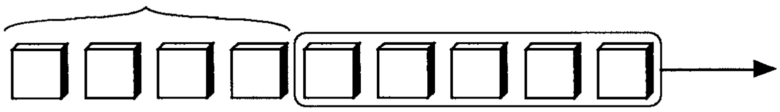
6. Most students will combine the groups of 9 and 5.

7. Write "9 – 5" on the board and ask your students to represent this expression with their tile. Discuss what happens.

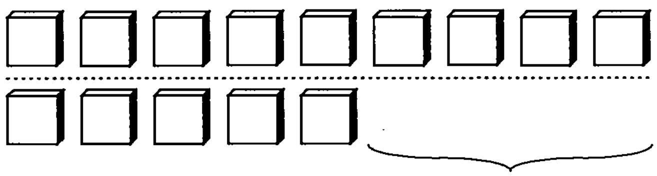
7. Many students will form a group of 9 tile and take 5 tile from this group. Other students may determine how many tile must be added to a group of 5 tile to match the group of 9 tile.

Actions

8. Discuss the *take-away* and *difference* methods of subtraction. Give examples, or ask your students to supply examples, of both uses of subtraction.



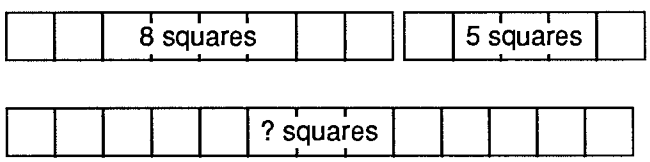
9 – 5 : take-away method



9 – 5 : difference method

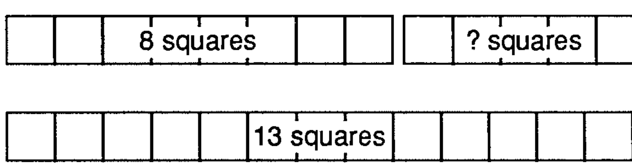
9. Show how the operations of addition and subtraction are related.

Figure C



? = 8 + 5

Figure D



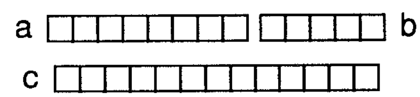
? = 13 – 8

Comments

8. Carrying out 9 – 5 by taking 5 tile from 9 tile is an example of the *take-away* method of subtraction. This method is appropriate in responding to the question: "If a student has 9 pencils and gives 5 of them away, how many pencils does the student left?"

Determining 9 – 5 by finding how many tile must be added to 5 tile to match a group of 9 tile is an example of the *difference* method of subtraction. This method is appropriate for the question: "If a student has 9 pencils and another has 5, how many more pencils will the first student have than the second."

9. Suppose there are 3 groups of squares *a*, *b* and *c* such that the number of squares in *a* and *b* together is the number of squares in *c*.



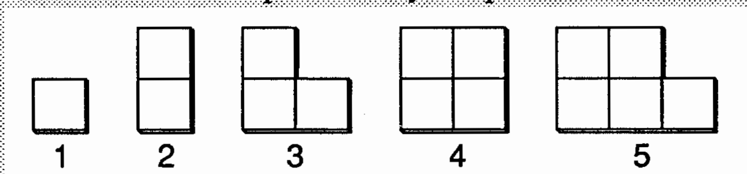
In the process of addition, one knows the number of squares in *a* and *b* and seeks the number of squares in *c* (Figure C).

In the process of subtraction, one knows the number of squares in *c* and one of the groups *a* or *b* and seeks the number of squares in the other group (Figure D).

Odd and Even Numbers

OVERVIEW

The concept of oddness and evenness of numbers becomes visually clear when numbers are represented by this pattern of tiles:



This activity will use this model and extend the idea to oddness and evenness of sums and differences.

Prerequisite Activity

None

Materials

At least 35 tiles for each student; odd and even demonstration cards (see Comment 8).

Actions

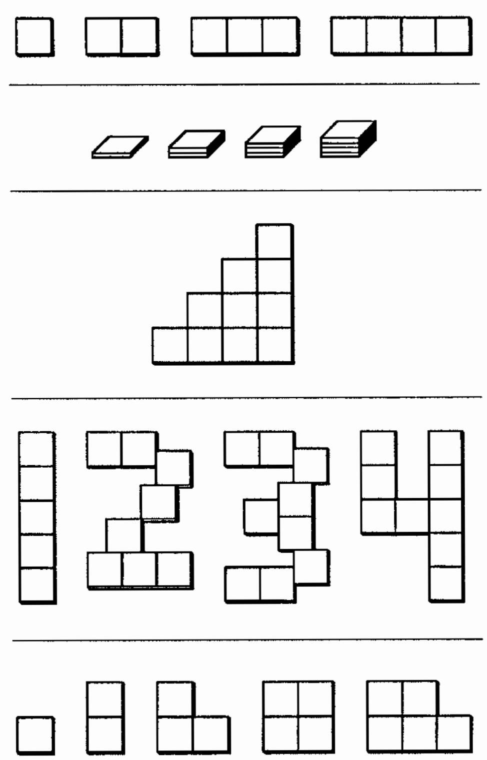
1. Write the first seven counting numbers on the chalkboard or overhead. Tell the students to use their tiles to represent each counting number.

2. Have the students observe the different ways that others have represented the numbers (or, you could summarize different ways on the overhead projector).

Comments

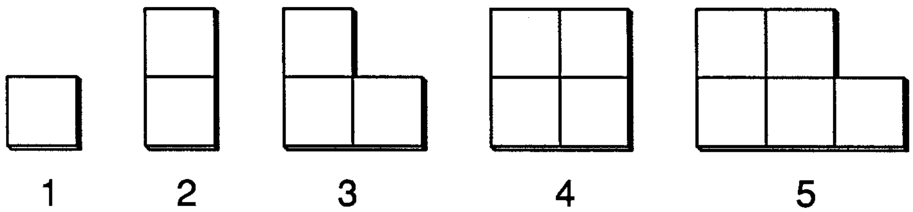
1. You may have to reword the directions so everyone understands. Circulate among the students making reassuring comments. There is no one "correct" way to do this activity.

2. Be prepared for variety. Here are some examples:



Actions

3. Acknowledge all the representations and select the following pattern for discussion.



Demonstrate this pattern on a table or put it on the overhead. Ask for volunteers to build representations for 6, 7 and 8.

4. Ask the students to represent a few more numbers in this pattern.

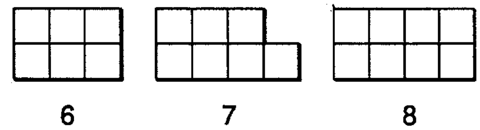
5. Point out that the numbers 1, 3, 5, 7, etc. are called *odd numbers* and the others (2, 4, 6, 8, etc.) are called *even numbers*. Comment on how these names (odd and even) are suggested by the tile patterns. Ask the students to picture in their minds, or draw a diagram of, the following numbers to determine if they are odd or even: 17, 42, 79, 106.

6. Mention to the students that adding can be thought of as combining. Demonstrate adding 5 and 7 by pushing their corresponding pattern members together to get 6 groups of 2, or 12. Have the students add 4 and 7 by pushing their pattern members together. Ask them if their answer is odd or even.

Comments

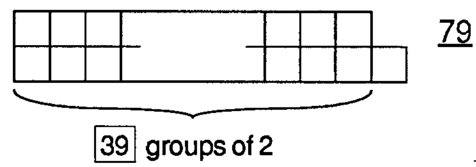
3. If no one forms this pattern, introduce it yourself. Put a numeral below each pattern member.

The next 3 are:



4. Encourage students to build these with tile.

5. This is intended to give students a visual way of distinguishing between odd and even numbers. The time you spend on this action will depend upon the level of your class. You may want to diagram some numbers as follows. The students can supply the number of groups of 2.



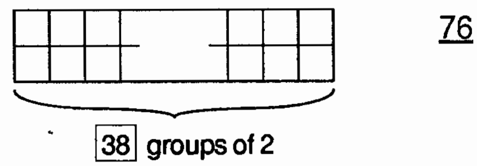
6. Be sure to use the odd and even tile representations of numbers:

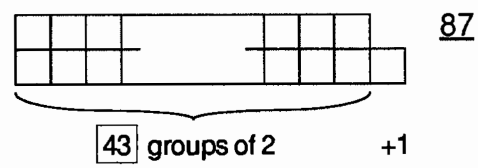
Actions

7. Tell the students that they will be asked to add some numbers and need not divulge the answer but, simply, whether the *answer is odd or even.* Try sums like: 11 + 3, 14 + 9, 16 + 18, etc.

Comments

7. The number of examples you use depends upon your class. The students need experiences visualizing so they can eventually generalize. Be sure to refer back to the odd and even patterns for these numbers. If you add large numbers like 76 + 87, it may help to diagram them both so it is visually clear that the sum will be odd.





Actions

8. Ask students to picture odd and even numbers in their minds as they reflect on the following questions. Have them explain or discuss the reasons for their answers.

(a) If you add two numbers will the sum be even or odd?

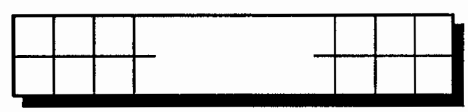
(b) If you add two even numbers will the sum be even or odd?

(c) If you add an even number and an odd number will the sum be even or odd?

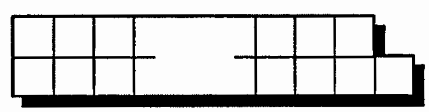
(d) What happens when you add three odd numbers ? Two odd numbers and an even number? Two evens and an odd? Five odds and one even?

Comments

8. It may be helpful to have some "generic" odd and even patterns made out of cardstock to illustrate or reinforce the students' conclusions. These large demonstration cards have no particular number assigned and look as follows:



even card



odd card

You may want several of each if you plan to discuss part (d).

Actions

9. Similar questions can be asked about subtraction. Is the result odd or even when you subtract: an odd number from an odd number? an even from an even? an odd from an even? an even from an odd?

Comments

9. If your class is ready to tackle this, proceed as in Action 8 to encourage visual thinking. Again, the demonstration cards may be helpful.

Unit II • Activity 3

Factors and Primes

O V E R V I E W

Factors and primes are introduced in a concrete setting by expressing counting numbers as rectangular arrays of tile. Prime numbers are those numbers which have exactly one nonsquare rectangle representation, while composite numbers have two or more such representations. The factors of a number are the dimensions of its rectangle representations.

Prerequisite Activity

None

Materials

140 tile for each group of 3 to 4 students, Activity Sheet II-3, a transparency of Activity Sheet II-3 and a grid transparency.

Actions

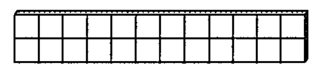
1. Divide students into groups of 3 or 4. Distribute tile to each group. Explain what is meant by a rectangular array of tile. Have each student in the group take 12 tile and form those tile into a rectangular array.

2. Make sure each group of students builds all three 12-tile rectangles. Now ask them to add to their collection all rectangles that can be made with fewer than 12 tile. Have them keep the entire collection on display and eliminate duplicates. Clarify directions as necessary.

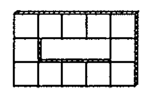
Comments

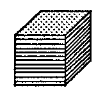
1. It will take a large space to arrrange the tile. If tables are not available you may wish to have students work on the floor.

This is a rectangular array.



These are not.

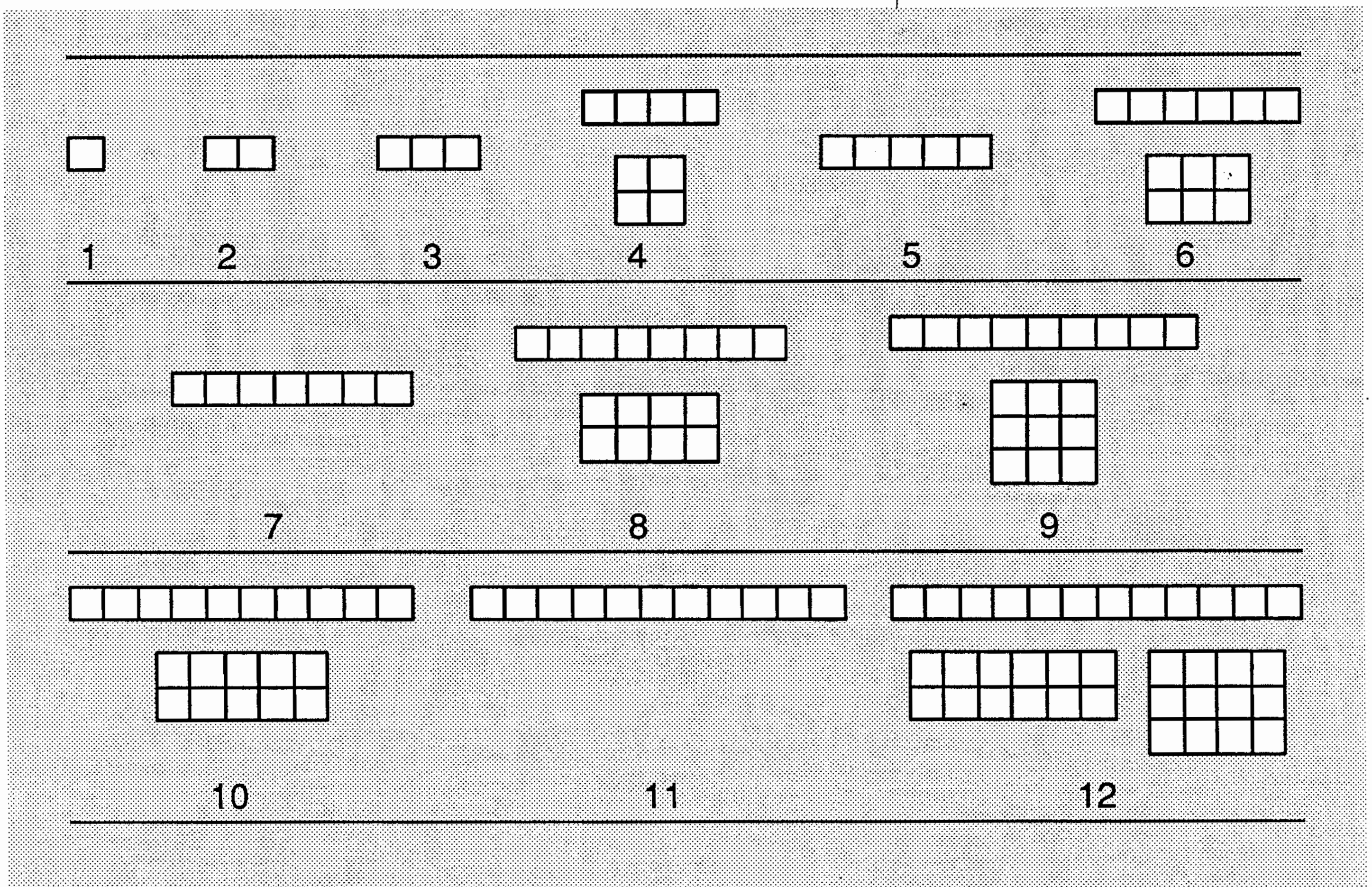




There are three possible rectangular arrays of 12 tiles: 1 by 12, 2 by 6, and 3 by 4. Some students may consider the 3 by 4 and the 4 by 3 arrays as being different. Tell them that for this activity they will be considered as the same rectangle because one can be made to fit exactly on top of the other.

2. There will be 19 different rectangles using a total of 139 tile (see picture next page).

Actions | **Comments**



3. If they have not done so already, ask each group to arrange their rectangles in numerical order.

3. Arranging the rectangles in order of the numbers they represent will make the following activities easier.

4. Give each student Activity Sheet II-3. Lead the students through the first few entries on the chart so there is no confusion. A transparency of the activity sheet can be used. Encourage them to refer to their rectangles.

Define a factor of a number as one of the dimensions of a rectangle for that number.

4. The first few entries are listed below.

No.	Number of Rectangles	Dimensions of Rectangles	Factors of the Number
1	*1*	*1 x 1*	*1*
2	*1*	*1 x 2*	*1, 2*
3	*1*	*1 x 3*	*1, 3*
4	*2*	*1 x 4, 2 x 2*	*1, 2, 4*

Actions

5. Ask the students for observations about the numbers and their rectangles. What can they observe about numbers by looking at their rectangular pictures?

6. Have the students rearrange their rectangles into three groups: (1) Those with exactly one factor, (2) Those with exactly two factors, and (3) those with more than two factors. Find other numbers whose rectangles belong in groups (2) and (3).

7. Ask the students if it is possible to find any other number, like 1, which has exactly one representation with identical dimensions.

8. Tell the students that numbers with exactly two factors are called *prime* numbers and that numbers with more than two factors are called *composite* numbers. The number one is therefore neither a prime number nor a composite number.

Have them complete Activity Sheet II-3 for the numbers 13-20 and then put a P or C in the last column to indicate if the number is prime or composite.

Comments

5. It is impossible to predict what students might observe. One way to give credibility to student observations is to summarize and list them on the overhead or chalkboard. In this activity we are particularly interested in observations like, "Some numbers have only 1 rectangle," and "Certain numbers have squares."

6. The intent of this activity is to classify counting numbers according to their number of factors.

7. There will be no other numbers grouped with 1. To be grouped with 1 and have identical dimensions the number would have to be square. But any other square number has at least two rectangles.

