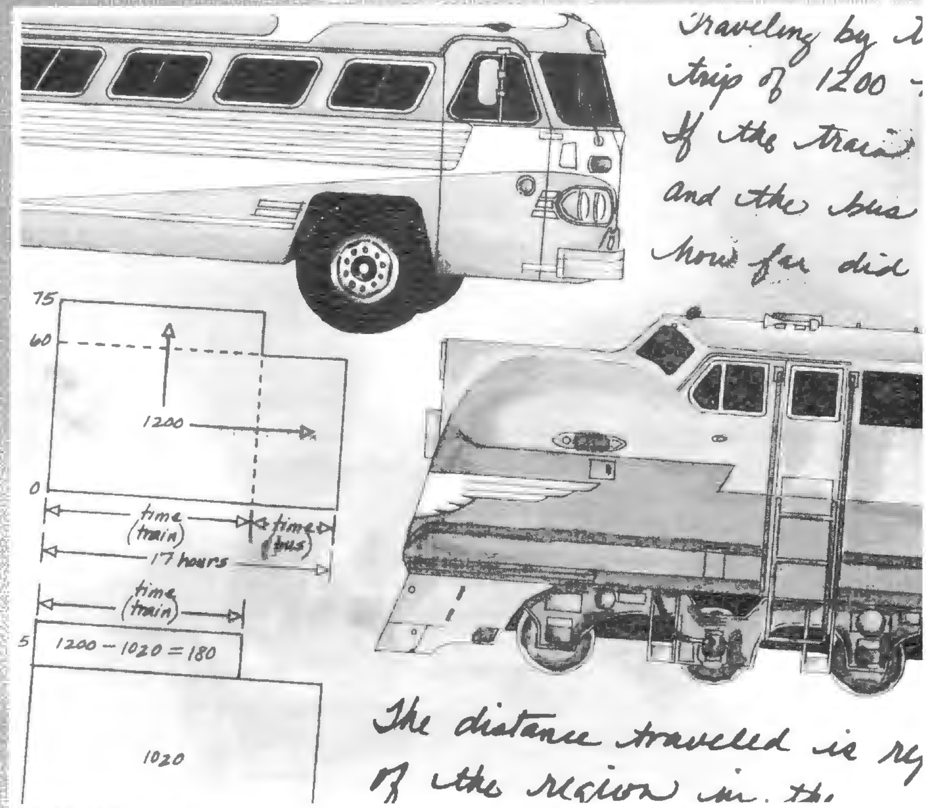

Sketching Solutions to Algebraic Equations

Eugene Maier

Sketching Solutions to Algebraic Equations

Math and the Mind's Eye materials are intended for use in grades 4-9. They are written so teachers can adapt them to fit student backgrounds and grade levels. A single activity can be extended over several days or used in part.

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Unit XIII • Activity 1

Sketching Solutions

O V E R V I E W

Sketches are used to solve standard algebra problems.

Prerequisite Activity

None.

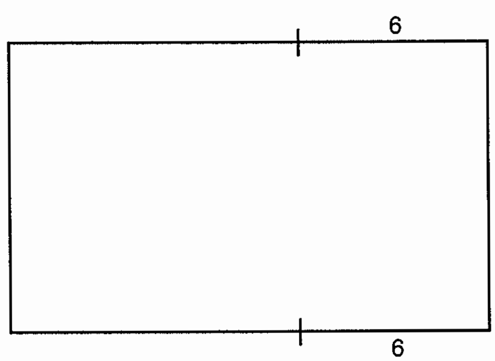
Materials

Copies of Puzzle Problems (see Comment 10).

Actions

1. Ask the students to draw a rectangle on a blank sheet of paper. Comment on the various rectangles drawn.

2. Show the students the following sketch. Ask them to describe what they see. After the students have had an opportunity to respond, ask them what more they can say about the rectangle if its perimeter is 56 units. Discuss the students' responses.

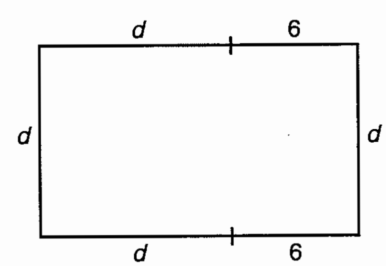


Comments

1. Students often have difficulty in drawing sketches that disclose the essential features of a problem situation. Thus, the beginning Actions in this activity focus on drawing sketches that require few, if any, words and symbols to convey information. If your students have had experience using diagrams and sketches to solve problems visually, you may want to skip to Action 9.

The Actions begin with one that almost every student will carry out rapidly. Most students will draw a rectangle that is wider than it is tall. Most, if not all, of the sketches will contain no words or symbols. Generally, it is unnecessary to label a sketch of a rectangle for the students to identify what has been drawn.

2. If the students simply reply, "A rectangle," ask them to tell you all they know about the rectangle. Most of the students will recognize that one dimension is 6 units longer than the other. Some may ask if the unlabeled segments are of equal length. If so, you can label the segments with the same letter, as shown in the sketch below.



Continued next page.

Actions

Comments

2. *Continued.* The perimeter of the rectangle is comprised of 2 segments of length 6, and 4 segments of unknown but equal length. The sum of the lengths of these latter 4 segments is 56 – 12 or 44. Hence, each segment is 11 and the dimensions of the rectangle are 11 and 11 + 6 or 17.

Alternately, a student may decide that half the perimeter is 28. Hence 2 of the segments total 28 – 6, or 22, inches.

If one wants, one can paraphrase a student's thinking while recording their thoughts in symbolic shorthand, as in the following example:

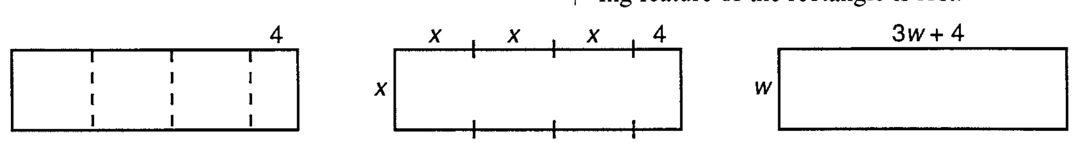
paraphrase:	*record*:
As I understand your argument, you say the perimeter consists of 2 segments of length 6, and 4 other segments all of the same length —let's call it *d*—and, since the perimeter is 56, these lengths total 56.	$12 + 4d = 56$
So the 4 segments have a total length of 56 – 12 or 44.	$4d = 56 - 12 = 44$
Thus, the length of each segment is 44 ÷ 4 or 11.	$d = 44 \div 4 = 11$
Hence, the dimensions of the rectangle are 11 and 11 + 6.	width $= d = 11$ length $= d + 6 = 17$

Notice, in this case, the algebraic equations become a symbolic way of recording one's thinking. In order to deal with the symbols, it is not necessary to have mastered a set of rules for their manipulation. Rather, the equations reflect a chain of thought based on the thinker's knowledge and insight.

3. Ask the students to draw a sketch, using as few words and symbols as possible, that portrays a rectangle of unknown dimensions whose length is 4 units longer than 3 times its width. Have several students replicate their drawings on the chalkboard. Discuss whether the drawings adequately convey the information given about the rectangle and whether the words and symbols used are essential.

3. Having the students draw sketches of a situation before a problem is posed focuses their attention on creating a sketch that portrays the essential features of the situation.

Below are some possible sketches. Notice that, in the last sketch shown, the essential information is carried in the symbols and not the sketch—that is, if the symbolic phrase "$3w + 4$" is erased, the distinguishing feature of the rectangle is lost.



Actions

4. Tell the students to suppose the perimeter of the rectangle they drew in Action 3 is 48 inches. Then ask them to determine the dimensions of the rectangle. Ask for volunteers to describe their thinking.

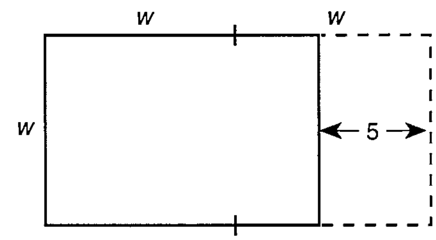
5. Repeat Action 3 for a rectangle whose length is 5 inches less than twice its width. Then ask the students to determine the dimensions of the rectangle if its perimeter is 32 inches. Have several students show their sketches and describe their thinking in determining the dimensions of the rectangle.

6. Ask the students to sketch a square. Then have them sketch an equilateral triangle whose sides are 2 feet longer than the sides of the square. Then ask the students to determine the length of the side of the square if the square and the triangle have equal perimeters. Ask for volunteers to show their sketches and describe their thinking.

Comments

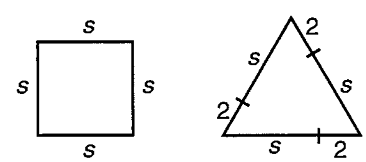
4. The students will use various methods to arrive at the dimensions. One way is to note that the perimeter of 48 inches consists of 2 segments of length 4 and 8 other segments of equal length. Hence, the lengths of the 8 segments total 40 inches, so each is 5. Thus, the dimensions of the rectangle are 5 inches and $3 \times 5 + 4$, or 19.

5. Here is one sketch:



The extended rectangle shown above has a perimeter of 42 inches—10 inches longer than the original rectangle. These 42 inches are composed of 6 equal lengths. So each of these lengths is 7 inches. The width of the original rectangle is one of these lengths, or 7 inches; the length of the rectangle is 5 inches less than 2 of these lengths, or 9 inches.

6. The perimeter of the square, in the following drawing, contains 4 segments of length s; that of the equilateral triangle contains 3 segments of length s and 3 of length 2. Thus, the 3 segments of length 2 must sum to s. So s is 6.

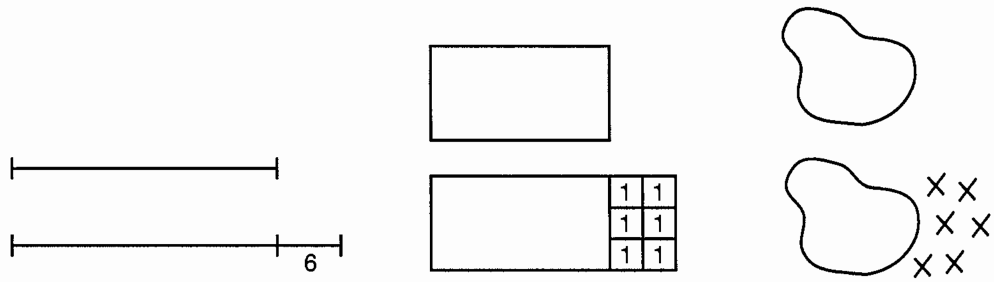


Actions

7. Ask the students to draw diagrams or sketches which represent a number and that number increased by 6. Show the various ways in which students have done this. Then ask the students to use one of the sketches to determine what the numbers are if their sum is 40.

Comments

7. The students may find this Action more difficult than the previous Actions in which they are asked to draw geometric figures. Since numbers have no particular shape, the students must invent a way of portraying number. They might do this in a variety of ways, e.g., as a length or as an area or as an amorphous blob.



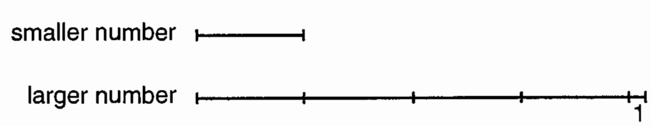
Three sketches of a number and that number plus 6.

Looking at the sketch on the left above, the sum of the lengths of the segments portraying the numbers is 40. The small segment has length 6. Hence, the sum of lengths of the other 2 segments is 34. Since these 2 segments are congruent, the length of each is $34 \div 2$, or 17. Hence the 2 numbers are 17 and $17 + 6$, or 23.

8. Ask the students to draw sketches that represent two numbers such that 4 times the smaller number is 1 less than the larger. Then ask them to use their sketches to determine the numbers if their sum is 36. Ask for volunteers to show their sketches and explain how they were used to arrive at their conclusion.