8. The factor approach is a precise way to define prime and composite numbers. Looking at rectangles and factors also makes it clear why the number one is neither prime nor composite and is the only such counting number.

Actions

9. Ask the students to sketch rectangles for these numbers: 24, 31, 72, 144; and

(a) determine which are prime,

(b) record all the factors of each number.

10. Looking at the tile rectangles (or diagrams) and the chart on Activity Sheet II-3, questions like the following may be posed as challenges or ending activities:

A. Some numbers are called *square* numbers. Looking at your tile rectangles, which numbers do you think they are? Will 72 be a square number? 48? 64?

B. What do you notice about the number of factors for any square number?

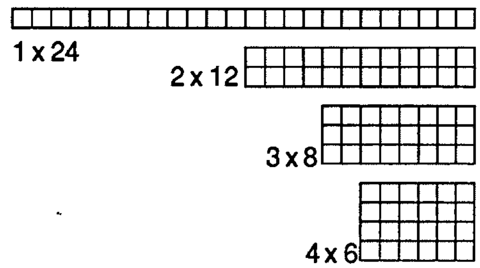
C. What is the first number that can be represented by exactly 4 different rectangles? 5 rectangles?

D. Look at the rectangle arrays of some numbers with exactly three factors. Can you find additional numbers with exactly three factors? What conclusions can you draw?

E. In your chart (Activity Sheet II-3) you have recorded numbers with 1, 2, 3, 4, 5 and 6 factors. Given any positive number, do you think it is possible to find a number that has that many factors? Try to find a number that has 8 factors, for example.

Comments

9. It may be helpful to do an example with the entire class. Trace around the desired region on a transparency grid sheet letting each square represent a tile. Start with 1-by rectangles and move systematically to 2-by, 3-by, etc.



10.

A. Those numbers which have a square among their rectanglur arrays are called square numbers: 1, 4, 9, 16, 25, etc.

B. Square numbers will always have an odd number of factors. Each rectangle, except the square, contributes two factors. The square array contributes only one factor.

C. Twenty-four is the first number that has four rectangular arrays. The first number with five rectangles is 36. There are other numbers with 4 or 5 rectangles.

D. Only square numbers have an odd number of factors. Numbers like 4, 9, 25 and 49 have exactly 3 factors but numbers like 1, 16, 36 and 64 do not. Only numbers which are the squares of primes have exactly 3 factors.

E. Given any positive counting number, it is possible to find numbers with that many factors. The following are all numbers with 8 factors: 24, 30, 40, 42, 54, and 56.

Name ______________________

Factors and Primes

No.	Number of Rectangles	Dimensions of Rectangles	Factors of the Number	
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				

Averaging

OVERVIEW

The *average* of two or more numbers is related to the process of *evening off* columns of cubes. Methods for arriving at the average are developed. Part I concerns the average of two numbers. Part II deals with the average of more than two numbers.

Prerequisite Activity
None

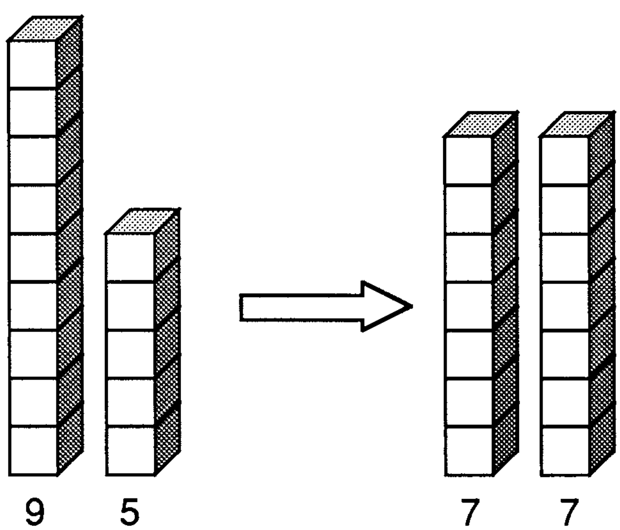
Materials
30 demonstration cubes and 4 half-cubes (see Comment 1); grid paper

Actions

Part I Averaging Two Numbers

1. Form two columns of cubes, one of height 9 and the other of height 5. Ask a student to even off the two columns so that both columns have the same height.

2. Ask if anyone knows the mathematical name for the evening-off process.



The average of 9 and 5 is 7.

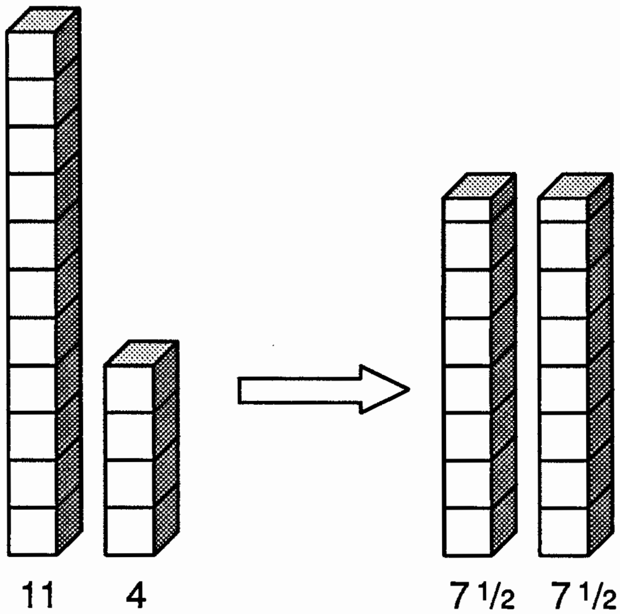
Comments

1. Use cubes that are large enough to be easily seen by the whole class. Half-cubes are needed in subsequent actions. They can be made by splitting a straight-grained wooden cube with a sharp instrument.

2. The evening-off process is called *averaging*. The height of the evened off columns is the *average* of the heights of the two original columns. Thus the average of 9 and 5 is 7.

Actions

3. Form two columns of cubes with heights of 11 and 4. Ask a student to even off these two columns.



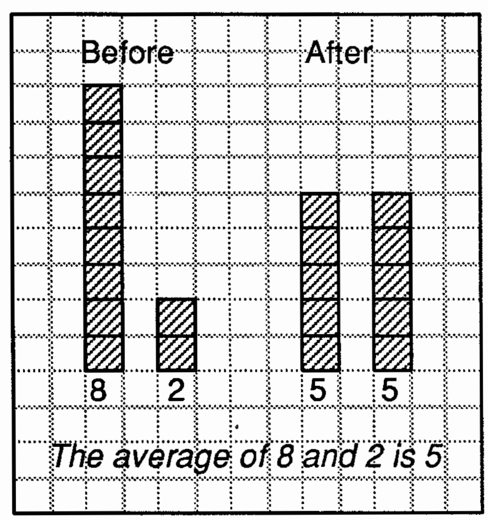
The average of 11 and 4 is 7 1/2.

4. Distribute grid paper to each student. Have students draw sketches of the front view of two columns of cubes of heights 8 and 2 before and after they are evened off. Then have them write a statement under their sketch concerning the average of 8 and 2.

Comments

3. The student may wonder what to do with the extra block. Suggestions may be sought from other students. If the half-cubes have not been noticed, call attention to them.

4. If a student has difficulty determining the height of the evened-off columns, ask them to use cubes to determine the height. A completed sketch might look as follows:



Actions

5. Repeat Action 4 for the following pairs of numbers:
(a) 9 and 5, (b) 6 and 1, (c) 7 and 4, (d) 8 1/2 and 3 1/2.

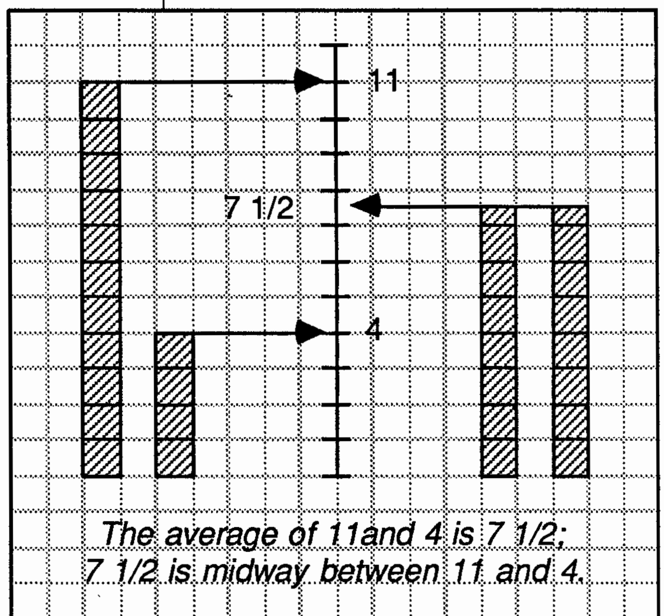
6. Ask students for any observations they have made about the average of two numbers. If no one notices it, point out the average of two numbers is midway between them.

7. Suppose columns of heights 23 and 13 have been evened off. Ask students to find the height of the evened-off columns. Ask several students to describe the methods they used.

Comments

5. For each pair of numbers, cubes may be used to find the height of the evened off columns.

6. One may illustrate that the average of two numbers is midway between them by drawing the following sketch on a grid on the board or on an overhead transparency.



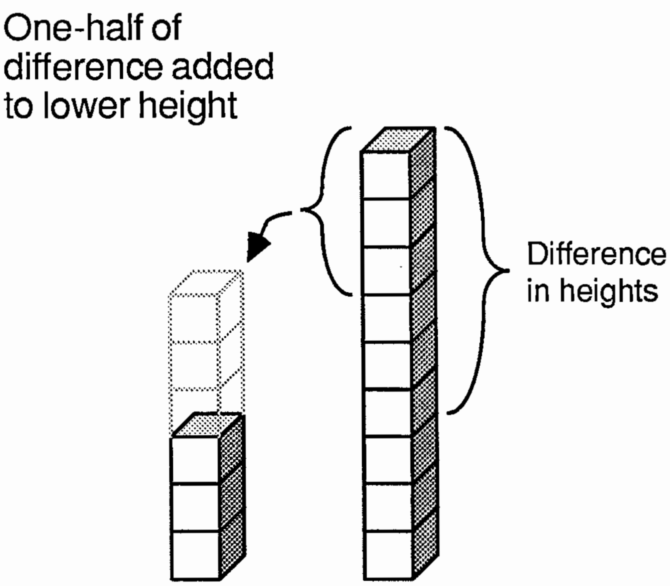
7. Students may use a variety of methods. Any method that yields a correct answer is acceptable.

Actions

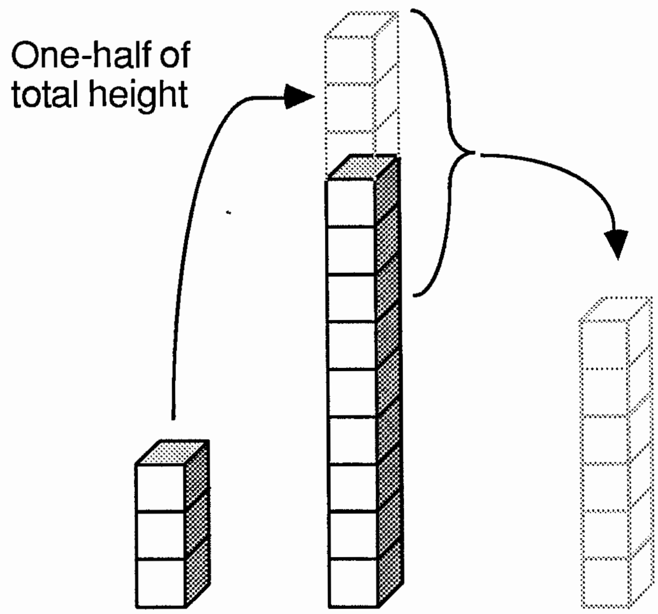
8. Discuss general methods for finding the height of two columns after they have been evened off. Use cubes to illustrate the methods suggested. If no one suggests them, introduce the two methods shown below.

Comments

8. In method A, half the difference between the original heights of the two columns is added to the original height of the lower column. In method B, the original height of the two columns are added together and then divided by two.



Method A



Method B

Actions

9. a) Ask the students to find the average of 42 and 26 by both methods. Discuss.

b) Have the students find the average of other pairs of numbers using whatever method they prefer.

Comments

9. The average of 42 and 26 will be the height when columns of 42 and 26 are evened off. By method A, the height of the evened-off columns will be 26 plus half the difference between 26 and 42, or $26 + {}^{1}/{2}\,(16) = 34$.

By Method B, the evened off height will be half of the combined heights of 26 and 42 or ${}^{1}/{2}\,(68) = 34$.

Actions	Comments

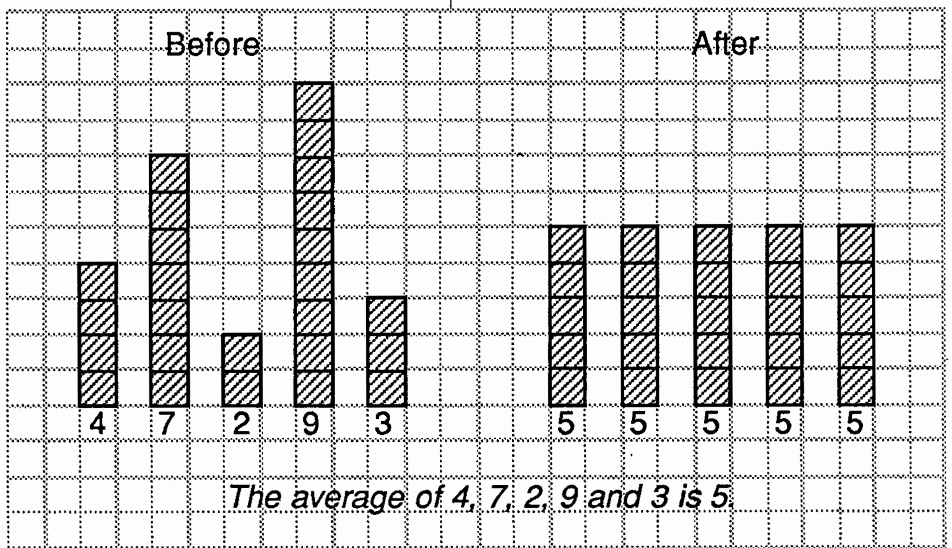
Part II Averaging Several Numbers

10. a) Form five columns of cubes with heights 4, 7, 2, 9 and 3. Ask each student to sketch the front view of these columns on grid paper.

10. Again, you may want to draw a sketch on a grid on the board or on an overhead transparency. A sketch might look as follows:

b) Ask a student to even off these columns (without changing the number of columns). Then ask the students to sketch the columns after they have been evened off.

c) Tell students that the height of the evened-off columns is the *average* of the height of the original columns. Since in this instance the height of the evened-off columns is 5, the average of 4, 7, 2, 9 and 3 is 5.



11. Now form four columns of heights 8, 5, 4 and 7. Ask each student to sketch the columns as they appear and then as they would appear after being evened off (without changing the number of columns).

11. The height of the evened off columns will be 6. This can be checked by asking a student to even off the columns of cubes.

12. Ask the students to write a statement about the average of 8, 5, 4 and 7.

12. One statement is, "The average of 8, 5, 4, and 7 is 6."

13. Repeat Actions 12 and 13 for each of the following situations:

a) 6 columns of height 3, 7, 9, 4, 6 and 1.

b) 4 columns of heights 9, 3, 6 and 8.

13. It is important that the number of columns remains unchanged in the evening-off process.

Half-cubes will be needed to even off the columns in part (b).

The heights of the evened-off columns are (a) 5 and (b) 6 1/2.

Actions

14. Ask students to consider the following. Suppose columns of 23, 30, 21, 34 and 22 are evened off. What is the height of the evened-off columns?

15. Discuss methods that might be used to find the evened-off height in Action 14.

16. For each of the following, ask students to find the average by both of the methods described above:

a) 27, 42, 39, 24

b) 31, 26, 41, 20, 27

Comments

14. The height is 26. Students may wish to discuss this question with each other. They may use a variety of methods to arrive at an answer. Any method that works is appropriate.

15. One method requires only simple computations. In this method, numbers are evened off in much the same manner as columns of cubes are evened off by taking cubes from one column and adding to another. For the numbers given in Action 14, one possible way of doing this is the following:

The Original Numbers:	23	30	21	34	22
Taking 5 from 2nd number and adding to 3rd:	23	25	26	34	22
Taking 4 from 4th number and adding to 5th:	23	25	26	30	26
Taking 3 from 4th number and adding to 1st:	26	25	26	27	26
Taking 1 from 4th number and adding to 2nd:	26	26	26	26	26

The average of 23, 30, 21, 34 and 22 is 26.

Another method of finding the height of the evened-off columns is to think of the five original columns stacked on top of one another to form a single column. Dividing this single column in five equal parts will produce the evened-off columns. Thus, the height of the evened-off columns will be the sum of the heights of the original columns divided by 5:
$(23 + 30 + 21 + 34 + 22)/5 = 130/5 = 26$.
This computation is readily performed on a calculator.