8. The students will represent the numbers in various ways.

If the sum of the two numbers is 36, in the sketch shown below, the sum of the lengths of the 5 congruent segments (1 in the smaller number and 4 in the larger number) is $36 - 1$, or 35. Hence, the length of each is $35 \div 5$, or 7. Thus the numbers are 7 and $4(7) + 1$, or 29.



Actions

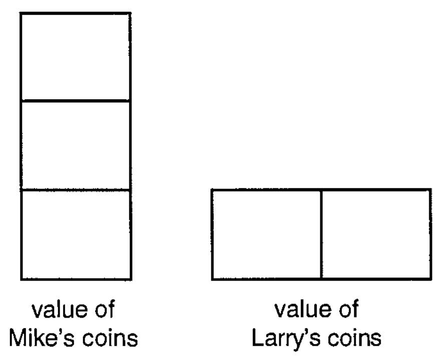
9. Tell the students that Mike has 3 times as many nickels as Larry has dimes. Ask them to draw sketches representing the value of their money. Then ask them to use their sketches to determine how much money Mike has if he has 45¢ more than Larry. Ask for volunteers to show their solutions.

Comments

9. Again, the students' sketches will vary.

In the following sketch, the value of Mike's and Larry's coins are represented by stacks of boxes, all of which have the same value. Since Mike has 3 times as many coins as Larry, his stack of boxes is 3 times as high as Larry's. Larry's stack is twice the width of Mike's since each of Larry's coins is worth twice as much as each of Mike's.

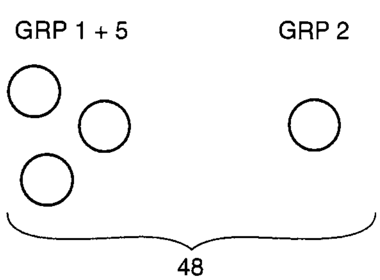
Mike's stack contains 1 more box then Larry's. Since Mike has 45¢ more than Larry, this box is worth 45¢. Thus Mike has 3 × 45¢ or $1.35.



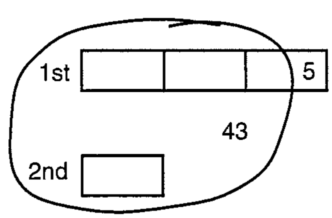
10. Ask the students to use sketches or diagrams to solve problems selected from the attached collection of puzzle problems.

ONE Separate 43 people into 2 groups so that the first group has 5 less than 3 times the number in the second group.

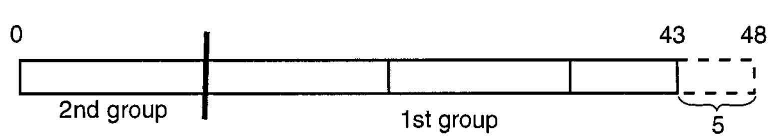
Sketch 1



Sketch 2



Sketch 3



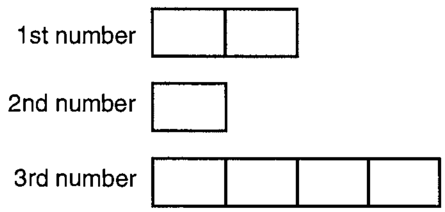
10. You may wish to select a problem or two for the students to work on in class, asking for volunteers to present their solutions. Others can be assigned as homework or the students can be asked to choose the problems they wish to work on.

One way to involve students in reflecting on other students work is to show a sketch a student used to solve a problem, omitting explanations, and ask the other students how they think the sketch was used to solve the problem. For example, shown on the left are three different sketches students used in problem 1 to determine that there are 31 people in the first group and 12 in the second. The explanations they gave are omitted.

Examples of solutions for the other problems are attached.

Puzzle Problems
Sample Sketches

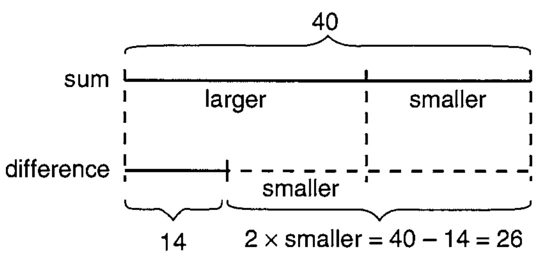
TWO There are 3 numbers. The first is twice the second. The third is twice the first. Their sum is 112. What are the numbers?



Each box represents 112 ÷ 7, or 16. Hence, the numbers are 32, 16 and 64.

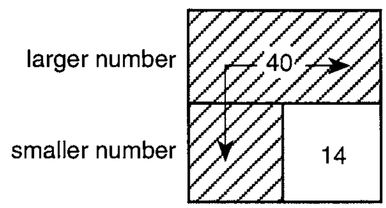
THREE The sum of 2 numbers is 40. Their difference is 14. What are the numbers?

Solution 1



The smaller number is 26 ÷ 2, or 13.
The larger number is 40 – 13, or 27.

Solution 2

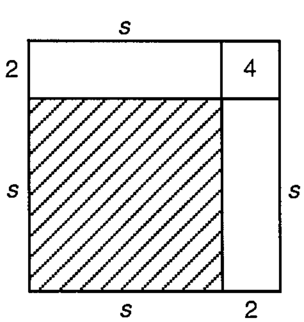


The area of the shaded region is the sum of the 2 numbers; the area of the unshaded region is the difference. The combined area of the shaded and unshaded regions is twice the larger number. Hence, the larger number is (40 + 14) ÷ 2, or 27. The smaller number is 27 – 14, or 13.

FOUR The sides of one square are 2 inches longer than the sides of another square and its area is 48 square inches greater. What is the length of the side of the smaller square?

In each of the following, s is the side of the smaller square.

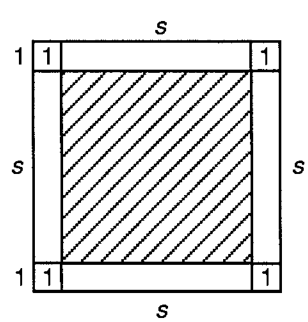
Solution 1



The area of the unshaded border is 48. Hence, the area of each of the two 2 × s rectangles is (48 – 4) ÷ 2, or 22. Thus, s is 11.

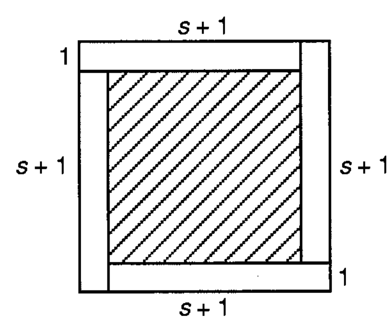
The area of the unshaded border is 48.

Solution 2



Hence, the area of each of the four 1 × s rectangles is (48 – 4) ÷ 4, or 11. Thus, s is 11.

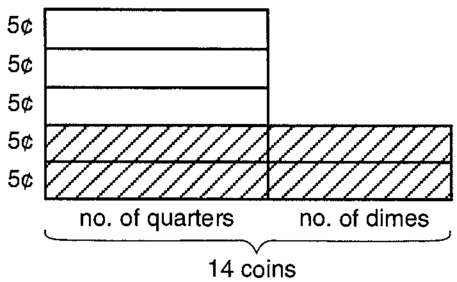
Solution 3



The area of the unshaded border is 48. Hence, the area of each of the four 1 × (s + 1) rectangles is 48 ÷ 4, or 12, and s is 11.

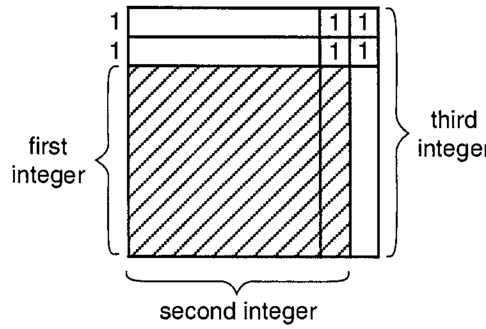
Puzzle Problems
Sample Sketches continued

FIVE Melody has $2.75 in dimes and quarters. There are 14 coins altogether. How many of each does she have?



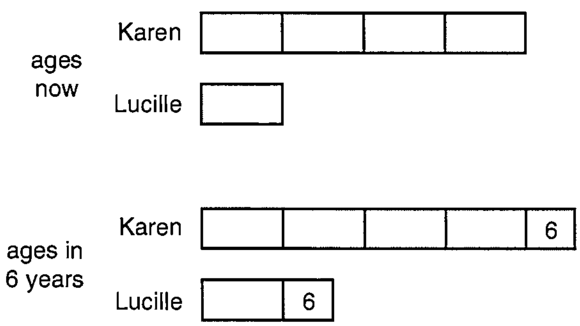
The value of each shaded bar is 5 × 14, or 70¢. Hence, the value of each unshaded bar is (275 – 140) ÷ 3, or 45¢. So, there are 9 quarters and 5 dimes.

SIX Find 3 consecutive integers such that the product of the first and second integers is 40 less than the square of the third integer.

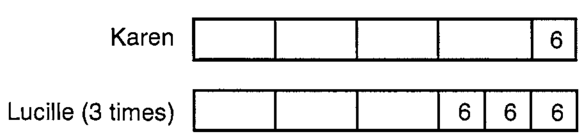


The area of the shaded rectangle is the product of the first two integers. The area of the unshaded region is the difference between that product and the square of the third integer which is given to be 40. Hence, each of the 3 unshaded rectangles has area (40 – 4) ÷ 3, or 12. Thus, the 3 numbers are 12, 13 and 14.

SEVEN Karen is 4 times as old as Lucille. In 6 years, she will be 3 times as old as Lucille. How old is Lucille?



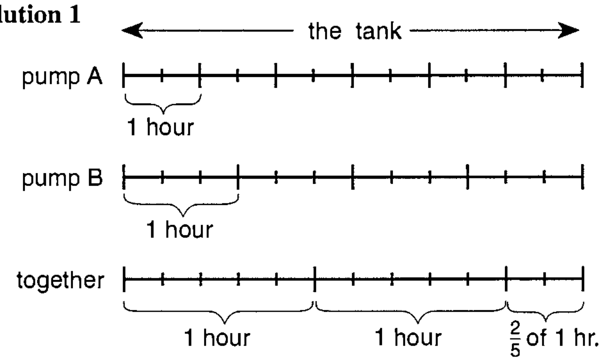
Comparison of Karen's age in 6 years with 3 times Lucille's age in 6 years:



These have the same value if each box represents two 6's, or 12. Hence, Karen is now 48 and Lucille is 12.

EIGHT One pump can fill a tank in 6 hours. Another pump can fill it in 4 hours. If both pumps are used, how long will it take to fill the tank?

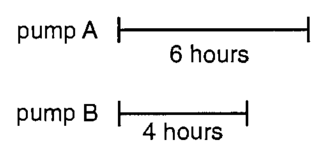
Solution 1



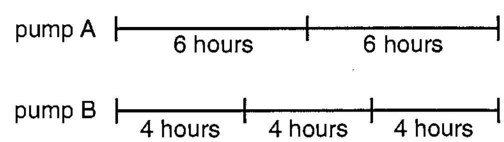
Together, the pumps take 2⅖ hours.

Solution 2

Time to fill 1 tank:



Tanks filled in 12 hours:



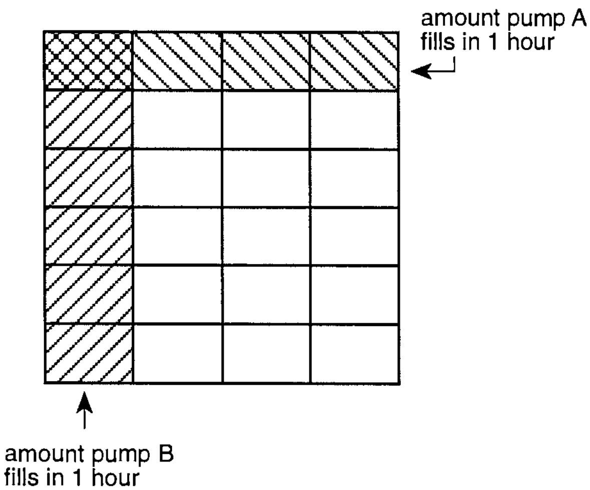
Together, pumps A and B fill 5 tanks in 12 hours; so they fill 1 tank in $^{12}/_{5}$ or 2⅖ hours.

Continued next page.

Puzzle Problems
Sample Sketches continued

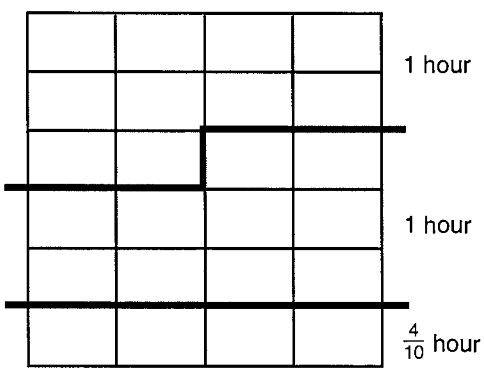
EIGHT (cont.)

Solution 3



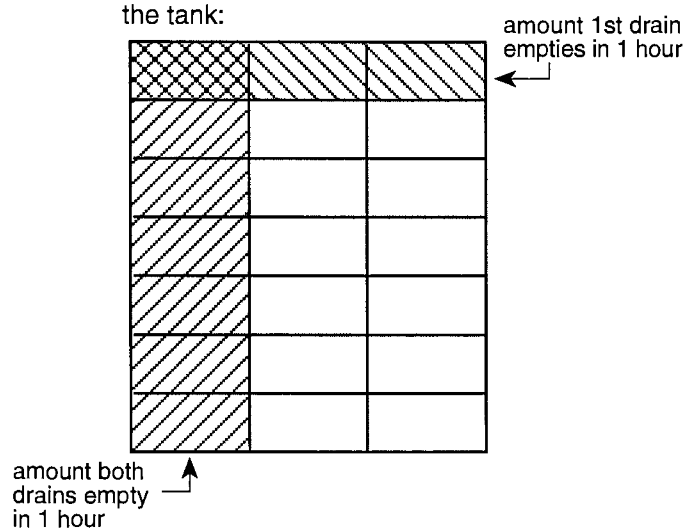
Pump A fills 4 subdivisions in 1 hour.
Pump B fills 6 subdivisions in 1 hour.

Together, they fill 10 subdivisions in 1 hour:

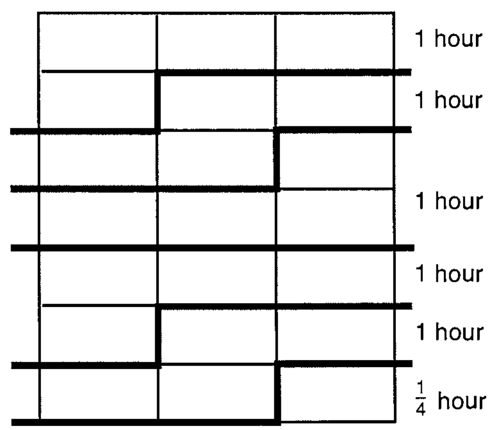


Together, pumps A and B fill the tank in $2\frac{4}{10}$ hours.

NINE A tank has 2 drains of different sizes. If both drains are used, it takes 3 hours to empty the tank. If only the first drain is used, it takes 7 hours to empty the tank. How long does it take to empty the tank if only the second drain is used?



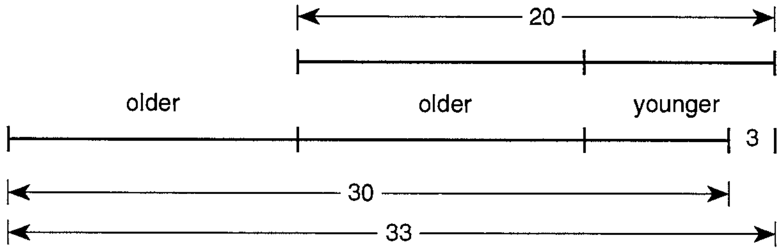
Working together, both drains empty 7 subdivisions in 1 hour. The first drain empties 3, so the second drain empties 4:



It takes the second drain $5^1/4$ hours to empty the tank.

Puzzle Problems
Sample Sketches continued

TEN Two sisters together have 20 books. If the younger sister lost 3 books and the older sister doubled the number she has, they would have a total of 30 books. How many books does each have?



The difference in length of the top and bottom arrows is the number of books the older sister has. Hence, she has 33 – 20, or 13, books. So, the younger sister has 20 – 13, or 7, books.

ELEVEN Of the children in a room, 3⁄5 are girls. There would be an equal number of boys and girls if the number of boys is doubled and 6 more girls are added. How many children are in the room?

Each of the 5 boxes below contains the same number of children; 3 of the boxes contain girls and 2 contain boys.

G
G
G
B
B

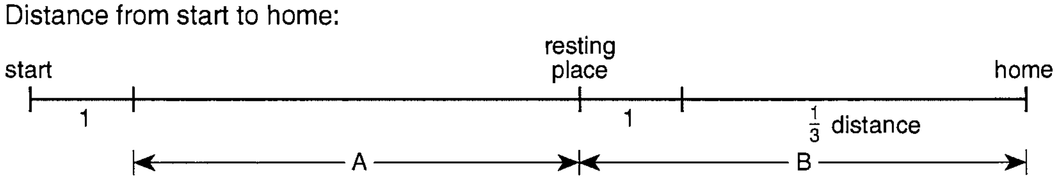
Doubling the boys gives 4 boxes of boys. Adding 6 to the girls, gives 6 more than 3 boxes of girls:

B	G
B	G
B	G
B	X X X X X X

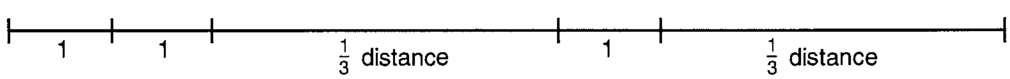
If the number of boys and girls are equal, the last box of boys most contain 6 boys. Thus, all boxes contain 6 children and, to begin with, there were 18 girls and 12 boys.

TWELVE Moe walked home. After he walked 1 mile, he decided to walk half the remaining distance before resting. When he reached his resting point, he still had 1⁄3 the distance to his home plus 1 mile left to walk. How far did Moe walk to get home?

Distance from start to home:



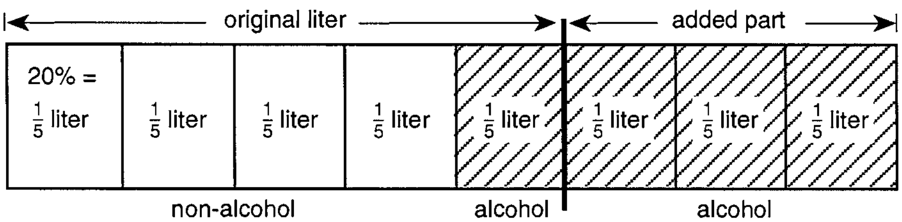
Segments *A* and *B* are equal. Thus, replacing *A* with *B*:



The 3 segments of length 1 comprise the other third of the distance. Hence, the distance from start to home is 9 miles.

THIRTEEN How much alcohol should be added to 1 L of a 20% solution of alcohol to increase its strength to 50%?

Solution 1

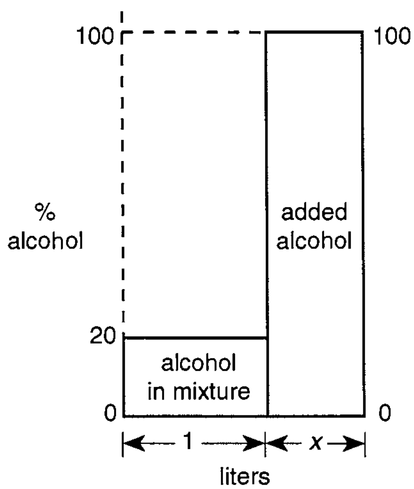


To make mixture half alcohol, 3⁄5 liter of alcohol must be added.

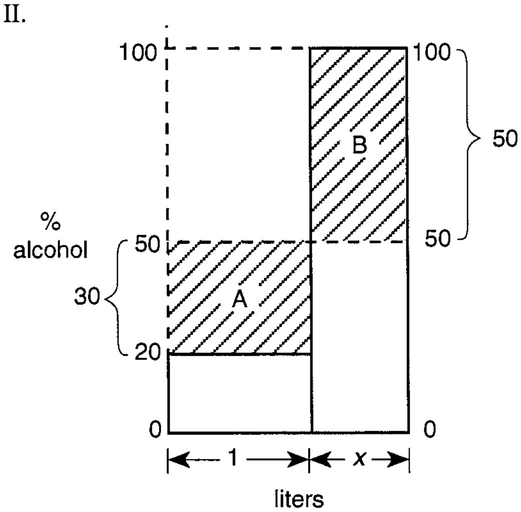
Solution 2

The areas of the rectangles in sketch I represent the amount of alcohol in 1 liter of the original mixture and x liters of added alcohol. If the resulting mixture is to be 50% alcohol, the two rectangles should "level off" at 50. This will be the case if, in sketch II, area A = area B. Since area A is 30 and area B is $50x$, the areas are equal if $30 = 50x$, that is, if $x = 3/5$. Hence, 3⁄5 liter of alcohol must be added.

I.



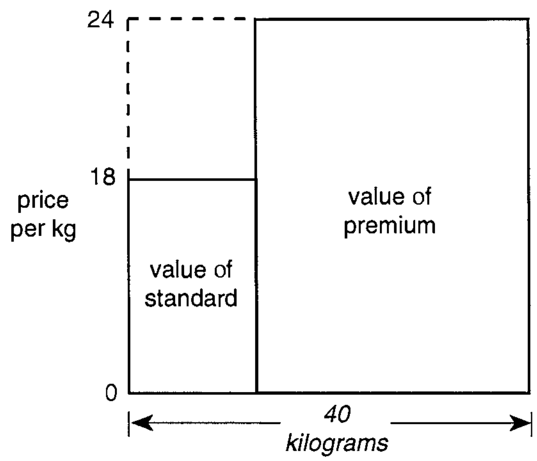
II.



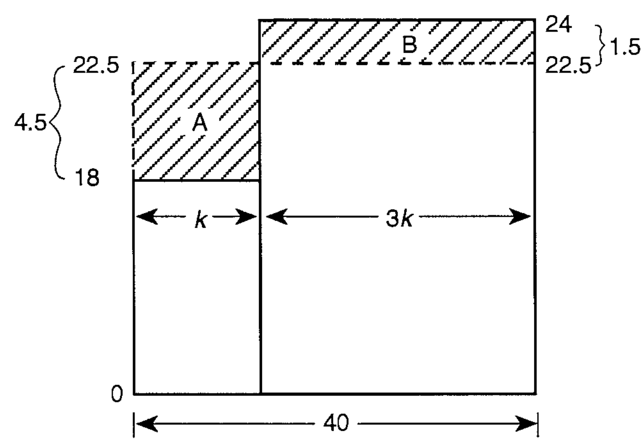
FOURTEEN Standard quality coffee sells for $18.00/kg and prime quality coffee sells for $24.00/kg. What quantity of each should be used to produce 40kg of a blend to sell for $22.50/kg?

The areas of the rectangles in sketch I represent the values of the coffees in the blend. If the blend is to sell for $22.50, the two rectangles should "level off" at 22.5. This will be the case if, in sketch II, area A= area B. Since the height of B is 1/3 the height of A, for the areas to be equal, the base of B must be 3 times the base of A. So, if the base of A is k, the base of B is $3k$. Thus, $4k = 40$ and $k = 10$. Hence, there should be 10 kg of standard coffee and 30 kg of premium coffee.