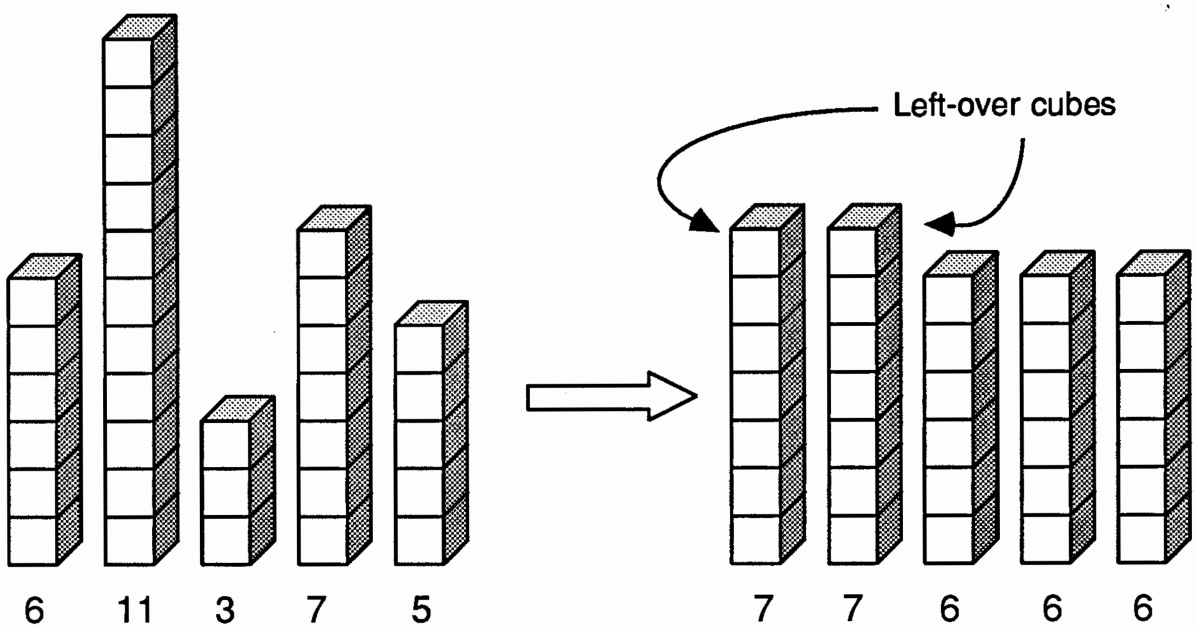
16. The averages are (a) 33 and (b) 29. Note that all of the computations avoid the use of fractions. The following Action is optional for students who have a rudimentary knowledge of fractions.

Actions

17. (Optional) Form five columns of blocks with heights of 6, 11, 3, 7, and 5. Ask a student to even off the columns. Discuss the average in this situation.

Comments

17. It will not be possible to even off the columns without cutting cubes. Since the columns, if evened off, would be more than 6 but less than 7 cubes high, the average is between 6 and 7.



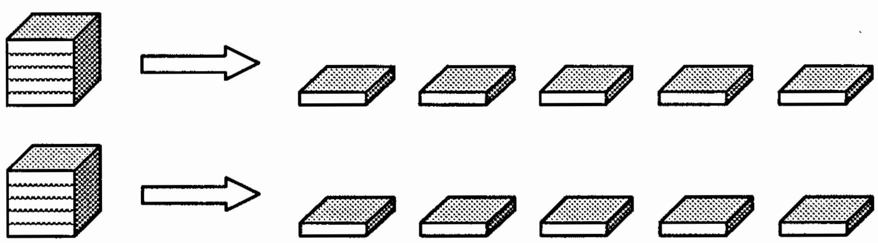


Figure 1

If cubes could be cut, one could cut each of the two left over cubes into 5 equal parts and distribute these parts among the columns. Each column would receive two parts of 1/5 each (see Figure 1). Thus, each evened off column would be 6 2/5 cubes high.

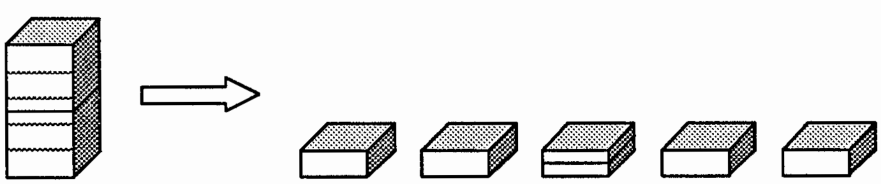


Figure 2

Alternately, the two left-over cubes could be glued together and then cut into 5 equal parts, which would then be distributed among the columns. Each of these parts would be 2/5 of a cube and each column would receive one of them (see Figure 2). Again, the height of the evened-off columns would be 6 2/5 cubes.

Greatest Common Divisors

O V E R V I E W

The greatest common divisor of two numbers is obtained by finding the largest square that tiles a rectangle which has those numbers as dimensions.

Prerequisite Activity

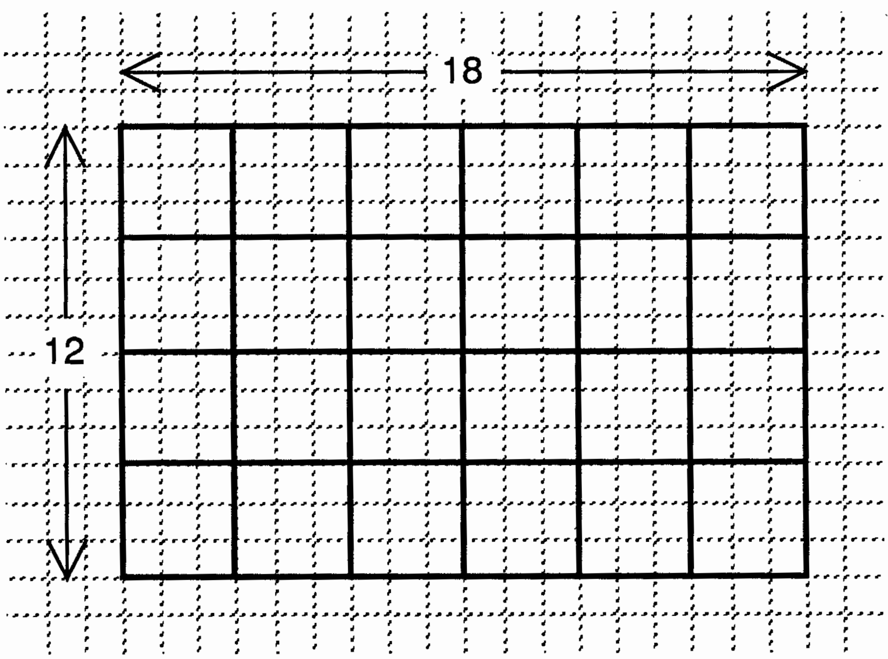
Unit II, Activity 1, *Basic Operations*; Unit II, Activity 3, *Factors and Primes*.

Materials

Copies of Activity Sheets II-5 and grid paper for each student; transparencies as indicated for teacher use (see Comments 1 and 4).

Actions

1. Place a transparency of page 10 of this activity on the overhead. As the students watch, subdivide one of the 12 x 18 rectangles into 3 x 3 squares as shown below. Ask the students to suggest other ways in which a 12 x 18 rectangle can be subdivided into squares all of the same size. As a suggestion is made, subdivide one of the rectangles in the manner suggested.



Comments

1. You may want to have student volunteers go to the overhead and divide the rectangles as suggested.

If one remains on grid lines, a 12 x 18 rectangle can be divided into 6 x 6, 3 x 3, 2 x 2, and 1 x 1 squares. If one doesn't remain on grid lines, smaller squares are possible, e. g. 1/2 x 1/2 squares or 1/4 x 1/4 squares.

Actions

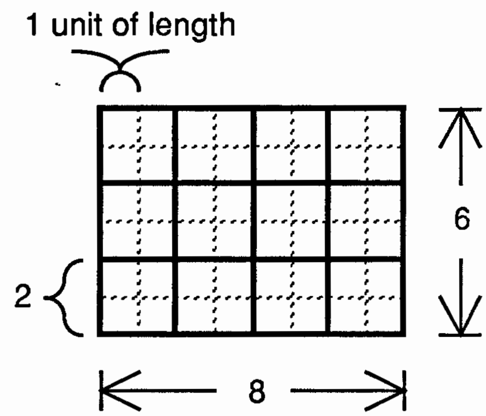
2. Give each student a copy of Activity Sheet II-5. For each rectangle, ask the students to divide it into squares, all of the same size, and, if possible, larger than 1 x 1 squares. Tell them the sides of the squares must coincide with grid lines. Have them record the information requested on the sheet.

Comments

2. A master of Activity Sheet II-5 is attached.

Make certain that the students understand that the *dimensions of a rectangle* are the lengths of its sides.

In this activity, the unit of length is the side of one square of the grid inscribed in the rectangles:



Dimensions of rectangle: 6 and 8
Length of side of square: 2

The rectangle in the upper right corner of the activity sheet can be divided into 3 x 3 squares. The rectangle in the middle can be divided into 2 x 2, 3 x 3 or 6 x 6 squares. If the edges of the squares coincide with grid lines, the rectangles at the bottom can only be divided into 1 x 1 squares.

If a rectangle can be divided into squares, all of the same size, we shall say a square of that size *tiles* the rectangle. For example, a 2 x 2 square tiles the 6 x 24 rectangle that appears in the middle of the activity sheet. That rectangle is also tiled by a 3 x 3 square and a 6 x 6 square.

Actions

3. Discuss the completed Activity Sheets with the students. Ask them for their observations relating the dimensions of a rectangle and the length of the side of a square which tiles it.

4. Tell the students that, in the remainder of this activity, all squares are to have sides whose lengths are whole numbers. Then, with the students help, list those squares which tile the 6 x 24 rectangle in the middle of the Activity Sheet. Ask the students to use this information to identify the greatest common divisor of 6 and 24.

Comments

3. The students will make various observations. If no one mentions it, point out that the side of a square that tiles a rectangle must fit evenly into both sides of the rectangle. Thus, both dimensions of a rectangle are *multiples* of the side a square that tiles it or, what is the same, the side of a square that tiles a rectangle is a *divisor* (or *factor*) of both dimensions of the rectangle. Hence, the side of the square that tiles a rectangle is a *common divisor* (or *common factor*) of the dimensions of the rectangle.

If a 1 x 1 square is the largest square that will tile a rectangle, then the dimensions of the rectangle are said to be *relatively prime* numbers. That is, two whole numbers are called *relatively prime* if 1 is their only common factor.

4. A 6 x 24 rectangle is tiled by 1 x 1, 2 x 2, 3 x 3 and 6 x 6 squares. Hence, the common divisors of 6 and 24 are 1, 2, 3 and 6. Thus the greatest common divisor of 6 and 24 is 6.

This information can be illustrated by showing the students an overhead transparency of page 11 of this activity.

In general, it is not a simple matter to determine which squares tile a rectangle. For example, it is not readily apparent which squares tile a 575 x 2185 rectangle.

In subsequent Actions in this activity, a procedure is developed for finding the largest square that tiles a rectangle. This enables one to find the greatest common divisor of two numbers. For example, the greatest common divisor of 575 and 2185 is 115 since, as one finds in Action 8, a 115 x 115 square is the largest square that tiles a 575 x 2185 rectangle.

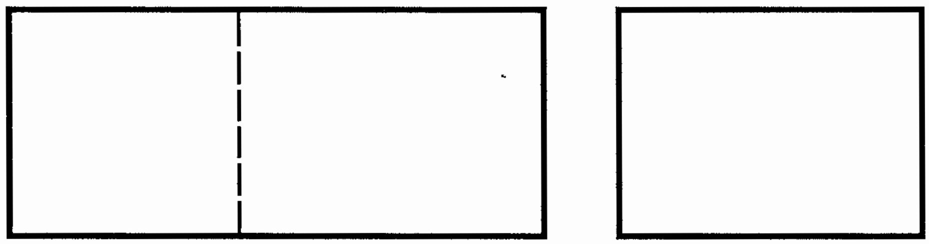
Actions

5. Distribute graph paper to the students. Have them draw pairs of rectangles that differ by a square. For each pair of rectangles they draw, ask them to determine the squares which tile each rectangle in the pair. Discuss their findings with the students.

Comments

5. A master for 1/2 cm grid paper is attached.

Two rectangles *differ by a square* if the smaller rectangle is obtained by cutting a square from the end of the larger, or, what is the same, the larger is obtained by adding a square to a side of the smaller. Rectangles A and B, shown below, differ by a square.

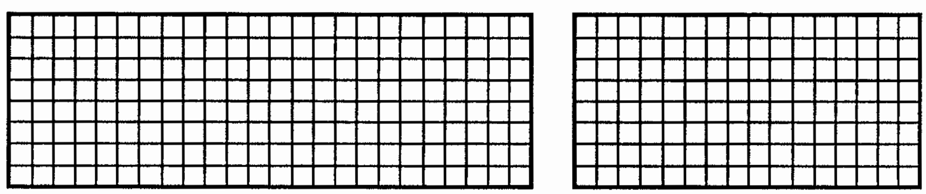


Rectangle A   Rectangle B

Here are the dimensions of several pairs of rectangles which differ by a square: 12 x 30 and 12 x 18, 9 x 8 and 1 x 8, 9 x 12 and 9 x 3, 5 x 10 and 5 x 5, 8 x 24 and 8 x 16.

The rectangles in each of these pairs are tiled by the same squares. For example, an 8 x 24 rectangle and an 8 x 16 rectangle are both tiled by 1 x 1, 2 x 2, 4 x 4, and 8 x 8 squares.

8 x 24   8 x 16



Both of these rectangles are tiled by 1 x 1, 2 x 2, 4 x 4, and 8 x 8 squares.

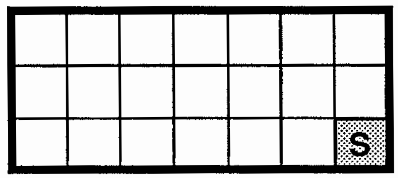
Continued next page.

Actions

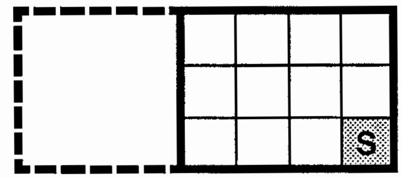
6. Discuss with the students how the result of Action 5 can be used to find the squares which tile a rectangle and, hence, the greatest common divisor of the rectangle's dimensions.

Comments

5. *(Continued)* It is always the case that two rectangles which differ by a square are tiled by the same squares. For, on the one hand, if a rectangle is tiled by square S, and then a square is cut off the end of the rectangle, the resulting rectangle is still tiled by S:

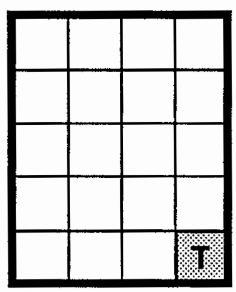


This rectangle is tiled by square S.

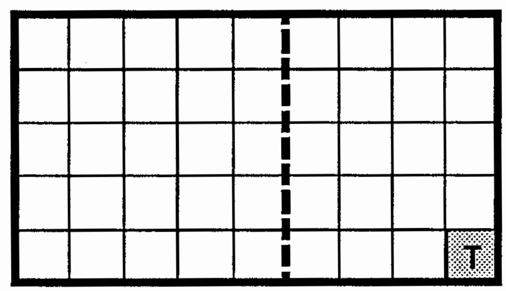


This rectangle is also tiled by square S.

On the other hand, if square T tiles a rectangle and the rectangle is extended by adding a square to the end of it, the tiling by T can also be extended:



This rectangle is tiled by square T.



This rectangle is also tiled by square T.

6. In Action 6, it was found that cutting a square off the end of a rectangle results in another rectangle that is tiled by the same squares. Hence, as illustrated below, the squares which tile a rectangle can be found by successively cutting a square off the end of the rectangle, the resulting rectangle and all subsequent rectangles until a square is obtained. (Once a square is obtained, the process stops since it is not possible to cut a square off the end of a rectangle which is itself a square.) The squares that tile this final square are those which tile the original rectangle.

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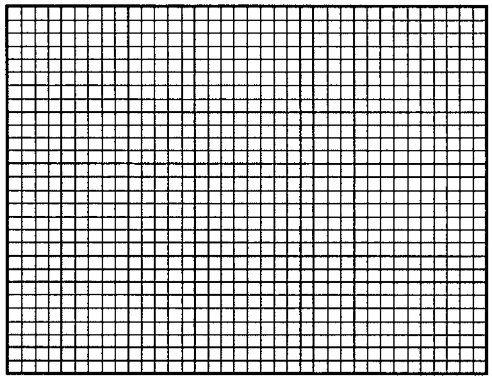
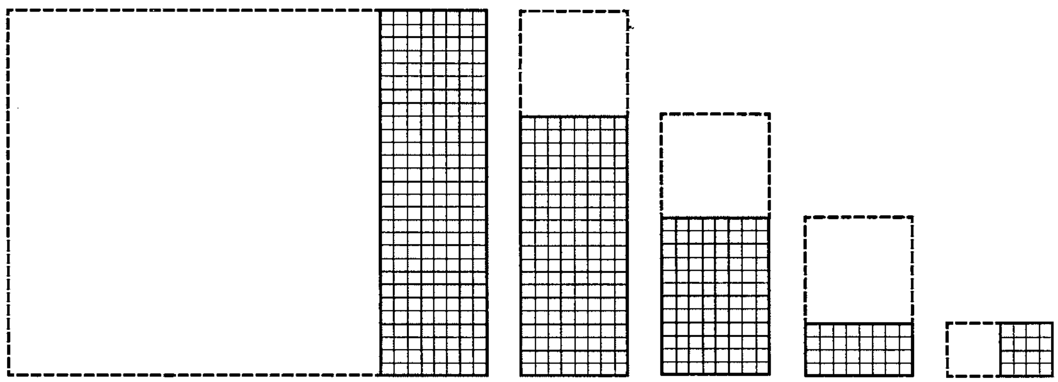
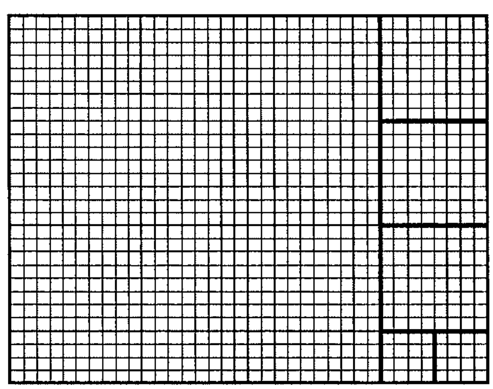


6. *(Continued)* The process is illustrated here for a 28 x 36 rectangle. You may want to have the students carry out the process. They can do this by drawing a series of sketches on grid paper or by actually cutting out a 28 x 36 rectangle and successively cutting off squares. In the latter case, you may want the students to cut out two 28 x 36 rectangles, one to use in the cutting process and one to retain to compare with the successive rectangles.