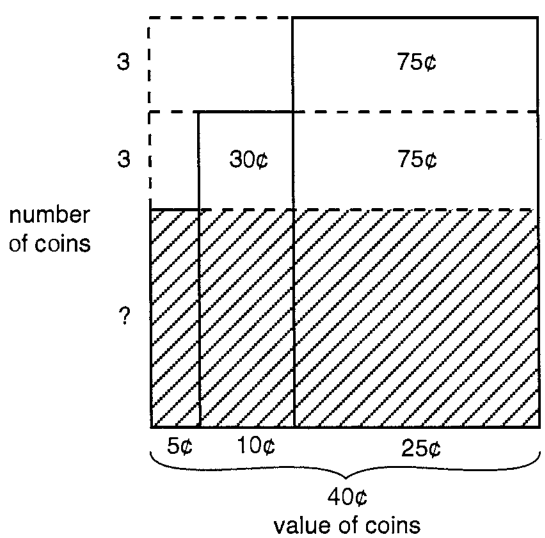
I.



II.



FIFTEEN A collection of nickels, dimes and quarters has 3 fewer nickels than dimes and 3 more quarters than dimes. The collection is worth $4.20. How many of each kind of coin are there?



The heights of the rectangles represent the number of coins and their bases the values, so the sum of the areas of the rectangles is the total value of the collection. The value of the unshaded portion is $1.80. Hence, the value of the shaded rectangle is $4.20 – $1.80, or $2.40. Since the value of its base is 40¢, its height is 2.40 ÷ 40, or 6. Thus, there are 6 nickels, 9 dimes and 12 quarters.

Puzzle Problems
Sample Sketches continued

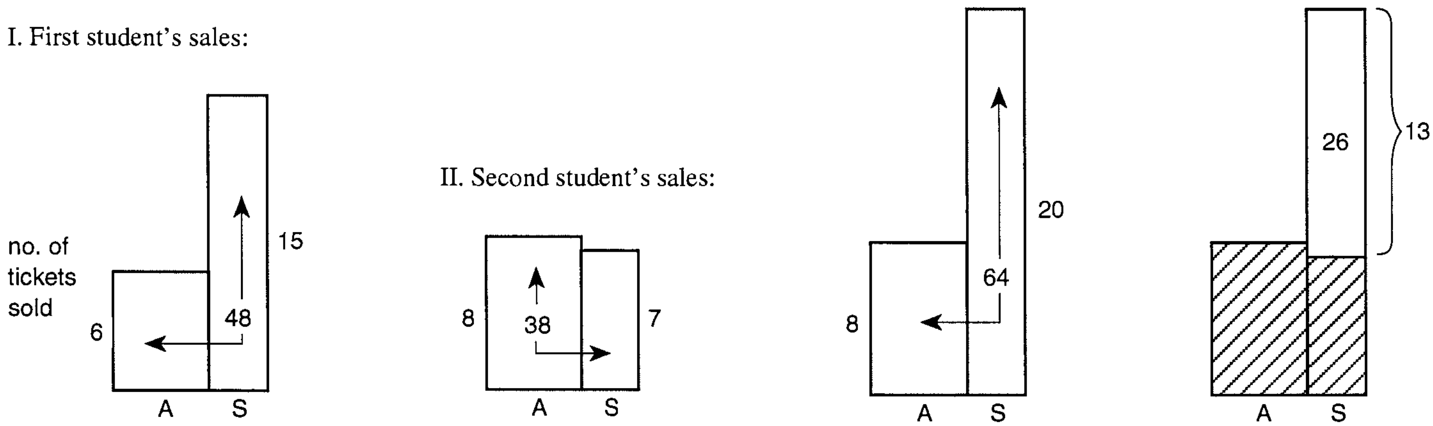
SIXTEEN For a school play, a student sold 6 adult tickets and 15 student tickets, and collected \$48. Another student sold 8 adult tickets and 7 student tickets, and collected \$38. Find the cost of adult and student tickets.

I. First student's sales:

II. Second student's sales:

III. First student's sales increased by a factor of one-third:

IV. Removing II from III:



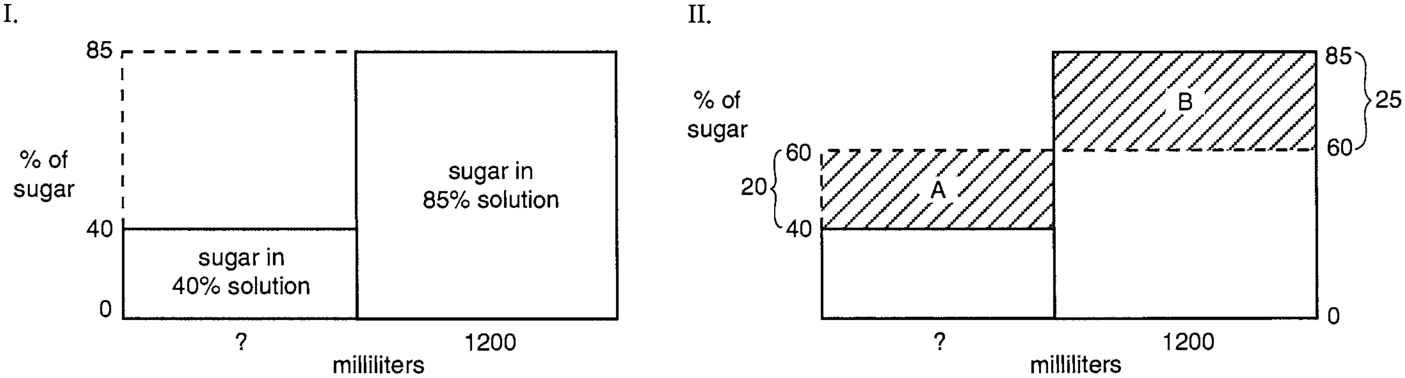
A is the cost of an adult ticket; *S* is the cost of a student ticket.

Increasing the first student's sales by a factor of one-third and removing the second student's sales from the result, as shown in sketch IV, shows that 13 student tickets cost \$26, so each cost \$2. Thus, in sketch I, the 15 student tickets cost \$30, so the 6 adult tickets cost \$18, and each costs \$3.

SEVENTEEN How much of a 40% sugar solution should be added to 1200 ml of an 85% sugar solution to create a 60% solution?

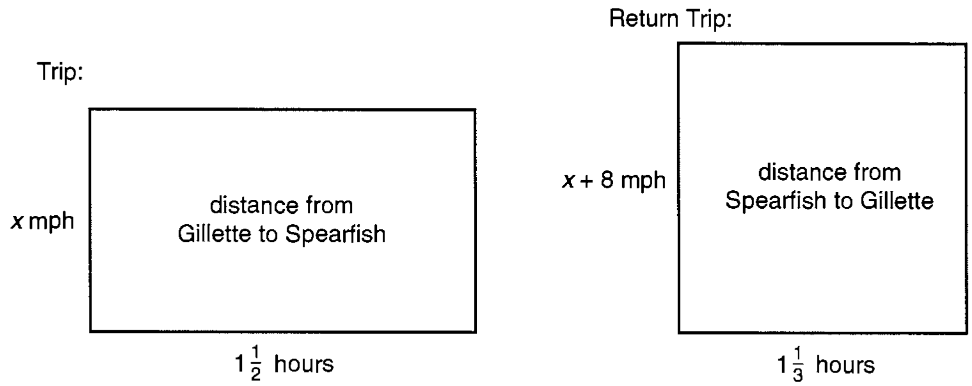
I.

II.

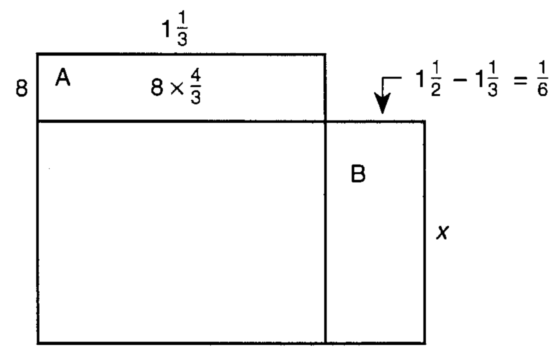


The areas of the rectangles in sketch I represent the amount of sugar in the solutions. If the resulting solution is to be 60% sugar, the two rectangles should "level off" at 60. This will be the case if, in sketch II, area A = area B. Since the area of B is 1200×25 and the height of A is 20, for the areas to be equal, the width of A must be $(1200 \times 25) \div 20$, or 1500. Hence 1500 ml of the 40% solution should be added.

EIGHTEEN A man drives from Gillette to Spearfish in 1 hour and 30 minutes. Driving 8 miles/hour faster, he makes the return trip in 1 hour and 20 minutes. How far is it from Gillette to Spearfish?



The areas of the above rectangles represent distances traveled. Since the distances are the same, the areas are equal. Thus, if one rectangle is superimposed on the other as shown on the right, the areas of *A* and *B* are equal.



Area B = Area A

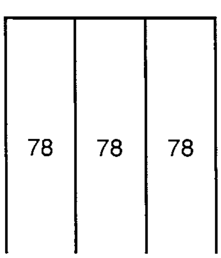
$\frac{x}{6} = \frac{32}{3}$,

$x = 64$ mph

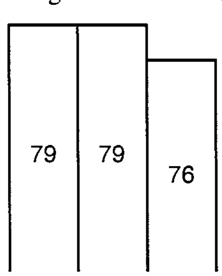
Thus the distance between Gillette and Spearfish is $64 \times (1\frac{1}{2})$, or 96, miles.

NINETEEN A student averaged 78 on three math tests. His score on the first test was 86. His average for the first two tests was 3 more than his score on the third test. What were his scores on the second and third tests?

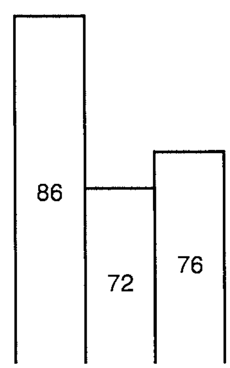
Average score is 78:



Moving 1 point from last score to each of first two scores, makes the average of first two scores 3 greater than third score:

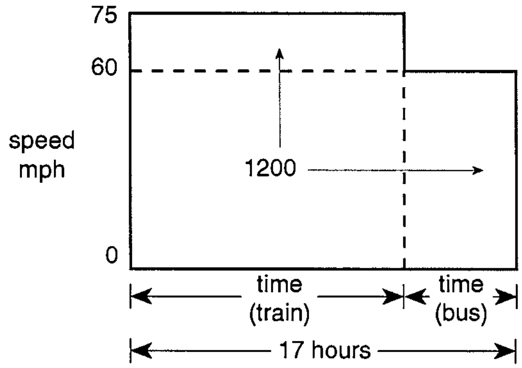


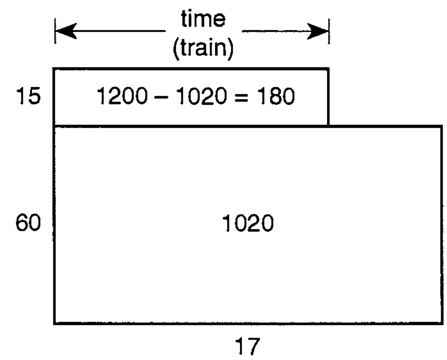
Moving 7 points from second score to first score, makes first score 86:



The 2nd and 3rd scores are 72 and 76, respectively.

TWENTY Traveling by train and bus, a trip of 1200 miles took 17 hours. If the train averaged 75 mi/hr and the bus averaged 60 mi/hr, how far did the train travel?





The distance traveled is represented by the area of the region in the first sketch. This region can be divided into the two rectangles shown in the second sketch. The area of the lower rectangle is 1020, Hence, that of the upper is 180, so its width is $180 \div 15$, or 12. Thus, 12 hours of the trip were by train, and the distance traveled by train was 75×12, or 900, miles.

ONE
Separate 43 people into 2 groups so that the first group has 5 less than 3 times the number in the second group.

TWO
There are 3 numbers. The first is twice the second. The third is twice the first. Their sum is 112. What are the numbers?

THREE
The sum of 2 numbers is 40. Their difference is 14. What are the numbers?

FOUR
The sides of one square are 2 inches longer than the sides of another square and its area is 48 square inches greater. What is the length of the side of the smaller square?

FIVE
Melody has $2.75 in dimes and quarters. There are 14 coins altogether. How many of each does she have?

SIX
Find 3 consecutive integers such that the product of the first and second integers is 40 less than the square of the third integer.

SEVEN
Karen is 4 times as old as Lucille. In 6 years, she will be 3 times as old as Lucille. How old is Lucille?

EIGHT
One pump can fill a tank in 6 hours. Another pump can fill it in 4 hours. If both pumps are used, how long will it take to fill the tank?

NINE
A tank has 2 drains of different sizes. If both drains are used, it takes 3 hours to empty the tank. If only the first drain is used, it takes 7 hours to empty the tank. How long does it take to empty the tank if only the second drain is used?

TEN
Two sisters together have 20 books. If the younger sister lost 3 books and the older sister doubled the number she has, they would have a total of 30 books. How many books does each have?

ELEVEN
Of the children in a room, $\frac{3}{5}$ are girls. There would be an equal number of boys and girls if the number of boys is doubled and 6 more girls are added. How many children are in the room?

TWELVE
Moe walked home. After he walked 1 mile, he decided to walk half the remaining distance before resting. When he reached his resting point, he still had ⅓ the distance to his home plus 1 mile left to walk. How far did Moe walk to get home?

THIRTEEN
How much alcohol should be added to 1 L of a 20% solution of alcohol to increase its strength to 50%?

FOURTEEN
Standard quality coffee sells for $18.00/kg and prime quality coffee sells for $24.00/kg. What quantity of each should be used to produce 40kg of a blend to sell for $22.50/kg?

FIFTEEN
A collection of nickels, dimes and quarters has 3 fewer nickels than dimes and 3 more quarters than dimes. The collection is worth $4.20. How many of each kind of coin are there?

SIXTEEN
For a school play, a student sold 6 adult tickets and 15 student tickets, and collected $48. Another student sold 8 adult tickets and 7 student tickets, and collected $38. Find the cost of adult and student tickets.

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How much of a 40% sugar solution should be added to 1200 ml of an 85% sugar solution to create a 60% solution?

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A man drives from Gillette to Spearfish in 1 hour and 30 minutes. Driving 8 miles/hour faster, he makes the return trip in 1 hour and 20 minutes. How far is it from Gillette to Spearfish?

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A student averaged 78 on three math tests. His score on the first test was 86. His average for the first two tests was 3 more than his score on the third test. What were his scores on the second and third tests?

TWENTY
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Sketching Quadratics, Part I

O V E R V I E W

Sketches are used to solve problems involving quadratic relationships.

Prerequisite Activity

Unit XIII, Activity 1, *Sketching Solutions*; Unit IX, *Picturing Algebra*; Unit XI, *Graphing Algebraic Relationships*; Unit XII, *Modeling Real & Complex Numbers*.

Materials

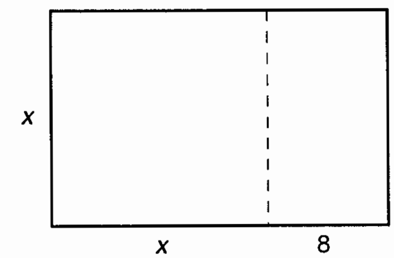
Master.

Actions

1. Have the students sketch a rectangle whose length is 8 units greater than its width. Then tell them the area of the rectangle is 1428 and ask them to find its dimensions.

Comments

1. Here is one sketch of the rectangle:



One way to determine the dimensions of the rectangle is to find two numbers which differ by 8 and whose product is 1428. If the rectangle were a square, its dimension would be $\sqrt{1428}$ which is about 38. Since its not a square, one dimension should be somewhat larger than this and one dimension somewhat smaller. If one guesses the dimensions are 34 and 42, a check will verify that this is correct. Making an educated guess and then checking to see if it is correct would be more difficult if the dimensions were not integers.

Another way to proceed is by "completing the square," as shown in the sketches on the left. If the strip of width 8 in the above sketch is split in two and half of it is moved to an adjacent side, as shown in Figure 1, the result is a square with a 4×4 corner missing. Adding this corner produces a square of area 1428 + 16, or 1444, and edge $x + 4$, as shown in Figure 2. Hence, $x + 4$ is $\sqrt{1444}$, or 38. Thus, x is 34. So the dimensions of the original rectangle are 34 and 34 + 8, or 42.

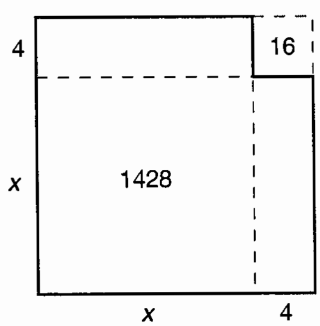


Figure 1

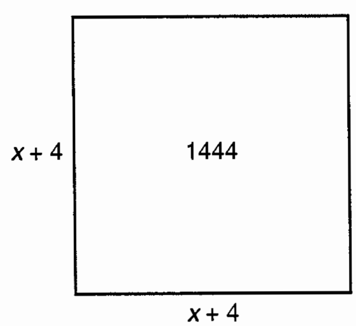


Figure 2

Actions

2. Ask the students to draw sketches to solve the following equations:

(a) $x^2 - 4x + 6 = 5$

(b) $x^2 - 4x + 13 = 5$

(c) $x^2 + 9x = 400$

(d) $x(3x - 4) = 4$

(e) $2x(3 - x) = 3$

(f) $4x^2 - 4x + 13 = 0$

Comments

2. All of these equations can be solved by completing the square. Collectively, they illustrate that a quadratic equation with integral coefficients can have solutions that are rational, irrational (see Activity 5 for a demonstration that $\sqrt{3}$ is irrational), or complex.

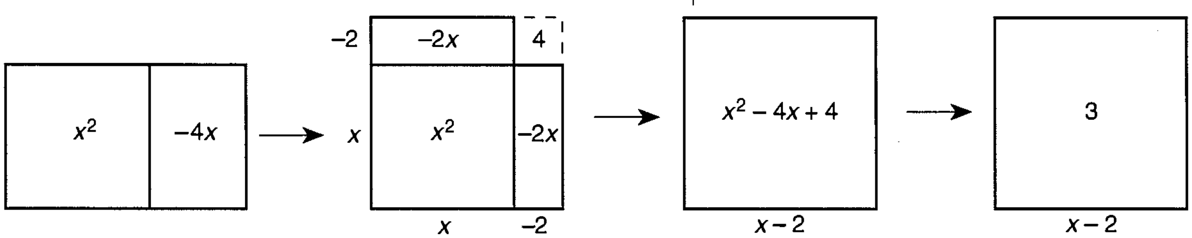
In the sketches that follow, differences are treated as sums, e.g., $x - 2$ is thought of as $x + (-2)$ and is portrayed by a line segment of value x augmented by a segment of value -2. For a discussion of the distinction between the length of a segment and its value, see the last paragraph of Comment 3 in Unit XI, Activity 3.

Sketches for each of the equations are shown below. In the sketches, some of the properties of rectangles discussed in Comment 1 of Unit XII, Activity 3, are used.

(a) $\mathbf{x^2 - 4x + 6 = 5}$

One can complete the square as shown in the following sequence of sketches. Note, in the third figure, that

$$x^2 - 4x + 4 =$$
$$(x^2 - 4x + 6) - 2 = 5 - 2 = 3.$$



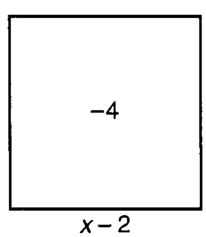
If a square region has value 3, its edges have value $\sqrt{3}$ or $-\sqrt{3}$. Hence,

$$x - 2 = \sqrt{3} \text{ or } x - 2 = -\sqrt{3},$$

and

$$x = 2 + \sqrt{3} \text{ or } x = 2 - \sqrt{3}.$$

(b) $\mathbf{x^2 - 4x + 13 = 5}$



In this case,

$$x^2 - 4x + 4 =$$
$$x^2 - 4x + 13 - 9 = 5 - 9 = -4$$

Proceeding as in (a) one obtains the square on the left. If a square region has value -4, its edges have value $2i$ or $-2i$. Hence,

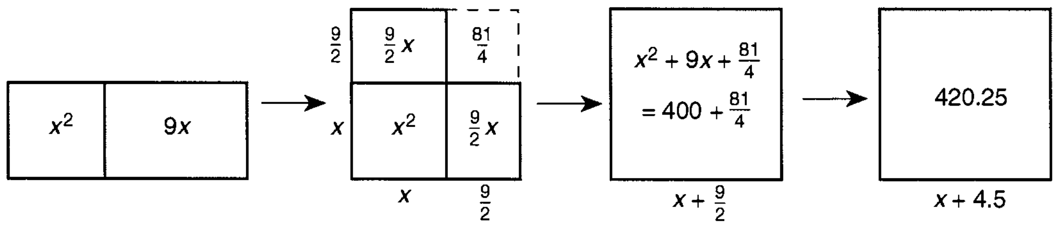
$x - 2 = 2i$ or $x - 2 = -2i$.
$x = 2 + 2i$ or $x = 2 - 2i$.

Continued next page.

2. (b) *Continued.* Some students may find it helpful to think in terms of the colors introduced in Unit XII, Activity 6, *Complex Numbers*: A red 2×2 square can have adjacent edges comprised of either 2 green pieces or 2 yellow pieces.

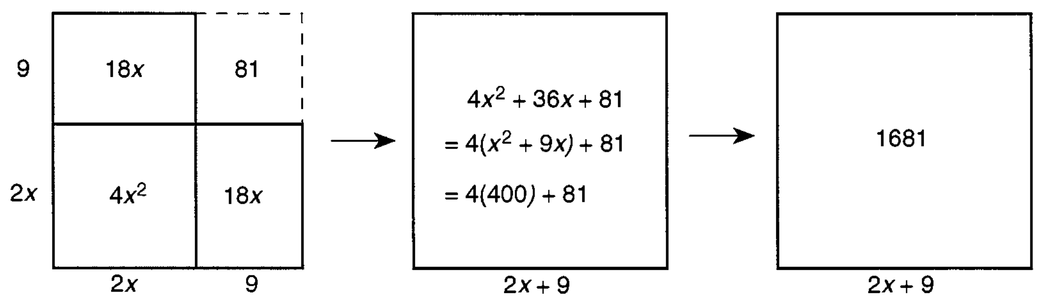
(c) $\mathbf{x^2 + 9x = 400}$

Completing the square gives the following sequence of sketches.



With the help of a calculator, one finds $\sqrt{420.25} = 20.5$. Thus, $x + 4.5 = \pm 20.5$ and $x = 16$ or $x = -25$.

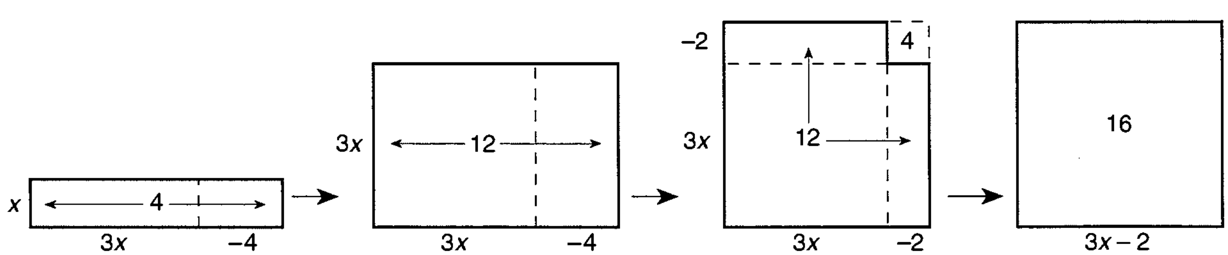
Fractions can be avoided by doubling dimensions as shown in the following sketches.



Since $\sqrt{1681} = 41$, $2x + 9 = \pm 41$ and the result follows.

(d) $\mathbf{x(3x - 4) = 4}$

In the following sequence, the second rectangular region is obtained from the first by increasing its height by a factor of 3.



$3x - 2 = \pm 4$,
$3x = 6$ or $3x = -2$,
$x = 2$ or $x = -2/3$.

Continued next page.