Cutting a square off the end of a 28 x 36 rectangle, and successive rectangles, leads to a 4 x 4 square.

The squares that tile a 28 x 36 rectangle are the same as those that tile a 4 x 4 square, namely a 1 x 1, 2 x 2 and 4 x 4 square. Since a 4 x 4 square is the largest square that tiles a 28 x 36 rectangle, the gcd of 28 and 36 is 4. Note that the largest square that tiles the rectangle is the final square obtained in the cutting-off process.



The above sequence of sketches can be replaced by the single sketch shown at left. This sketch was drawn by starting with a sketch of the original rectangle and marking off a square at the end of this rectangle, and subsequent rectangles, until only a square remained. This last square is the largest square that will tile the original rectangle.

Actions

7. Distribute grid paper to each student. Ask the students to use the process of Action 6 to find the greatest common divisor of the following pairs of numbers:

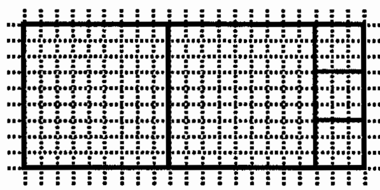
(a) 9 and 21 (b) 15 and 26

(c) 20 and 34 (d) 30 and 42

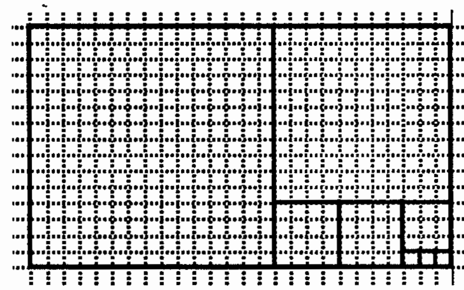
Comments

7. A master for 1/2 cm grid paper occurs at the end of this activity. Each student will need at least two sheets.

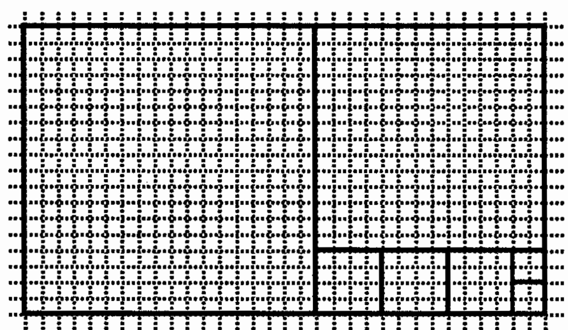
The gcd of a pair of numbers can be found by sketching a rectangle with those numbers as dimensions, and then successively marking squares off the end of this rectangle, and subsequent rectangles. The last square is the largest square that tiles the rectangle. The length of the side of this square is the gcd of the pair of numbers.



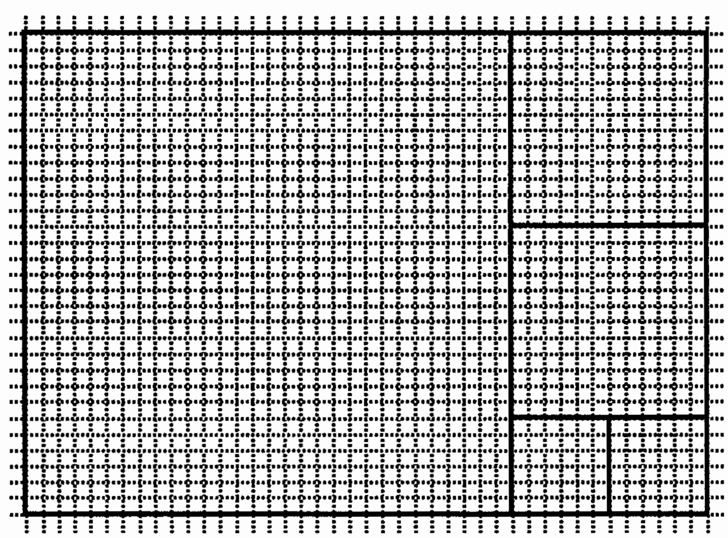
(a) the gcd of 9 and 21 is 3



(b) the gcd of 15 and 26 is 1



(c) the gcd of 20 and 34 is 2



(d) the gcd of 30 and 42 is 6

Actions

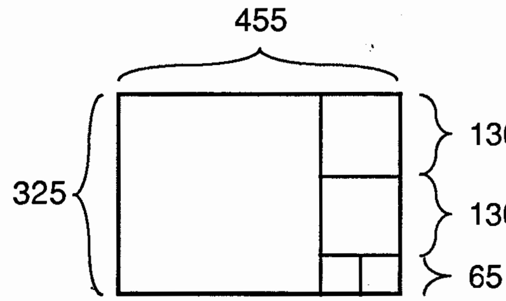
8. Ask the students to draw sketches to find the greatest common divisor of the following pairs:

(a) 325 and 455 (b) 420 and 714

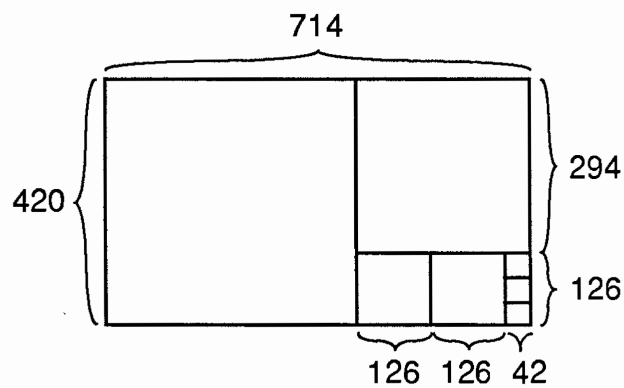
(c) 575 and 2185

Comments

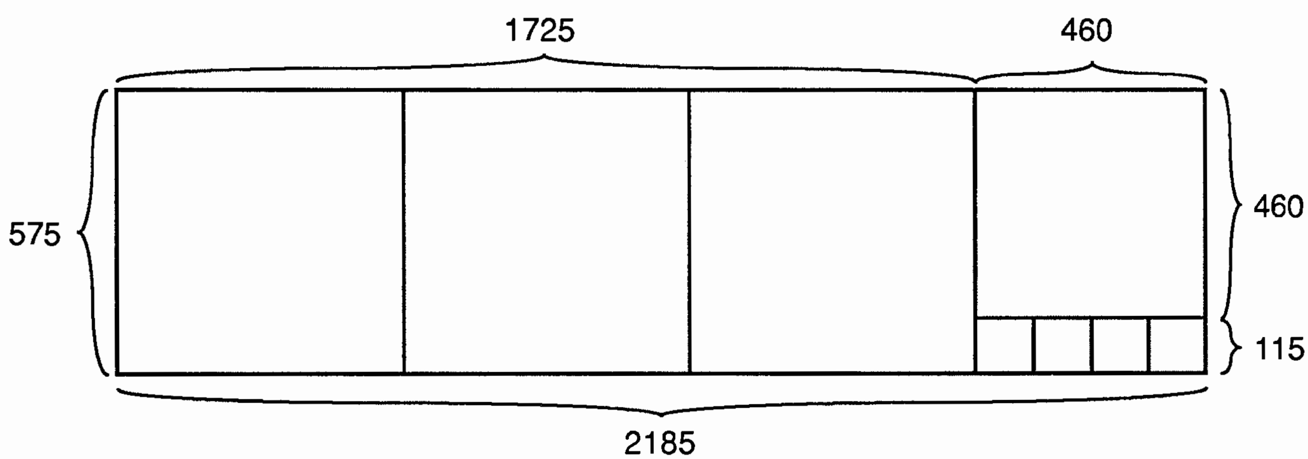
8. The numbers are too large to draw grid paper sketches. However rough sketches can be drawn:



(a) the gcd of 325 and 455 is 65



(b) the gcd of 420 and 714 is 42



(c) the gcd of 575 and 2185 is 115

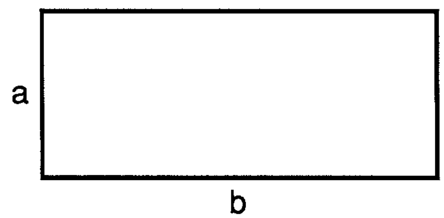
Actions

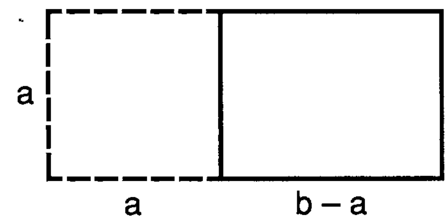
9. (Optional) Discuss a computational procedure for finding greatest common divisors. Have the students use the procedure to find the greatest common divisor of several pairs of numbers.

Comments

9. One computational procedure is based on the conclusion reached in Comment 5 that two rectangles that differ by a square are tiled by the same squares. Hence the greatest common divisors of the dimensions of the two rectangles are equal.

Note that cutting a square from a rectangle leaves the smaller dimension unchanged and replaces the larger by the difference of the dimensions. (See the figure.)





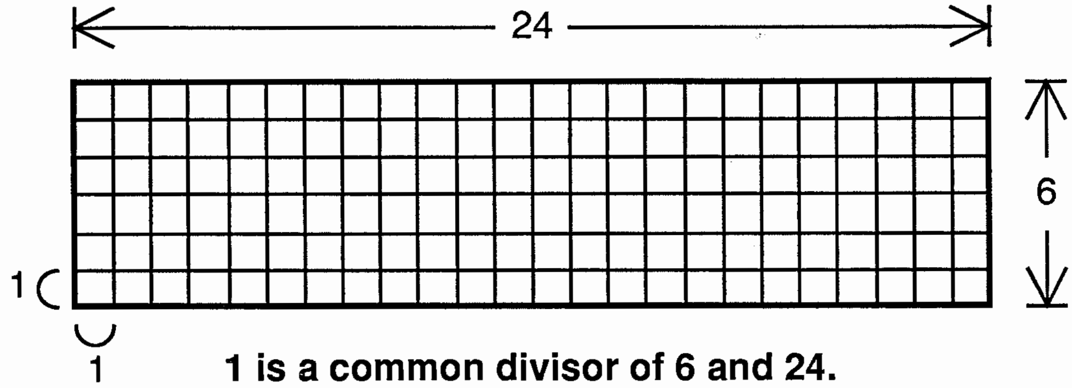
This means that the greatest common divisor of a pair of numbers remains the same if the larger of the two numbers is replaced by their difference.

For example, starting with the pair 16 and 38 and successively replacing the larger of the two numbers by their difference, one obtains the following sequence of pairs:

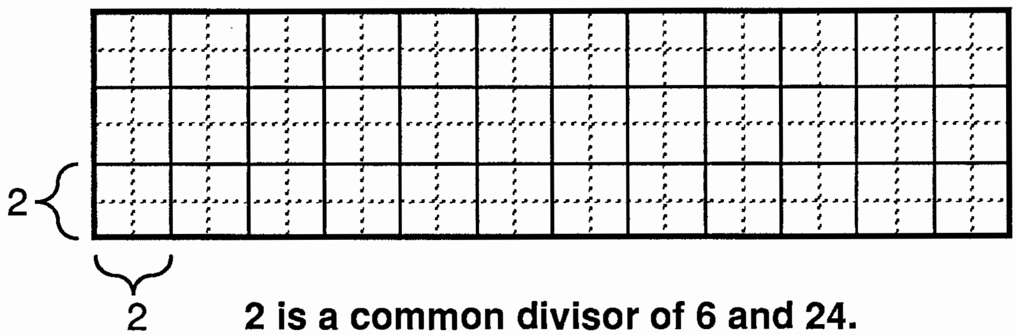
16, 38
16, 22
16, 6
10, 6
4, 6
4, 2
2, 2

All of these pairs have the same greatest common divisor. Hence, the greatest common divisor of 16 and 38 is the same as the greatest common divisor of 2 and 2, which is clearly 2.

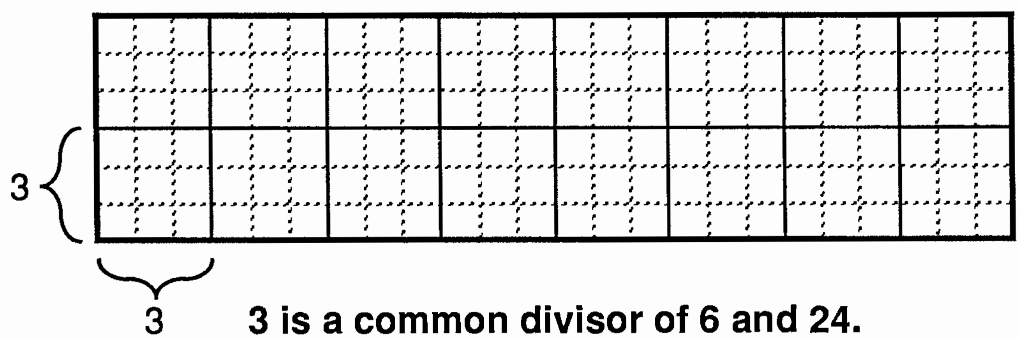
For larger numbers, calculators are helpful in making the computations.



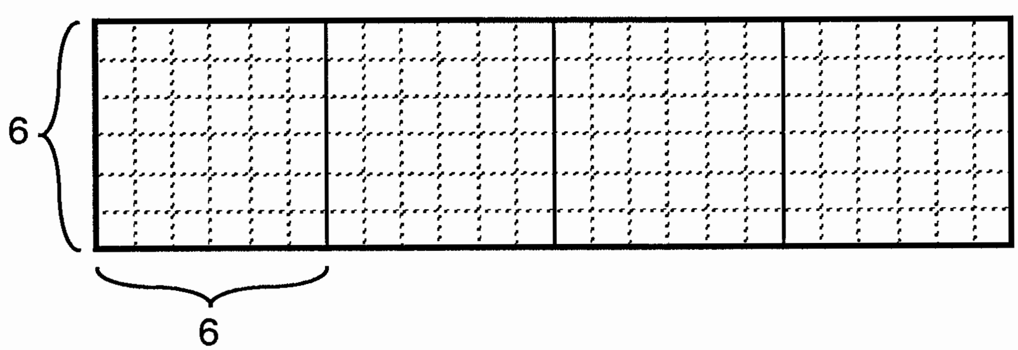
1 is a common divisor of 6 and 24.



2 is a common divisor of 6 and 24.

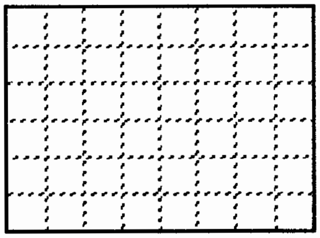


3 is a common divisor of 6 and 24.



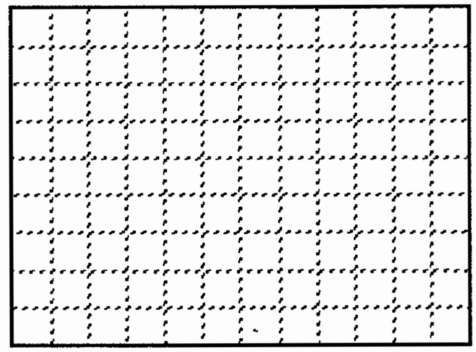
6 is the greatest common divisor of 6 and 24.

Greatest Common Divisors



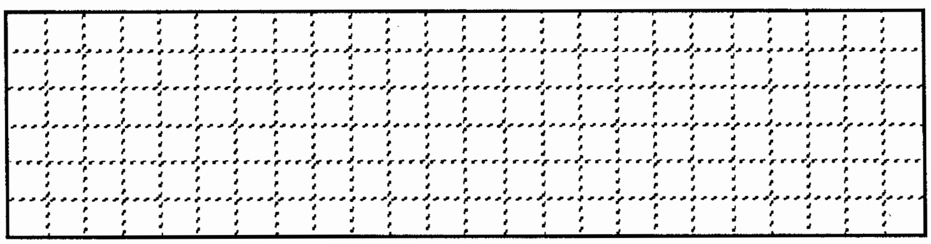
Dimensions of Rectangle: ______ and ______

Length of side of square: ______



Dimensions of Rectangle: ______ and ______

Length of side of square: ______

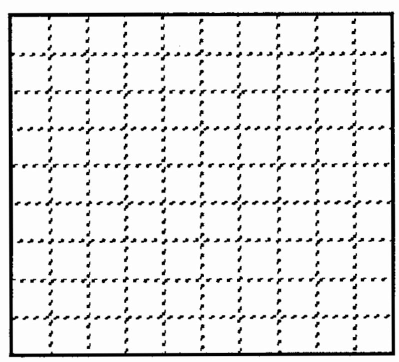


Dimensions of Rectangle: ______ and ______

Length of side of square: ______

Dimensions of Rectangle: ______ and ______

Length of side of square: ______



Dimensions of Rectangle: ______ and ______

Length of side of square: ______

Least Common Multiples

OVERVIEW

Finding the least common multiple of two numbers is accomplished by finding the smallest square composed of rectangles having these numbers as dimensions.