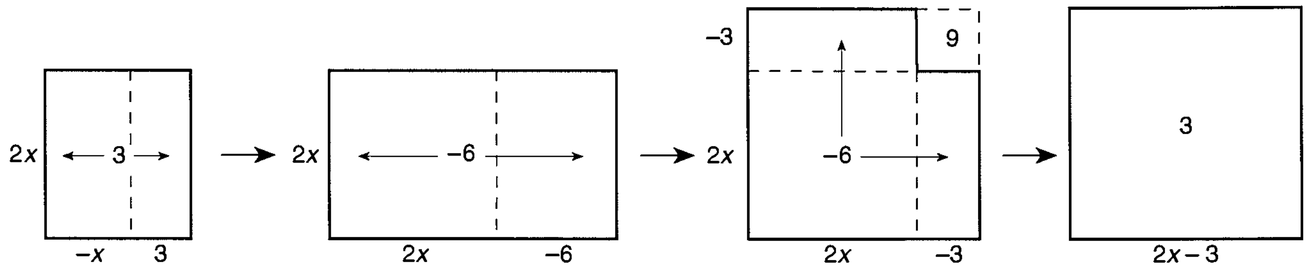
Actions

Comments

2. *Continued.*

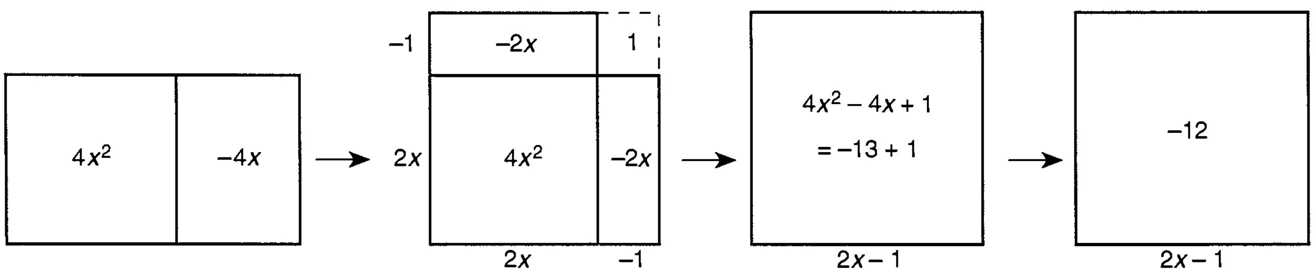
(e) $\mathbf{2x(3 - x) = 3}$

In the following sequence of sketches, the second rectangular region is obtained from the first by changing the value of its base by a factor of –2.



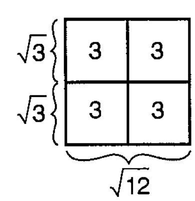
$$2x - 3 = \pm\sqrt{3},$$
$$2x = 3 \pm\sqrt{3},$$
$$x = \tfrac{1}{2}(3 \pm\sqrt{3}).$$

(f) $\mathbf{4x^2 - 4x + 13 = 0}$



$$2x - 1 = \pm i\sqrt{12},$$
$$x = \tfrac{1}{2}(1 \pm i\sqrt{12}).$$

Since $\sqrt{ab} = \sqrt{a}\sqrt{b}$ (see Comment 16 in Unit XII, Activity 5, *Squares and Square Roots*), one may write $\sqrt{12}$ as $\sqrt{4}\sqrt{3}$, or $2\sqrt{3}$. One can also see that $\sqrt{12} = 2\sqrt{3}$ by dividing a square of area 12 into fourths, as shown below, and looking at the lengths of the edges.



Many sequences of sketches shown in the solutions above, and elsewhere in the unit, contain more figures than may be in the sketches the students draw. In a number of instances, several figures shown in a sequence of sketches could be combined into a single figure, especially if an oral presentation is being made concurrently, or if solutions are being developed for private use and not for the benefit of a reader.

Actions

3. Ask the students to use sketches in solving the following problems:

(a) The difference of two numbers is 6. The sum of their squares is 1476. What are the numbers?

(b) The length of a rectangle is 6 units less than twice its width. Its area is 836. What are its dimensions?

(c) The product of two consecutive even numbers is 2808. What the numbers?

(d) What are the dimensions of a rectangle whose perimeter is 92 and area is 493?

(e) The sum of two numbers is 32 and the sum of their squares is 520. What are the numbers?

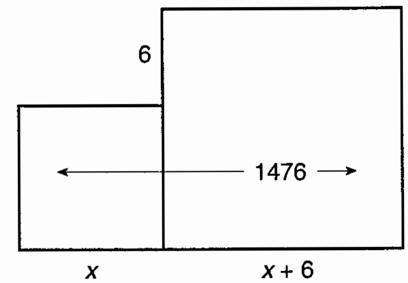
(f) A 40 foot by 60 foot rectangular garden is bordered by a sidewalk of uniform width. If the area of the sidewalk is 864 square feet, what is its width?

Comments

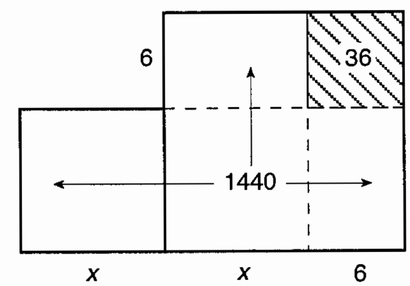
3. A master of the problems is attached.

One solution for (a) is given below. A sample solution for each of the other problems is also attached.

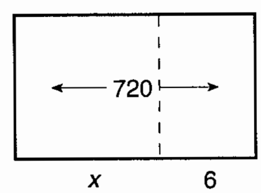
(a) The squares of two numbers whose difference is 6 and sum is 1476:



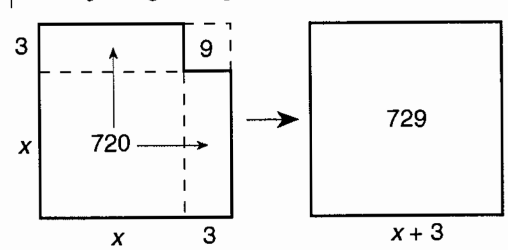
The shaded area below is 36; so the unshaded area is 1476 – 36, or 1440:



Half the unshaded area is 720:



Completing the square:

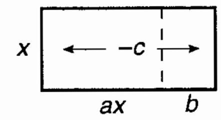


Since $\sqrt{729} = 27$, $x + 3 = \pm 27$. Thus, x is either 24 or –30 and $x + 6$ is, respectively, 30 or –24. Hence, the two numbers are 24 and 30 or –24 and –30.

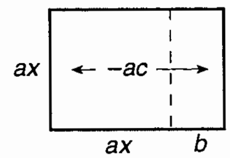
4. (Optional) Ask the students to find x (in terms of a, b and c) if $ax^2 + bx + c = 0$.

4. One way to find x is shown below. In the solution shown, the size of regions are adjusted to avoid fractions.

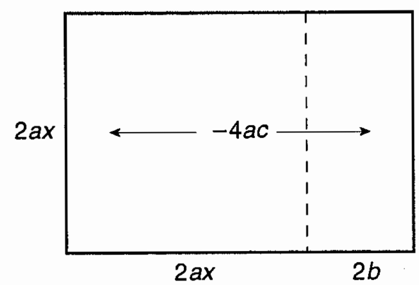
If $ax^2 + bx + c = 0$, then $ax^2 + bx = -c$, and $x(ax + b) = -c$:



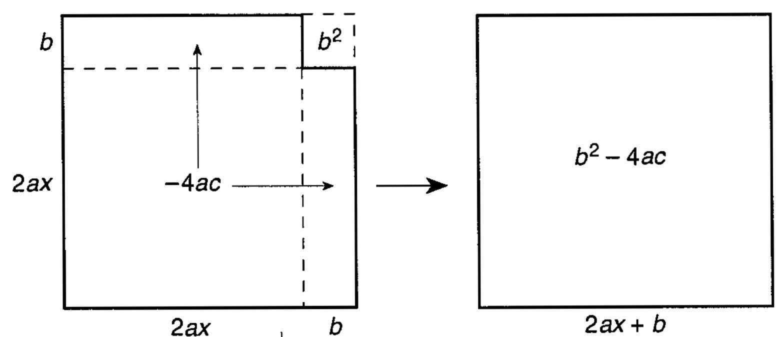
Changing the value of the height, and hence that of the region, by a factor of a:



Changing the values of the base and the height by factors of 2 and, hence, that of the region by a factor of 4:



Completing the square:



$$2ax + b = \pm\sqrt{b^2 - 4ac}.$$

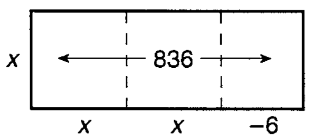
$$\text{So, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This result is known as the *Quadratic Formula.*

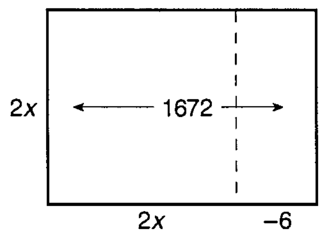
Solutions to Action 3 Problems

b The length of a rectangle is 6 units less than twice its width. Its area is 836. What are its dimensions?

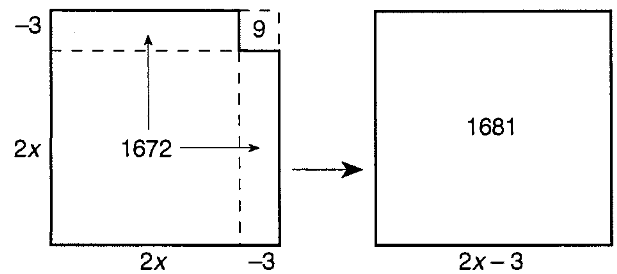
A rectangular region whose value is 836 with one edge whose value is 6 less than twice the value of the other edge:



Doubling this region:



Completing the square:



$2x - 3 = \sqrt{1681} = 41,$
$x = 22.$

The dimensions of the rectangle are 22 and 2(22) – 6, or 38.

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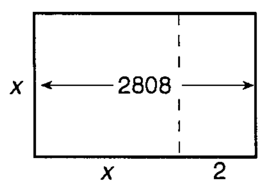
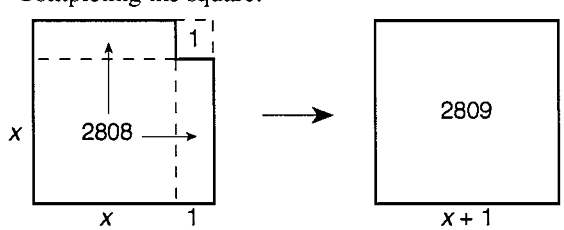
c The product of two consecutive even numbers is 2808. What are the numbers?

Two consecutive even numbers whose product is 2808:



Completing the square:



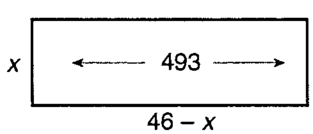
$x + 1 = \pm\sqrt{2809} = \pm 53,$
$x = 52$ or $x = -54.$

The two numbers are 52 and 54 or –54 and –52.

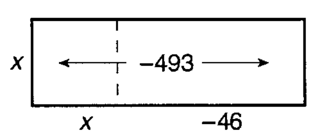
---

d What are the dimensions of a rectangle whose perimeter is 92 and area is 493?

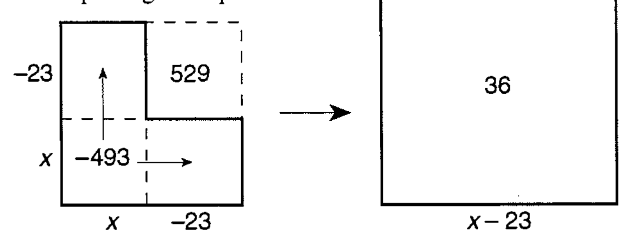
A rectangle whose perimeter is 92 and area is 493:



Changing the value of the base, and, hence, the value of the region, by a factor of –1:



Completing the square:



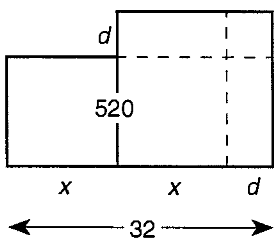
$x - 23 = \sqrt{36} = 6,$
$x = 29.$

The dimensions are 29 and 46 – 29, or 17.