Prerequisite Activity

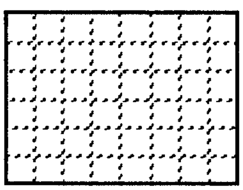
Unit III/Activity 6, *Greatest Common Divisors*

Materials

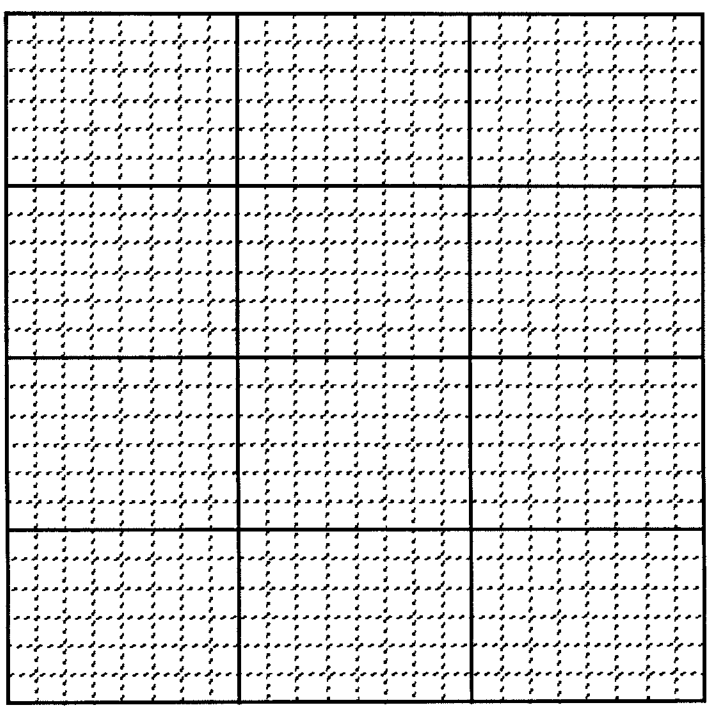
Overhead transparencies for teacher use (see Comments 1 and 6), grid paper and cardstock rectangles for each student (see Comment 2 and 5)

Actions

1. Show the students a 6 x 8 rectangle. Then show them a square composed of 6 x 8 rectangles such as the 24 x 24 square shown below.



A 6 x 8 Rectangle



A Square Composed of 6 x 8 Rectangles

Comments

1. Page 13 of this activity is a master for making an overhead transparency showing a 6 x 8 rectangle and a 24 x 24 square composed of 6 x 8 rectangles.

Actions

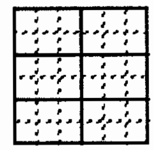
2. Distribute grid paper to the students. Ask them to draw sketches of squares composed of 2 x 3 rectangles. Using the students' sketches as a starting point, discuss questions associated with composing squares out of 2 x 3 rectangles.

Comments

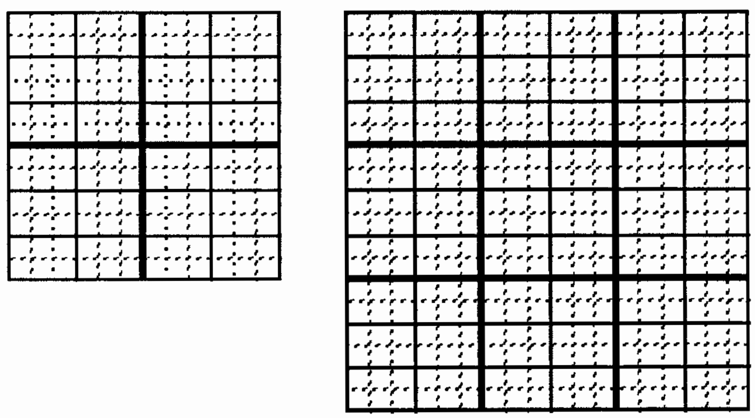
2. A master for 1/2 cm grid paper is attached to Unit II, Activity 5, *Greatest Common Divisors*. As an alternative to grid paper, 2 x 3 rectangles can be cut from cardstock and a supply given to each student or group of students. The students can be asked to use these rectangles to construct squares.

Some possible discussion questions: What size squares can be composed of 2 x 3 rectangles? What is the size of the smallest square? The largest? How many rectangles are used? Can a square be composed of 12 2 x 3 rectangles? Can a square be composed of 2 x 3 rectangles in which not all rectangles have the same orientation? (Answers to most of these questions follow.)

The smallest square that can be composed is a 6 x 6 square:

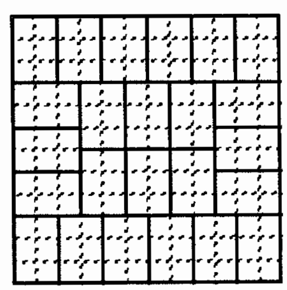


There is no largest square, but using the 6 x 6 square shown above as a basis, larger and larger squares can be composed of 2 x 3 rectangles:



Since the area of a 2 x 3 rectangle is 6, the area of 12 rectangles of this size is 72. This is not a square number, so a square cannot be composed of 12 2 x 3 rectangles.

The rectangles do not have to have the same orientation:



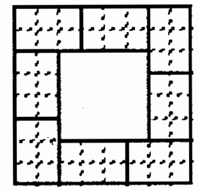
Continued next page.

Actions

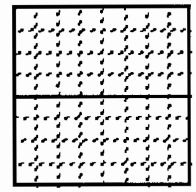
3. Tell the students that for the remainder of this activity all rectangles used in composing a square are to have the same orientation and the same dimensions. Then ask the students to sketch two squares, one composed of 4 x 8 rectangles and one composed of 4 x 10 rectangles. Discuss the relationship between the dimensions of a square and those of the rectangles used to compose it.

Comments

2. *(Continued.)* You may also want to discuss what is meant by a square composed of 2 x 3 rectangles. Some students may compose a square border like the one shown.

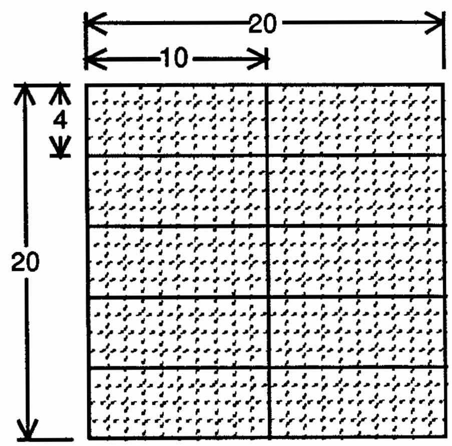


3. A square composed of 4 x 8 rectangles.



The length of the side of a square is a multiple of both dimensions of the rectangle used to compose it. For example, the length of the side of a square composed of 4 x 10 rectangles, all oriented the same way, is a common multiple of 4 and 10. The length of the side of the *smallest* square composed of 4 x 10 rectangles is the *least common multiple* of 4 and 10.

A 20 x 20 square is the smallest square that can be composed of 4 x 10 rectangles:



The lcm of 4 and 10 is 20.

Actions

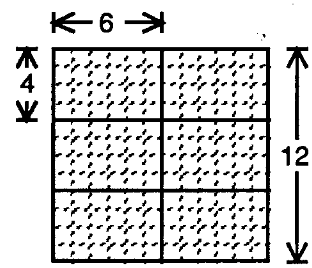
4. Ask the students to draw the smallest square that can be composed of rectangles with the following dimensions:

(a) 4 x 6 (b) 4 x 5 (c) 6 x 9 (d) 5 x 15

For each sketch, have the students write an appropriate statement about the least common multiple of the dimensions of the rectangle. Discuss the methods the students use.

Comments

4. While the students are working at their seats, you may want to illustrate this Action by drawing a sketch and writing a statement for part (a) on the overhead:

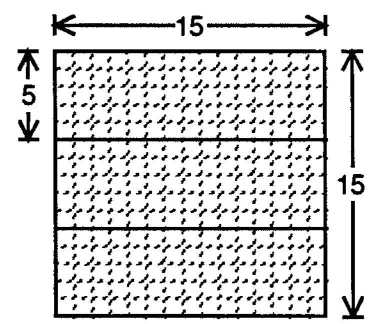


The lcm of 4 and 6 is 12.

(b) The smallest square composed of 4 x 5 rectangles has side 20. Hence the lcm of 4 and 5 is 20.

(c) The smallest square has side 18, so the lcm of 6 and 9 is 18.

(d) The lcm of 5 and 15 is 15:



Some students may use trial and error methods. Others may use their knowledge of multiplication facts to find the least common multiple of a rectangle's dimensions.

The size of the smallest square composed of rectangles of a given size may not be readily apparent. For example, it is not immediately seen that 1560 is the side of the smallest square composed of 260 x 312 rectangles.

For those who are interested, a process for finding the size of the smallest square that can be composed of rectangles of a given size is developed in the next Actions. This development begins with an investigation of the relationship between tiling a rectangle with a square (see Unit II, Activity 5, Greatest Common Divisors) and composing a square out of copies of the rectangle.

Actions

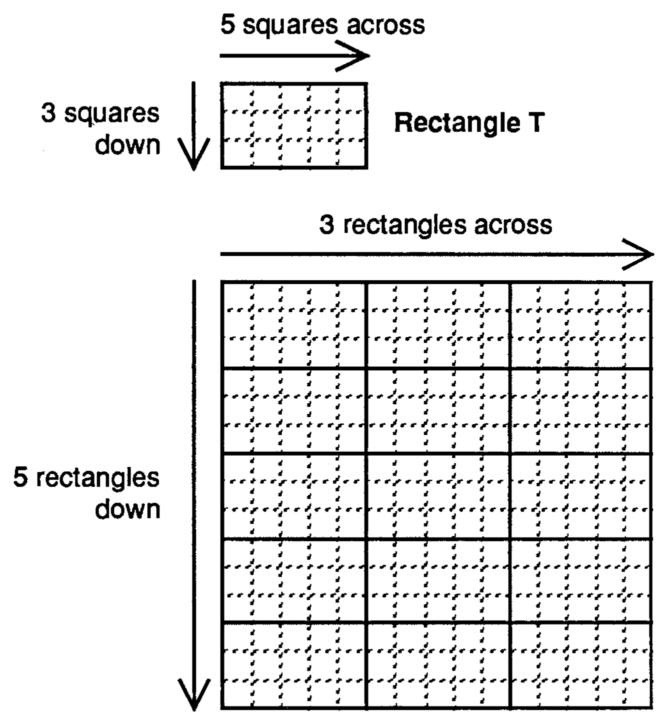
5. (Optional) Distribute a supply of 3 x 5 rectangles to each student or group of students. Ask the students to build squares composed of 3 x 5 rectangles. Discuss with the students how they know that the figures constructed are squares. Then discuss with them a process for constructing a square out of copies of a tiled rectangle.

Comments

5. A supply of 3 x 5 rectangles can be obtained by making copies on cardstock of page 14 of this activity and cutting on the heavy lines. The students can assist in cutting out the rectangles. There are enough rectangles on one page for two students or two groups of students.

An array of 3 x 5 rectangles, 3 across and 5 down, contains 15 squares across and 15 down and hence is itself a square.

If a rectangle is tiled by a square, the tiling provides information that can be used to find a square composed of copies of the rectangle. As shown below, if a tiling of a rectangle contains 5 squares across and 3 down, then an array of these rectangles, 3 across and 5 down, is a square.
Page 15 of this activity is a master for a transparency of this diagram. It can be shown to the students to illustrate this process for constructing a square out of 3 x 5 rectangles.



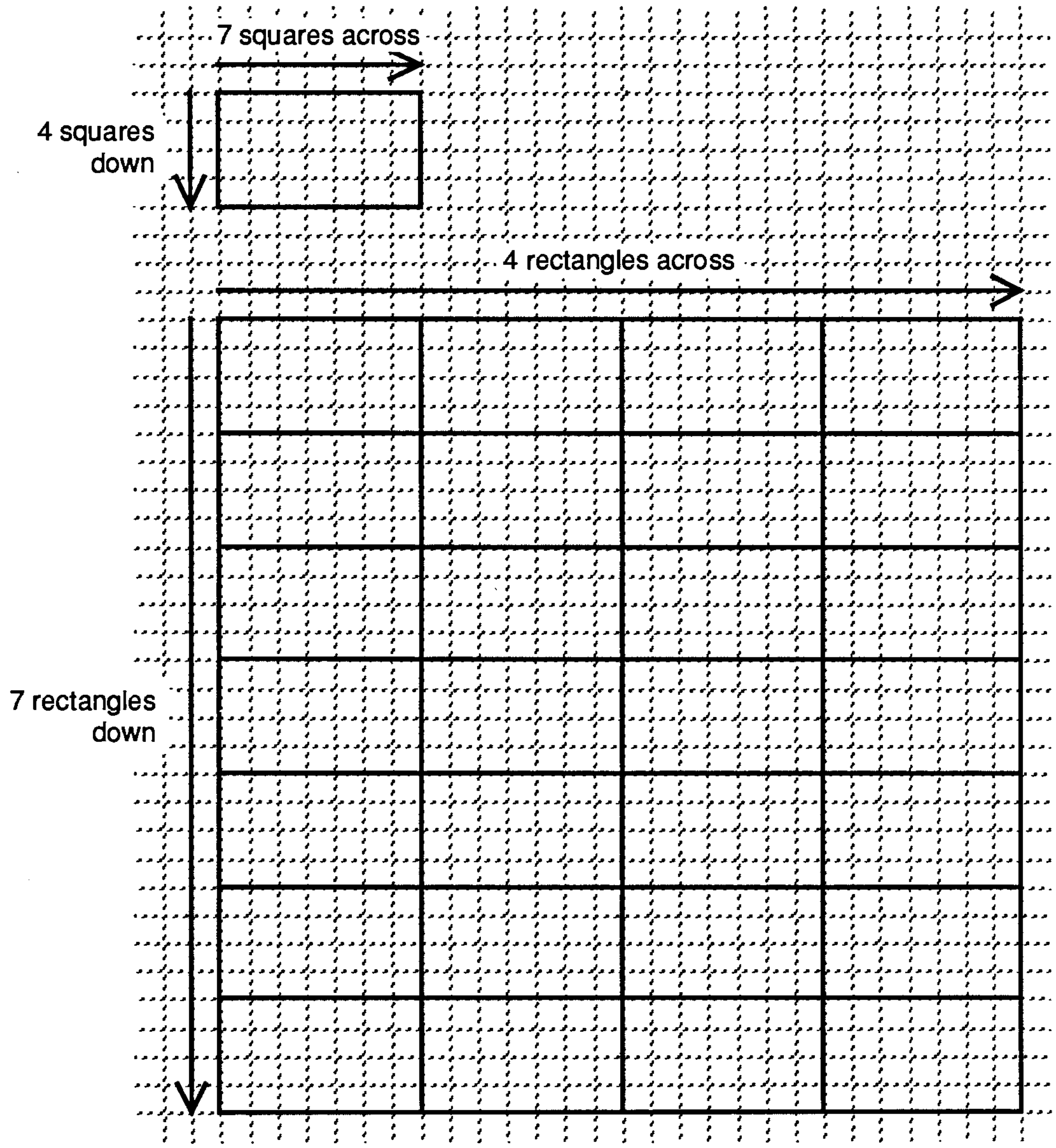
A Square Composed of Coples of Rectangle T

Continued next page.

Actions

Comments

5. *(Continued.)*You may want to show the students other illustrations of this process. For example, you can draw a 4 x 7 rectangle on a 1/2 cm grid transparency and display it on the overhead. You can ask the students to describe an array of these rectangles that will form a square and then sketch the array, indicating the number of rectangles across and the number down, as shown below.



In general, if a rectangle is tiled by a square and the tiling contains k squares across and m squares down, then an array of these rectangles, m rectangles across and k rectangles down, is a square.

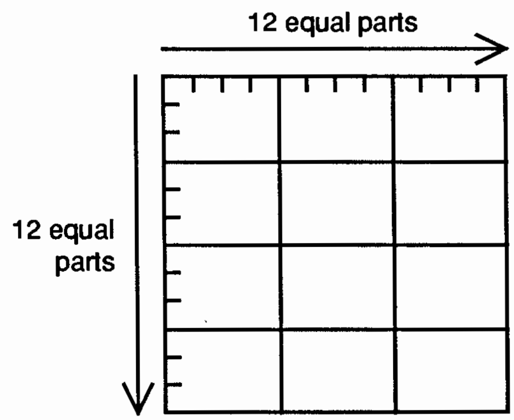
Actions

6. (Optional) Show the students a transparency of page 15 of this Activity. Then show them a transparency of page 16. Tell them that this sketch is similar to the first, except the tiling of the rectangle has been removed. Ask the students to suggest how the rectangle was tiled. Repeat, using a transparency of page 17.