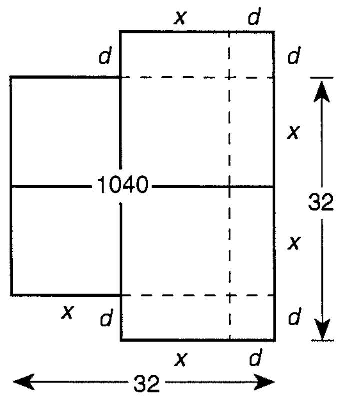
Solutions to Action 3 Problems
continued

e The sum of two numbers is 32 and the sum of their squares is 520. What are the numbers?

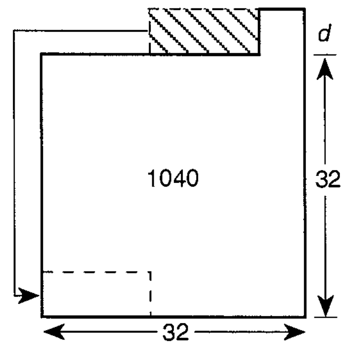
Two numbers, x and $x + d$, whose sum is 32 and whose squares add to 520:



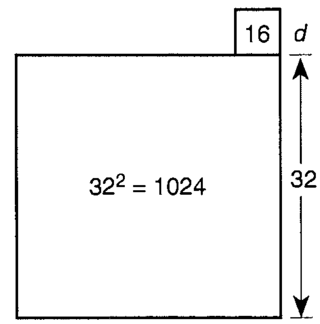
Double the figure:



Move the shaded region from the top to the bottom of the figure:



The result is two squares, one 32×32 and one $d \times d$, which sum to 1040:



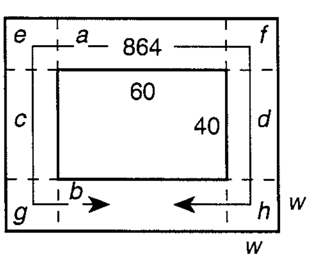
$d^2 = 1040 - 1024 = 16$,
$d = 4$.

Since $d = 4$ from the first sketch, $2x + 4 = 32$. Thus, $x = 14$ and the two numbers are 14 and 18.

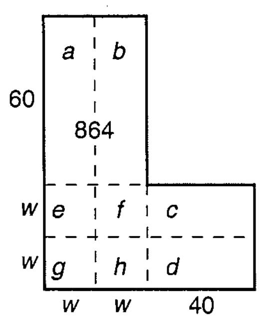
---

f A 40 foot by 60 foot rectangular garden is bordered by a sidewalk of uniform width. If the area of the sidewalk is 864 square feet, what is its width?

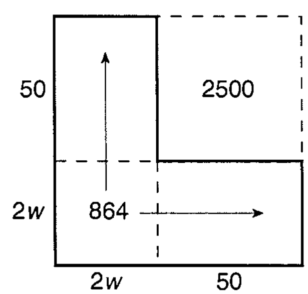
A 40×60 rectangular garden with a 864 square foot border of uniform width, w:

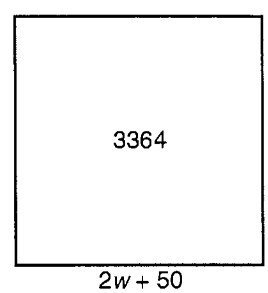


Rearranging the border:



Moving 10 feet from a and b to c and d, and then completing the square:





$2w + 50 = \sqrt{3364} = 58$,
$w = 4$.

The width of the border is 4 feet.

ACTION 3

a The difference of 2 numbers is 6. The sum of their squares is 1476. What are the numbers?

b The length of a rectangle is 6 units less than twice its width. Its area is 836. What are its dimensions?

c The product of 2 consecutive even numbers is 2808. What are the numbers?

d What are the dimensions of a rectangle whose perimeter is 92 and area is 493?

e The sum of 2 numbers is 32 and the sum of their squares is 520. What are the numbers?

f A 40 foot by 60 foot rectangular garden is bordered by a sidewalk of uniform width. If the area of the sidewalk is 864 square feet, what is its width?

Sketching Quadratics, Part II

O V E R V I E W

Further ways of using sketches to solve quadratics are discussed.

Prerequisite Activity

Unit XIII, Activity 2, *Sketching Quadratics, Part I.*

Materials

Master.

Actions

1 . Ask the students to draw sketches to solve the following equations:

(a) $(x + 3)^2 + 4(x + 3) = 45$

(b) $(x + 3)^2 + 4x = 65$

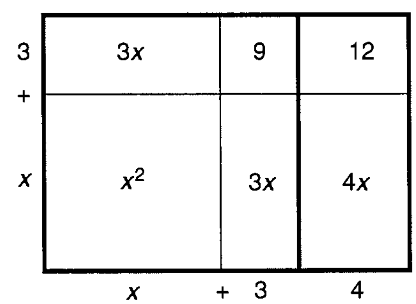
(c) $(x + 3)(x - 5) = 20$

(d) $(x + 3)^2 = 16x$

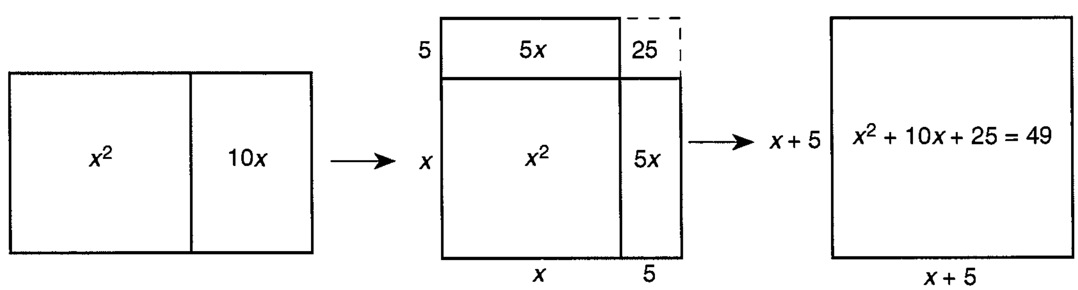
(e) $(x + 3)(x - 4) = 3x$

Comments

1. The equations in Action 1 can be changed into equivalent equations which are similar to those solved in Action 2 of Unit XIII, Activity 2, *Solving Quadratics, Part I.* For example, since $(x + 3)^2 + 4(x + 3) = x^2 + 10x + 21$ (see the sketch), equation (a) is equivalent to $x^2 + 10x = 24$. This



latter equation can be solved by completing the square, as shown below.



$(x + 5)^2 = 49$
$x + 5 = \pm 7$
$x = 2$ or -12

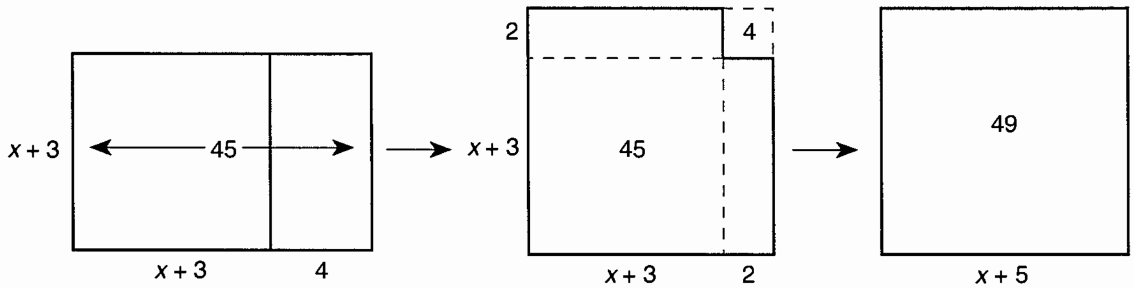
Continued next page.

Actions

Comments

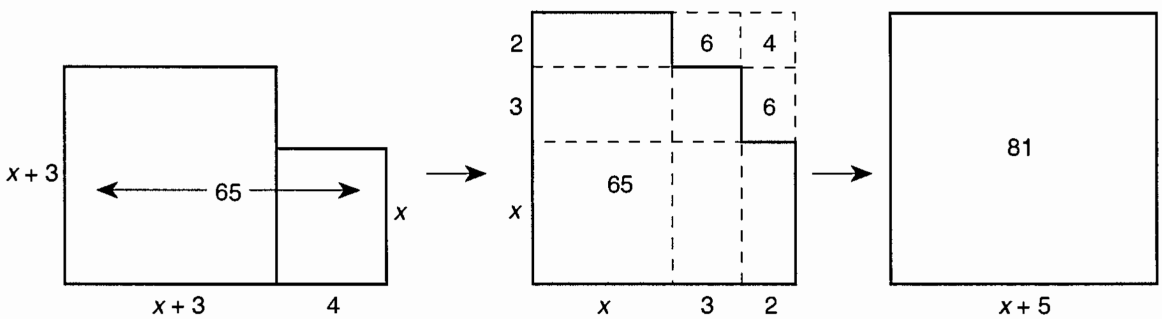
1. *Continued.* Alternatively, the equations can be solved by drawing sketches suggested by their existing forms and then adjusting these sketches to complete squares. Some ways of doing this are shown in the following sample solutions. The students may devise other ways.

(a) $(x + 3)^2 + 4(x + 3) = 45$



$(x + 5)^2 = 49$
$x + 5 = \pm 7$
$x = 2$ or -12

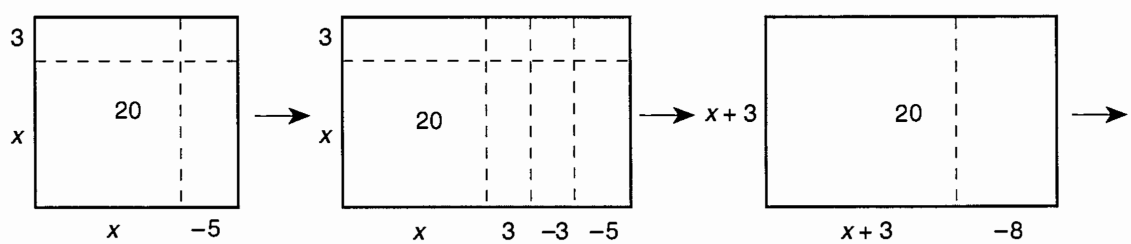
(b) $(x + 3)^2 + 4x = 65$

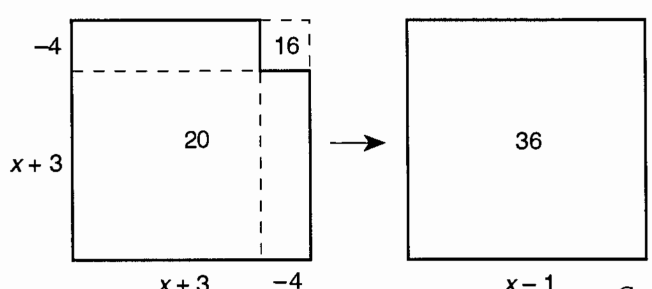


$(x + 5)^2 = 81$
$x + 5 = \pm 9$
$x = 4$ or -14

(c) $(x + 3)(x - 5) = 20$

In the second sketch, below, to get a square whose edge has value $x + 3$, two regions are added—one whose base has value 3 and one whose base has value -3. This does not change the values of either the region or its edges.





$(x - 1)^2 = 36$
$x - 1 = \pm 6$
$x = 7$ or -5

Continued next page.