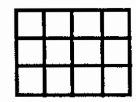
Comments

6. This Action is the converse of Action 5. In Action 5, one begins with a square tiling of a rectangle and, using information provided by the tiling, constructs an array of these rectangles that forms a square. In this Action, one starts with an array of rectangles that forms a square and, using information about the array, obtains a square tiling of one of these rectangles.

Dividing the horizontal side of each copy of Rectangle A into 3 equal parts and the vertical sides into 4 equal parts, as shown on the left below, divides both sides of the square into 12 equal parts. Hence the horizontal and vertical divisions are equal and lead to a tiling of Rectangle A which has 4 squares across and 3 down.

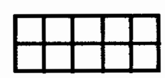


A square composed of copies of rectangle A.



Rectangle A tiled by a square.

A square tiling of Rectangle B is obtained by dividing its length into 5 equal parts and its width in two:



Rectangle B

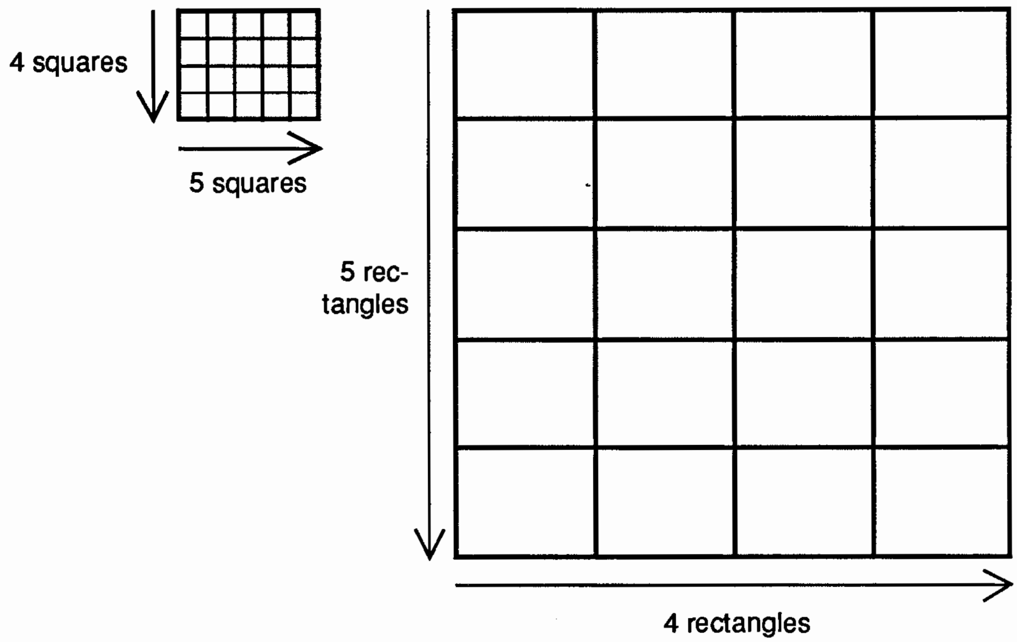
Actions

7. (Optional) Summarize the results of Actions 5 and 6, emphasizing the relationship between the largest square which tiles a rectangle and the smallest square composed of copies of the rectangle.

Comments

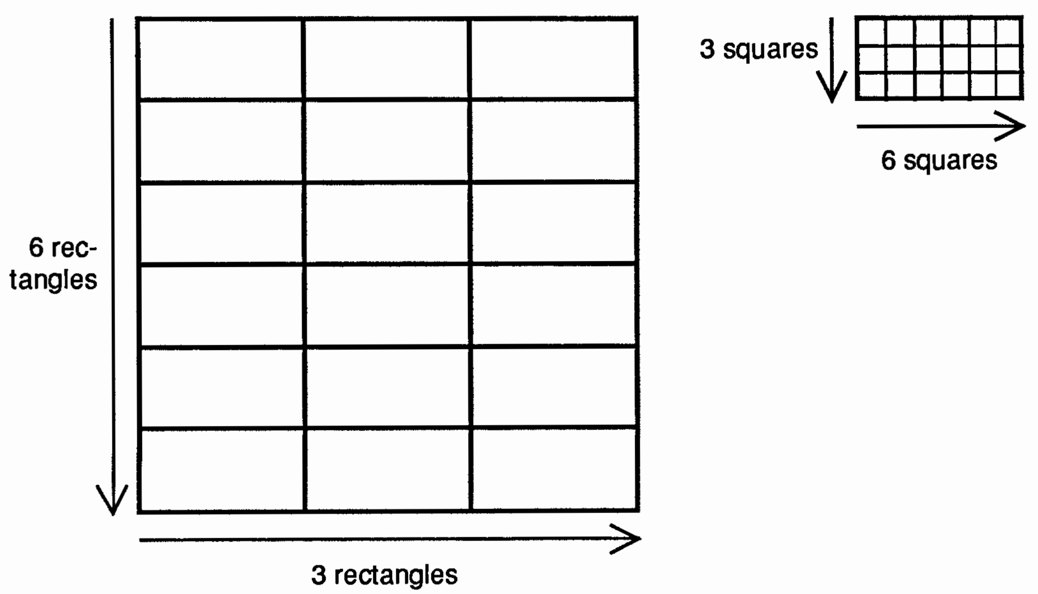
7. Each tiling of a rectangle by a square leads to a square composed of copies of the rectangle:

This tiling ⟶ leads to this square



Conversely, each square composed of copies of a rectangle leads to a tiling of the rectangle by a square:

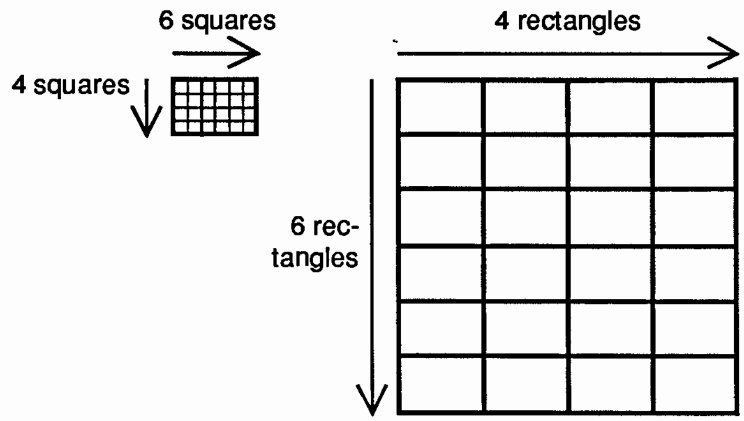
This square ⟶ leads to this tiling

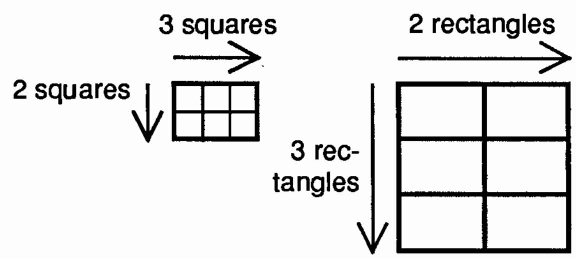


Continued next page.

Actions	Comments

7. *(Continued)*. Note that the fewer the number of squares in a tiling of a rectangle, the fewer the number of copies of the rectangle in the corresponding square. For example, as shown in the sketches below, a tiling of a rectangle that contains 4 squares down and 6 squares across leads to a square array that contains 24 rectangles, while a tiling of the same rectangle that has 2 squares down and 3 across leads to an array of 6 rectangles.





Thus, the larger the square that tiles a rectangle, the smaller the corresponding square composed of copies of the rectangle. Consequently, the largest square that tiles a rectangle corresponds to the smallest square composed of copies of that rectangle.

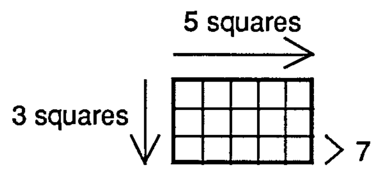
Actions

8. (Optional) Ask the students to use the information obtained in Action 7 to find the least common multiples of the following pairs of numbers:

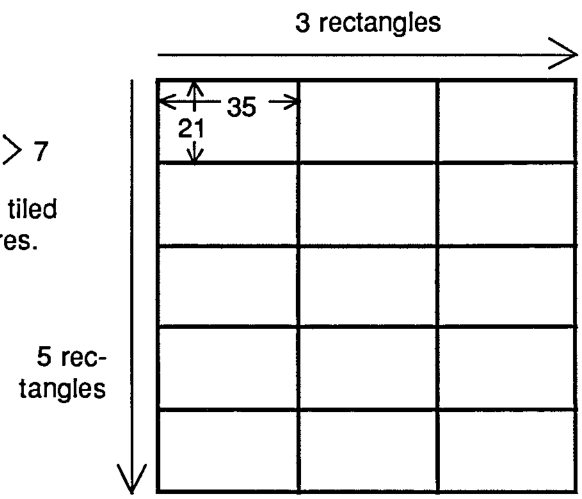
(a) 21 and 35 (b) 15 and 27 (c) 18 and 42

Comments

8. (a) Using methods developed in Unit II, Activity 5, *Greatest Common Divisors*, one finds that a 7 x 7 square is the largest square that tiles a 21 x 35 rectangle. This tiling corresponds to the smallest square composed of 21 x 35 rectangles. The length of the side of this square is the lcm of 21 and 35. As seen in the illustration, this length is 105.

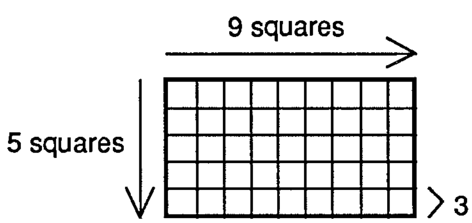


21 x 35 rectangle tiled with 7 x 7 squares.

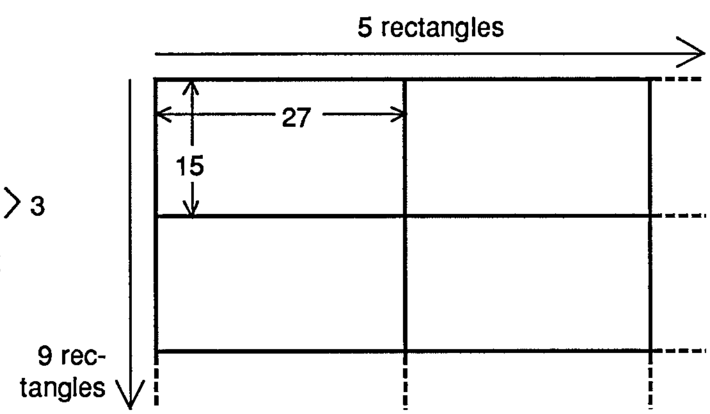


Smallest square composed of 21 x 35 rectangles. The lcm of 21 and 35 = length of side of square = 5 x 21 = 3 x 35 = 105.

(b) A 3 x 3 square is the largest square that tiles a 15 x 27 rectangle. The length of the side of the corresponding square is 105.



15 x 27 rectangle tiled with 3 x 3 squares.



The lcm of 15 and 27 = 5 x 27 = 9 x 15 = 135

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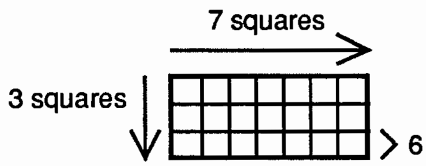
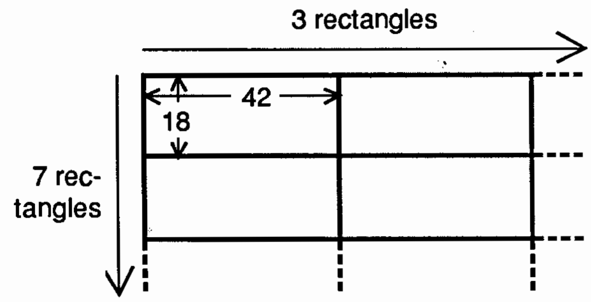
Actions

Comments

8. *(Continued.)*
(c) The lcm of 18 and 42 is 126:



A 6 x 6 square is the largest square that tiles a 18 x 42 rectangle.



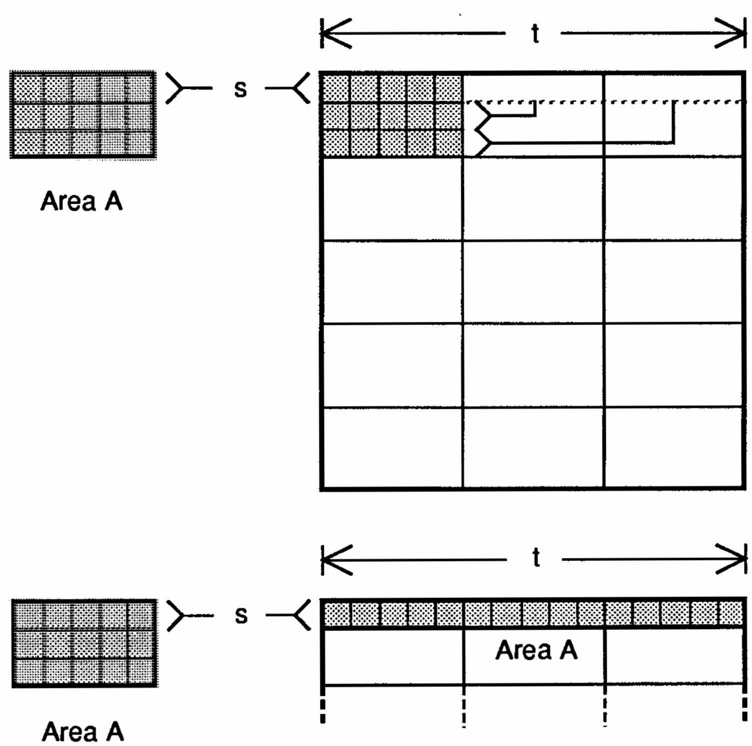
The lcm of 18 and 42 = 3 x 42 = 7 x 18 = 126

9. (Optional) Develop the relationship between the least common multiple and the greatest common divisor of two numbers. Have the students use this relationship to find the least common multiple of the following pairs of numbers:

(a) 87 and 111 (b) 134and 185 (c) 390 and 465

9. A useful relationship between the gcd and lcm of two numbers can be obtained from the following observation concerning areas.

Shown below is a rectangle that has been tiled by a square and the corresponding square composed of copies of the rectangle. Let A be the area of the rectangle, let s be the length of the side of a square in the tiling and let t be the side of the square array of rectangles. One of the rectangles of area A can be dissected and rearranged to form a rectangle whose dimensions are s and t as shown. The product of its dimensions is its area. Hence s x t = A.



A = s x t

Continued next page.

Actions

Comments

9. *(Continued.)* Suppose that the original rectangle has dimensions a and b so that its area A is a x b. Suppose also that it has been tiled by the largest possible square. Then the corresponding square composed of these rectangles is the smallest possible. In this case, s is the gcd of a and b and t is the lcm of a and b and the equation s x t = A becomes:

(gcd of a and b) x (lcm of a and b) = a x b.

Thus, the product of the gcd and lcm of two numbers is the same as the product of the numbers.

If two numbers are given, their gcd can be found by one of the methods developed in Unit II, Activity 5, *Greatest Common Divisors*. Then their lcm can be found using the above relationship.

For example, the gcd of 124 and 440 is the same as the gcd of the following pairs (see the last Action of Unit II, Activity 5, *Greatest Common Divisors*):

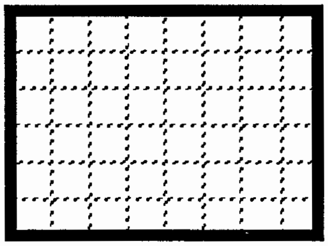
124, 440
124, 316
124, 192
124, 68
56, 68
56, 12
44, 12
32, 12
20, 12
8, 12
8, 4
4, 4

Hence the gcd of 124 and 440 is 4. Thus, 4 x (lcm of 124 and 440) = 124 x 440 and, hence, lcm of 124 and 440 = (124 x 440) ÷ 4 = 13,640.

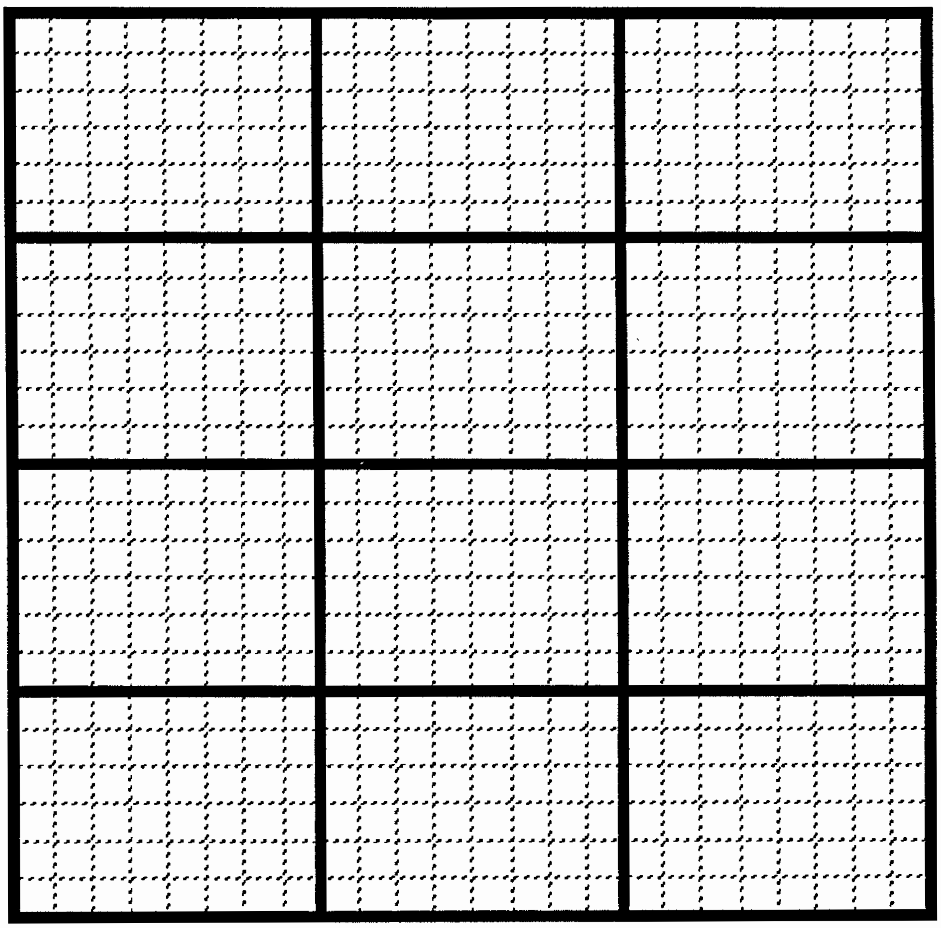
(a) lcm of 87 and 111 = (87 x 111) ÷ 3 = 3219

(b) lcm of 134 and 185 = (134 x 185) ÷ 1 = 24,790

(c) lcm of 390 and 465 = (390 x 465) ÷ 15 = 12,090.

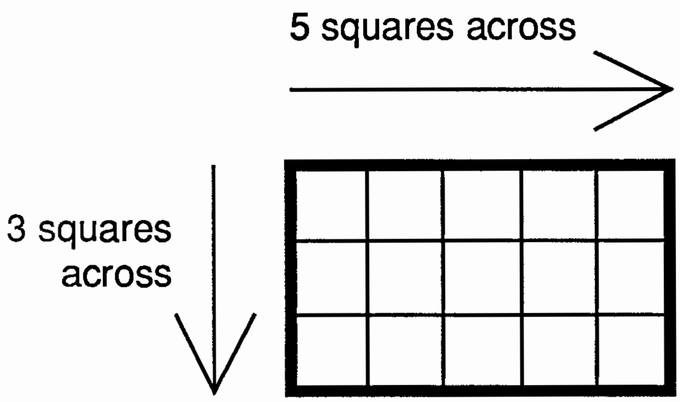


A 6 x 8 Rectangle

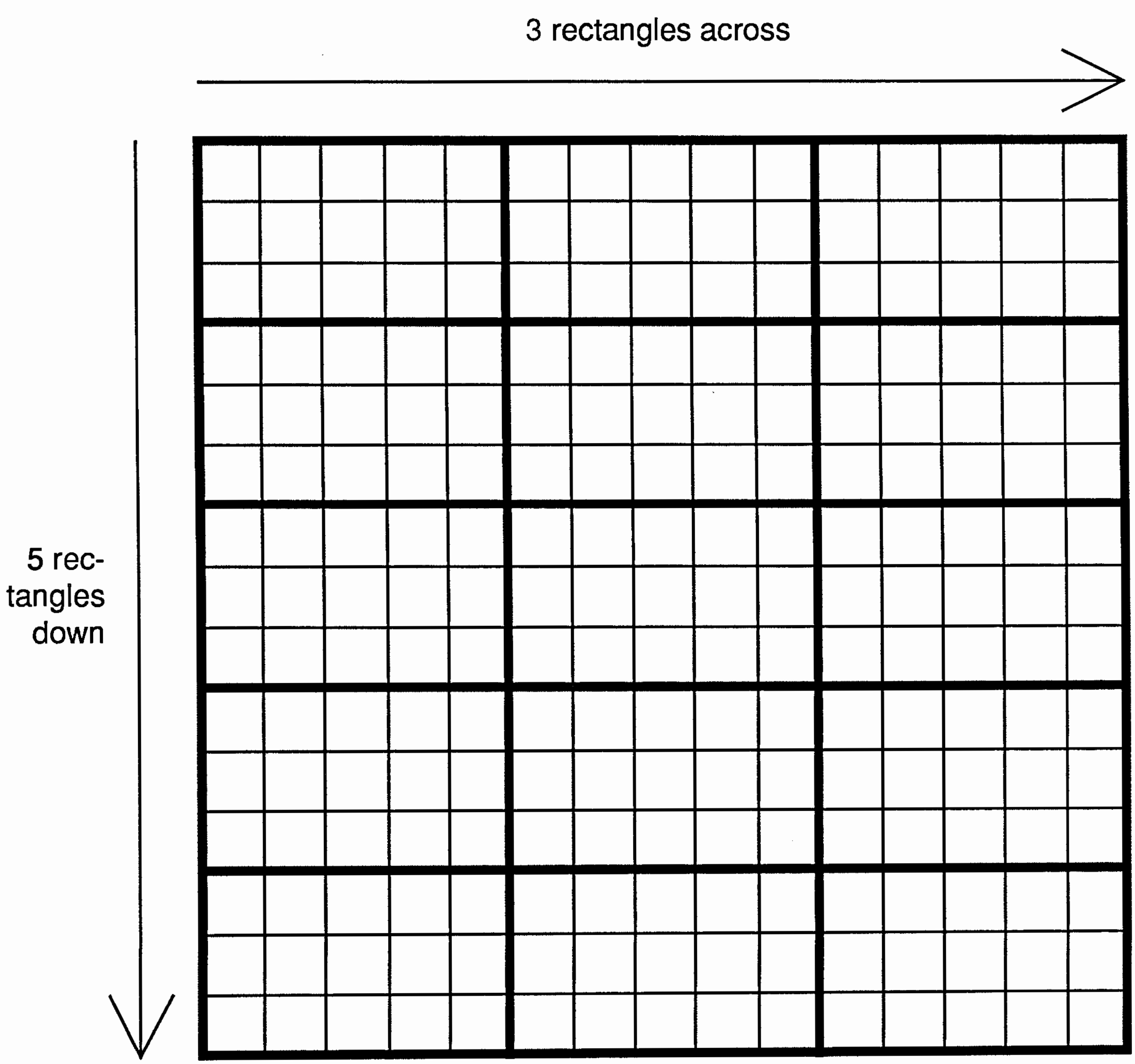


A Square Composed of 6 x 8 Rectangles

Cut on Heavy Lines

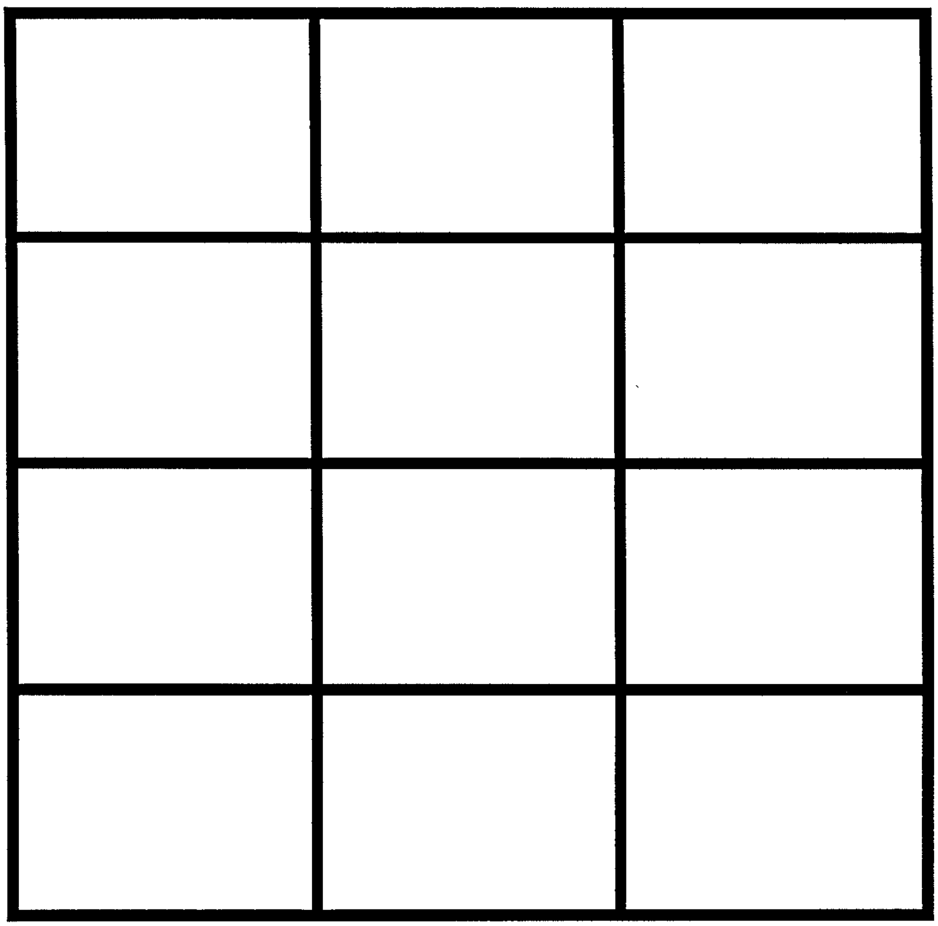


Rectangle T



A Square Composed of Copies of Rectangle T

Rectangle A

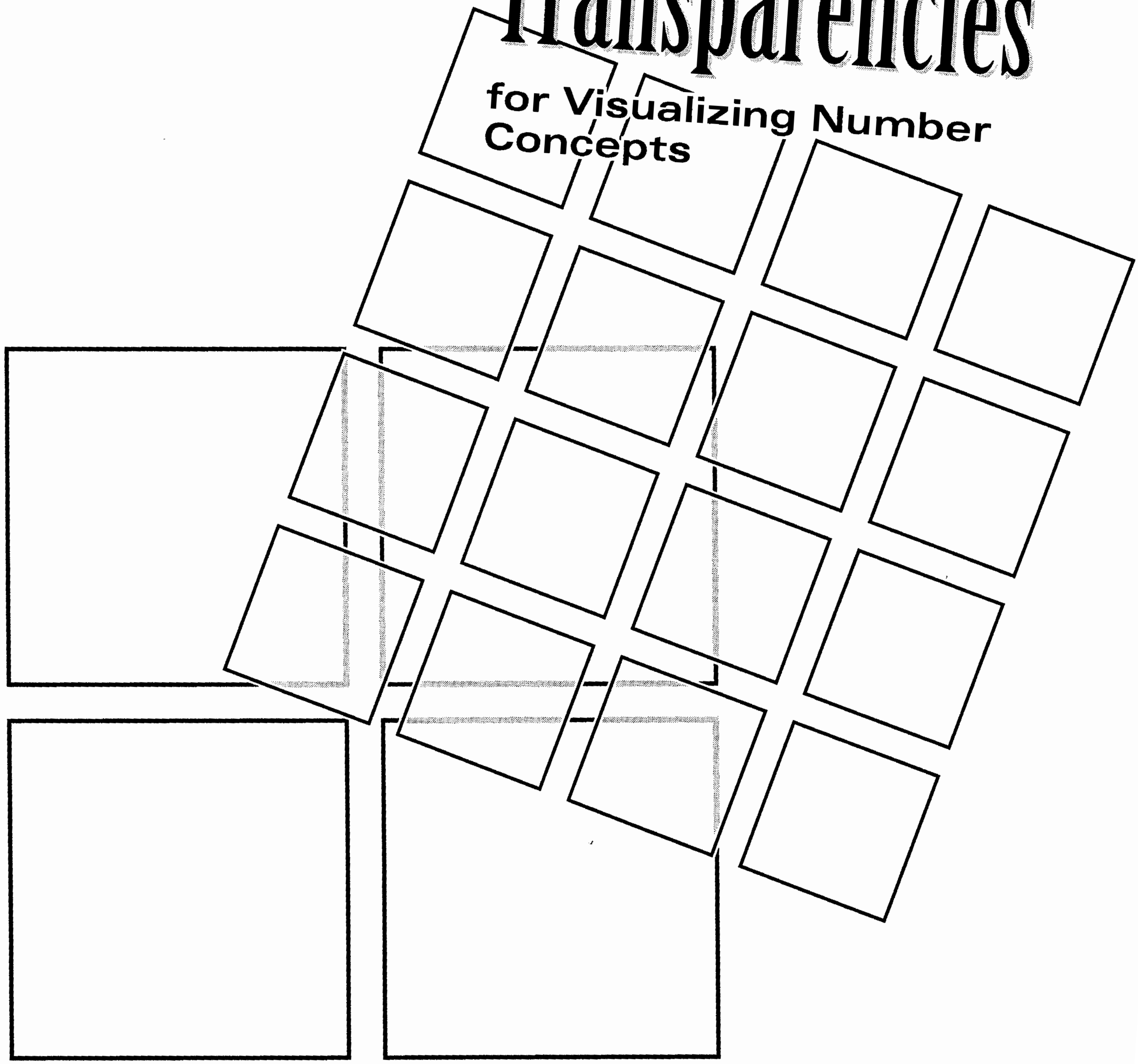


A Square Composed of Copies of Rectangle A

Unit II / Math and the Mind's Eye

Transparencies

for Visualizing Number Concepts

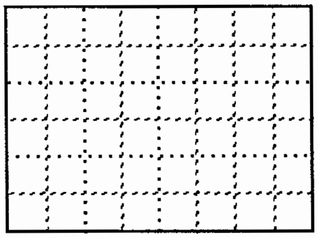


The Math Learning Center
PO Box 3226
Salem, Oregon 97302

Catalog #MET2

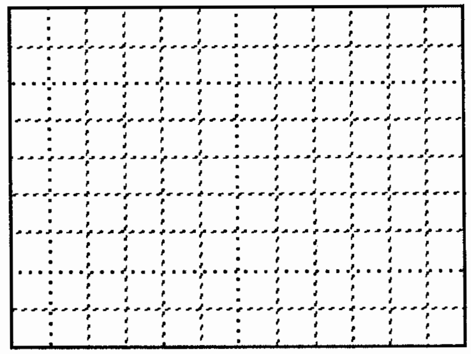
No.	Number of Rectangles	Dimensions of Rectangles	Factors of the Number	
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				

Activity Sheet II-3



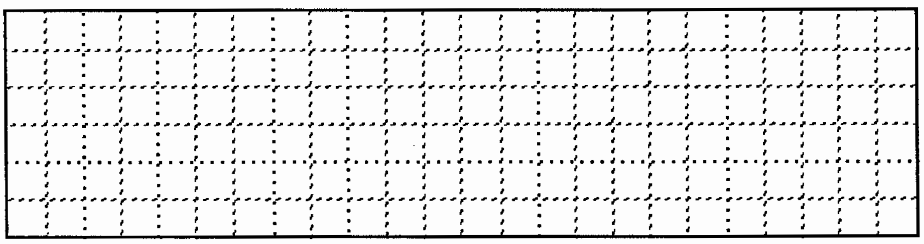
Dimensions of Rectangle: ______ and ______

Length of side of square: ______



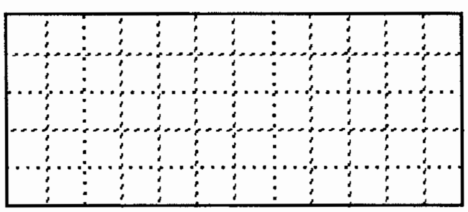
Dimensions of Rectangle: ______ and ______

Length of side of square: ______



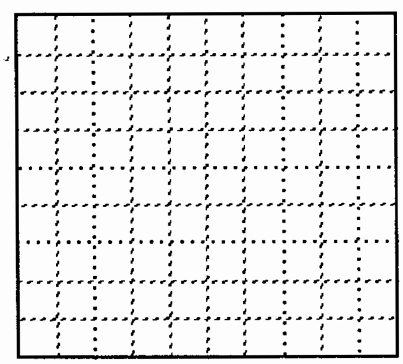
Dimensions of Rectangle: ______ and ______

Length of side of square: ______



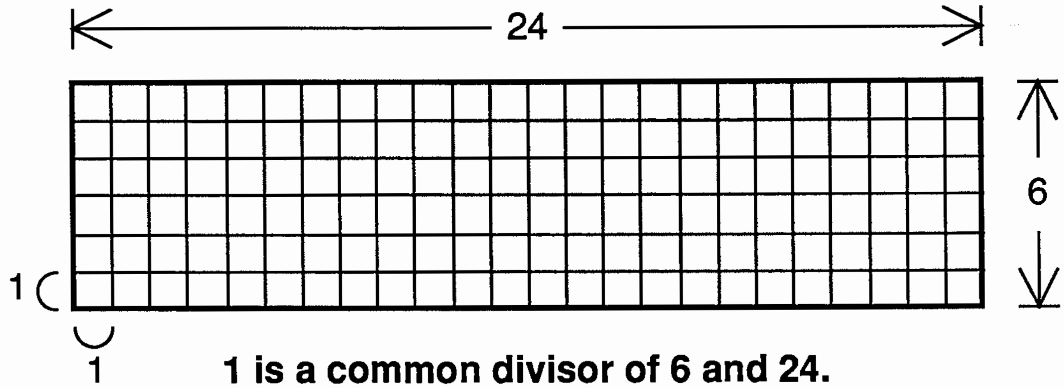
Dimensions of Rectangle: ______ and ______

Length of side of square: ______

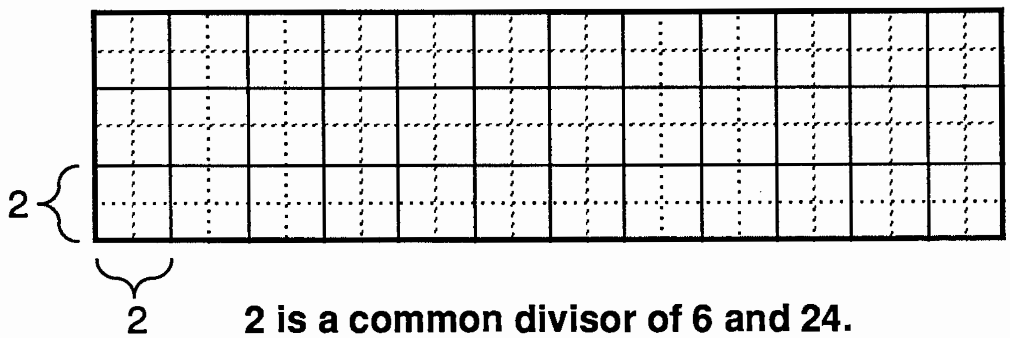


Dimensions of Rectangle: ______ and ______

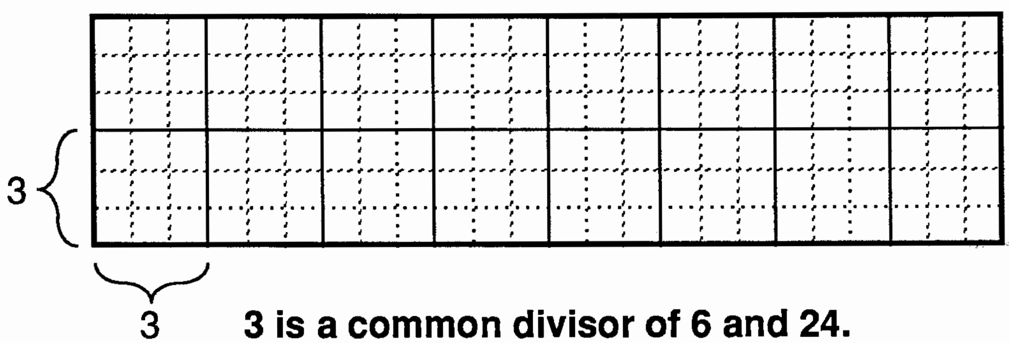
Length of side of square: ______



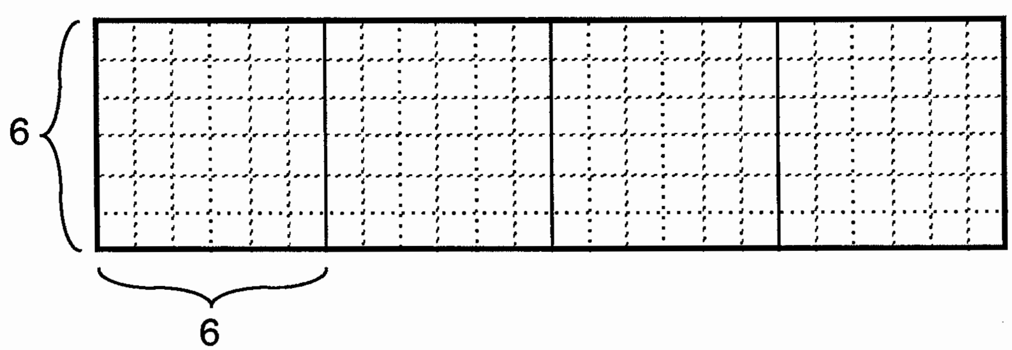
1 is a common divisor of 6 and 24.



2 is a common divisor of 6 and 24.

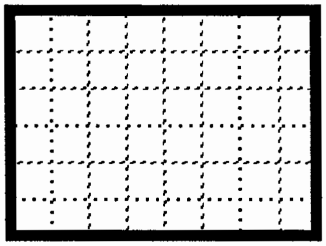


3 is a common divisor of 6 and 24.

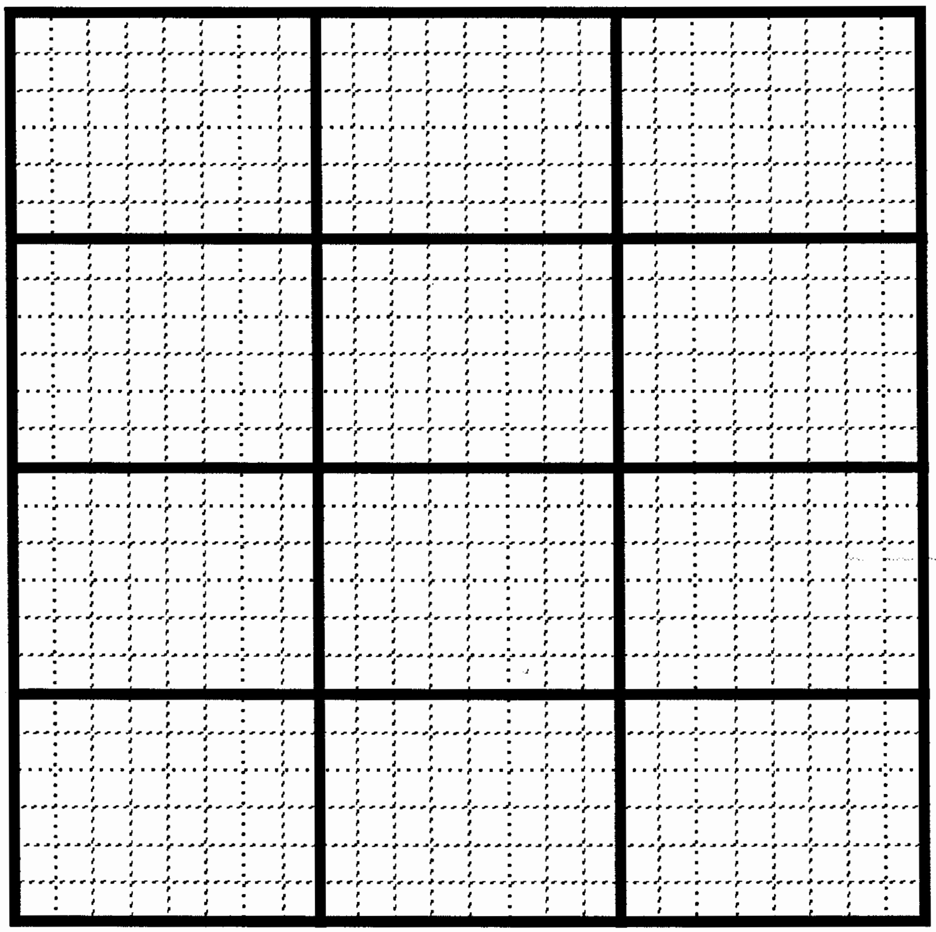


6 is the greatest common divisor of 6 and 24.

1/2 Centimeter Grid Paper

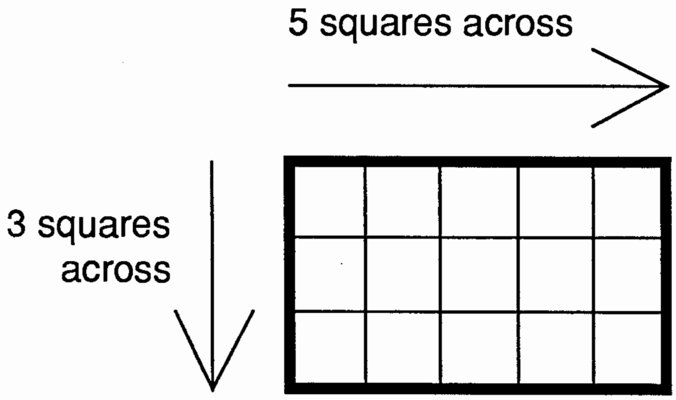


A 6 x 8 Rectangle

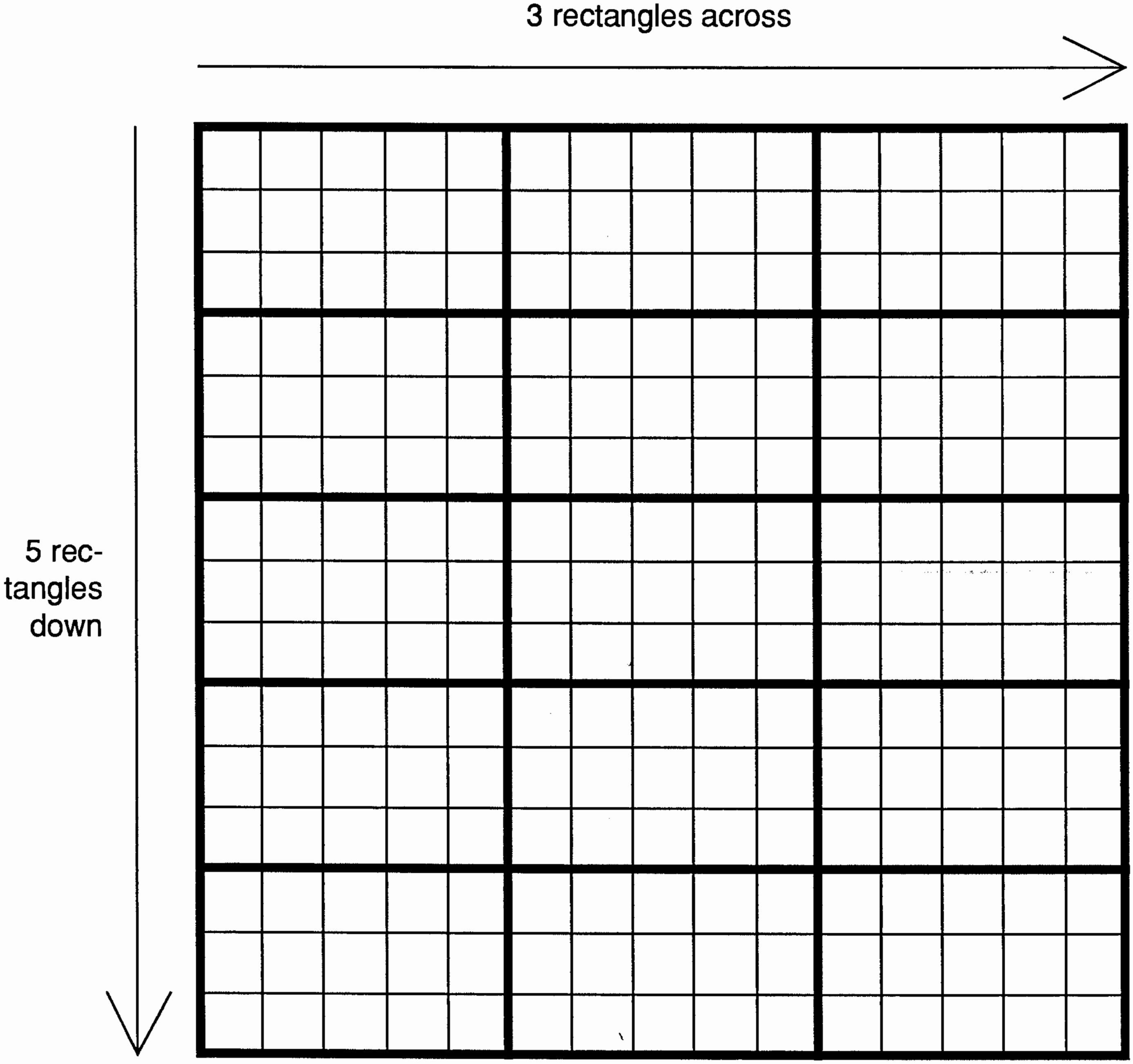


A Square Composed of 6 x 8 Rectangles

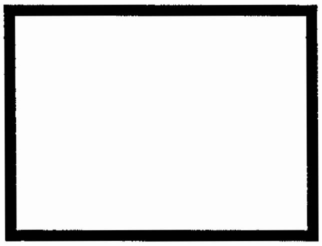
Cut on Heavy Lines



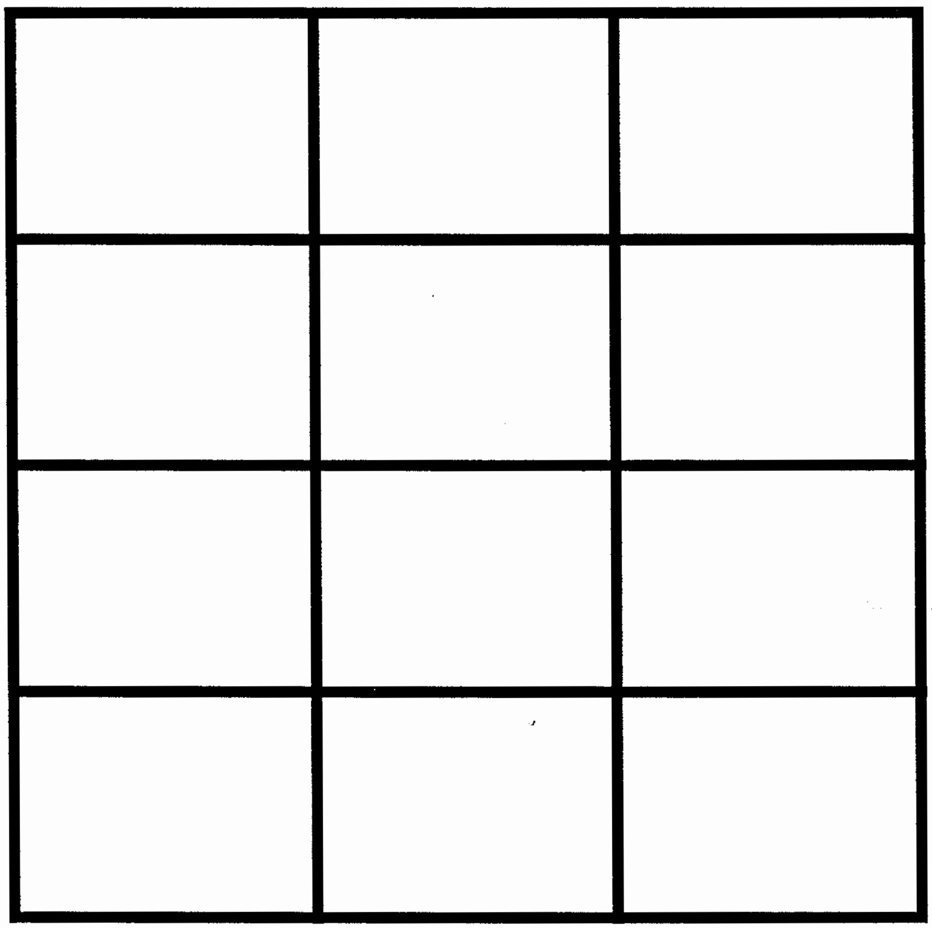
Rectangle T



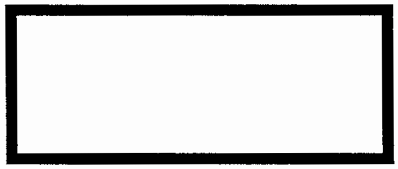
A Square Composed of Copies of Rectangle T



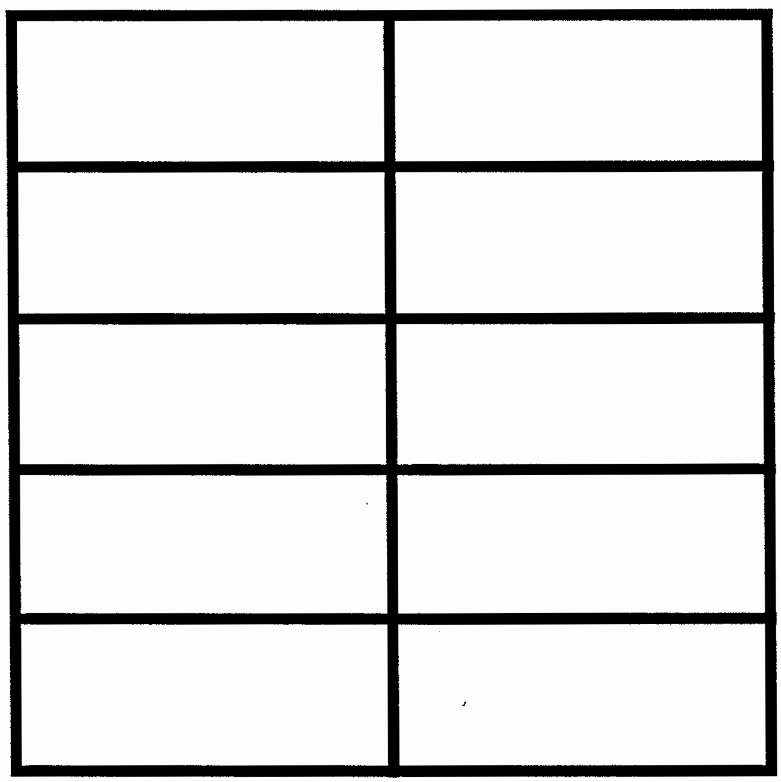
Rectangle A



A Square Composed of Copies of Rectangle A



Rectangle B



A Square Composed of Copies of Rectangle B