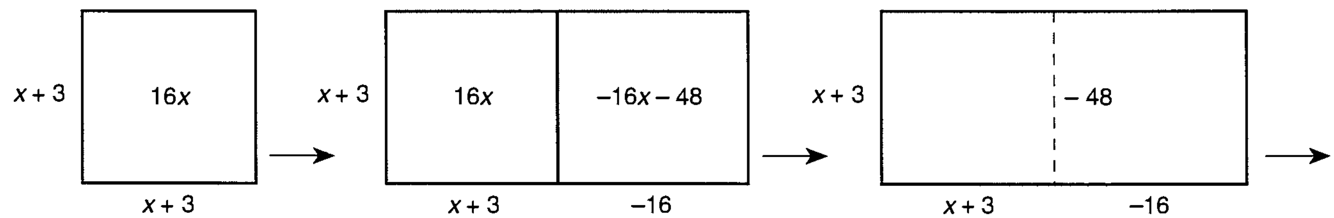
Actions

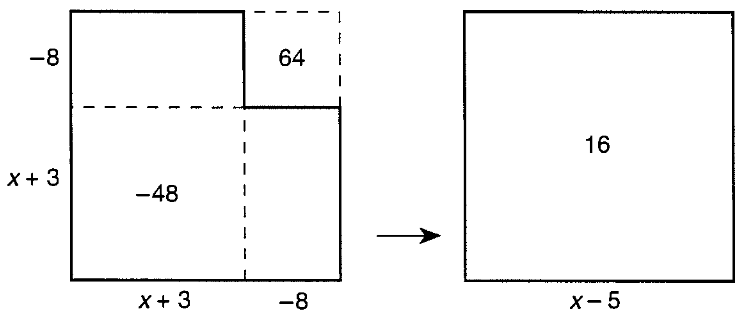
Comments

1. *Continued.*

(d) $\boldsymbol{(x+3)^2 = 16x}$

In the second sketch, a region is added whose base has value -16. The added region has value $-16x - 48$. The added region was chosen so that when its value is added to the value, $16x$, of the original region, the x terms are eliminated.

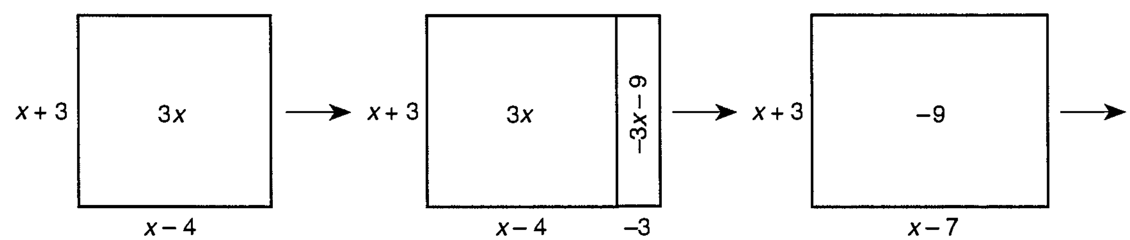


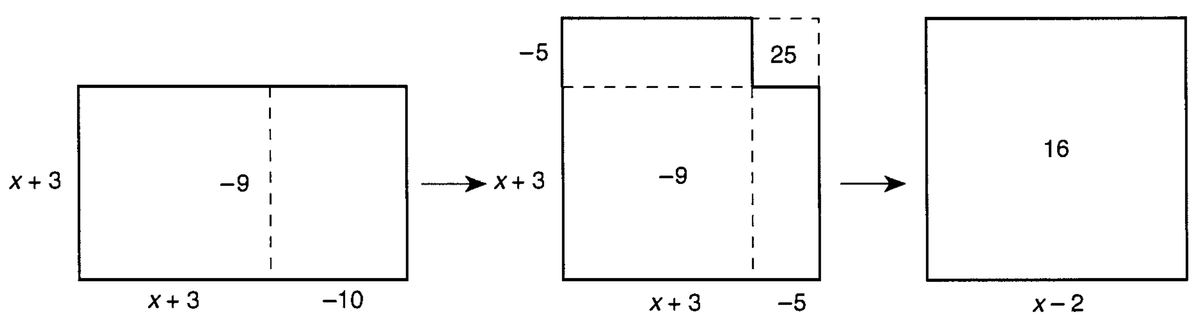


$(x-5)^2 = 16$
$x - 5 = \pm 4$
$x = 9$ or 1

(e) $\boldsymbol{(x+3)(x-4) = 3x}$

As in part (d), a region, chosen to eliminate the x term, is added in the second sketch.





$(x-2)^2 = 16$
$x - 2 = \pm 4$
$x = 6$ or -2

Continued next page.

Actions

2. Ask the students to solve the following equations:
   (a) $(3x - 2)(3x + 4) = 352$
   (b) $(3x - 2)(x + 4) = 325$
   (c) $(2x + 3)(3x - 5) = 130$
   (d) $(x - 2)(x + 3) = 6x + 8$
   (e) $(5x - 2)(2x + 3) = 20x + 300$

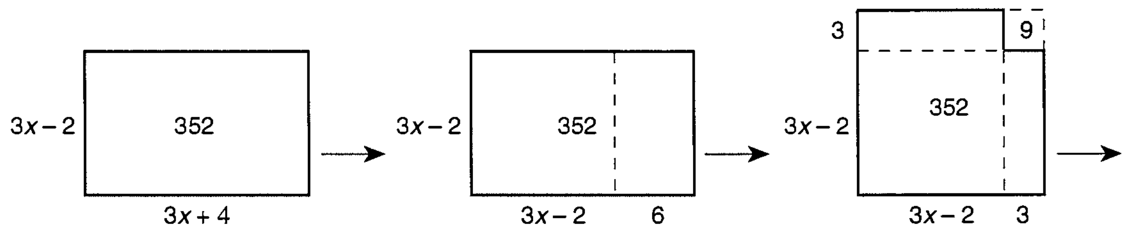
Comments

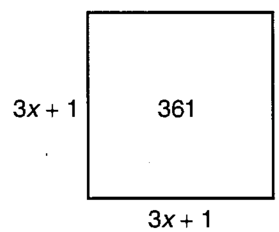
2. These equations are similar to some arising in the next Activity, *Equations Involving Rational Expressions*. A master of the equations is attached.

The sample solutions shown below are patterned after the solutions shown in Comment 1. The students may devise solutions other than those shown.

(a) **$(3x - 2)(3x + 4) = 352$**

In the second sketch, below, $3x + 4$ is converted to the sum of $3x - 2$ and 6, preparatory to completing the square.





$3x + 1 = \pm 19$
$3x = 18 \text{ or } -20$
$x = 6 \text{ or } -20/3$

Continued next page.

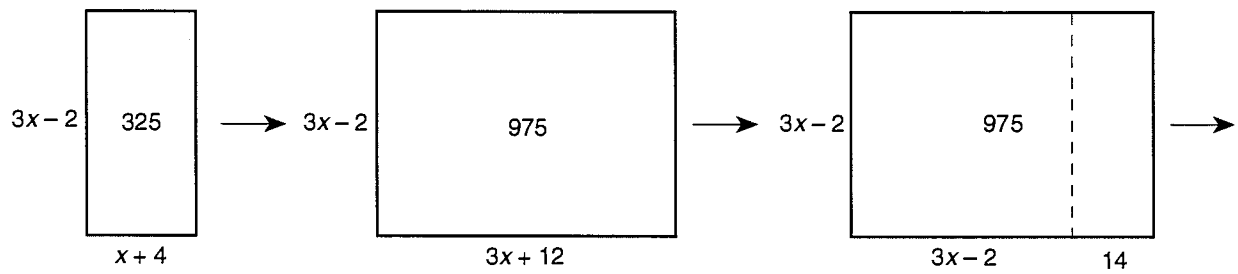
Actions

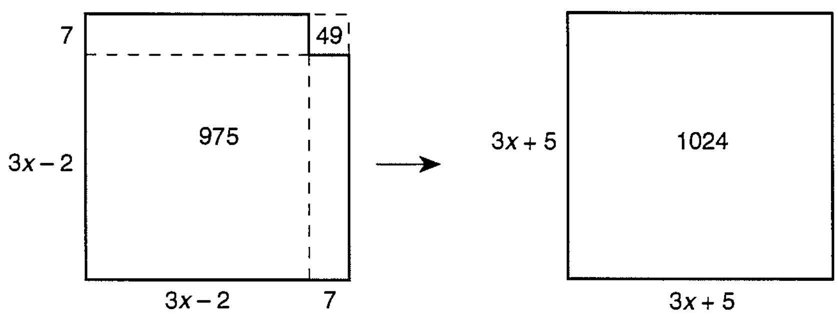
Comments

2. *Continued.*

(b) **$(3x - 2)(x + 4) = 325$**

In the second sketch below, in order that the values of the base and height have the same coefficient of x, the value of the base, and hence that of the region, is increased by a factor of 3.





$$3x + 5 = \pm\sqrt{1024} = \pm 32$$
$$3x = 27 \text{ or } -37$$
$$x = 9 \text{ or } -{}^{37}\!/_{3}$$

Continued next page.

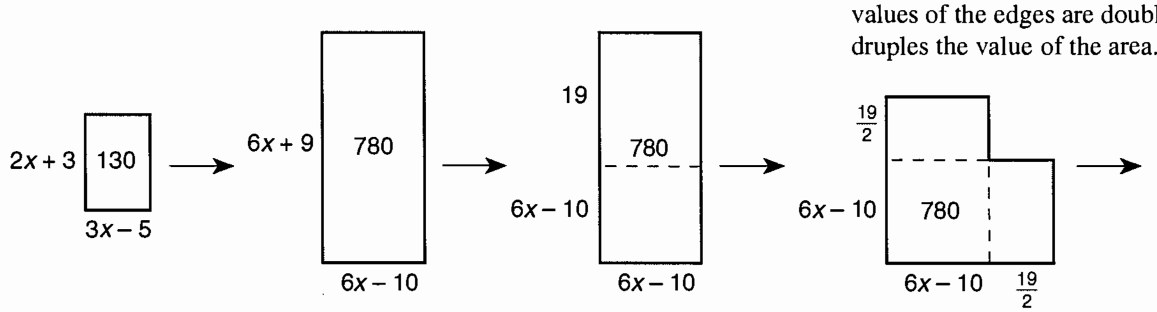
Actions

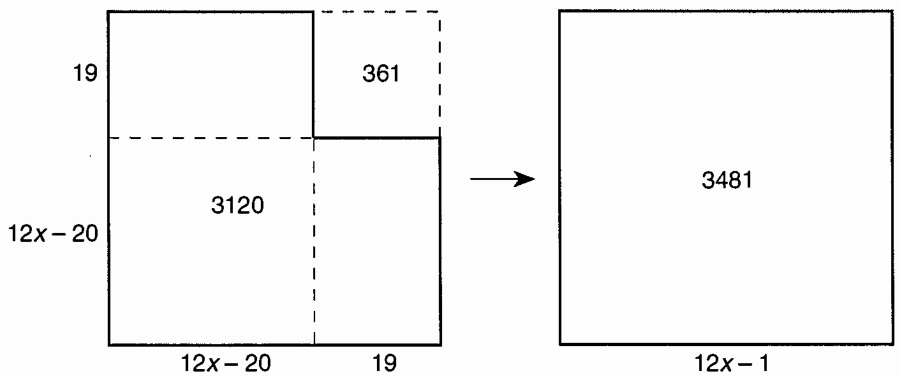
Comments

2. *Continued.*

(c) $(2x + 3)(3x - 5) = 130$

In the second sketch below, in order that the values of the base and height have the same coefficient of x, their values are increased by factors of 2 and 3, respectively. Hence, that of the region is increased by a factor of 6. Later, to avoid fractions, the values of the edges are doubled, which quadruples the value of the area.

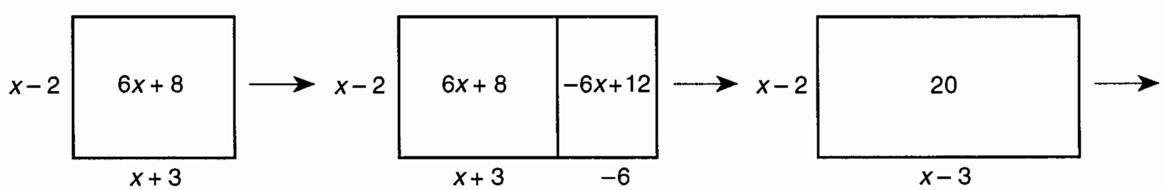


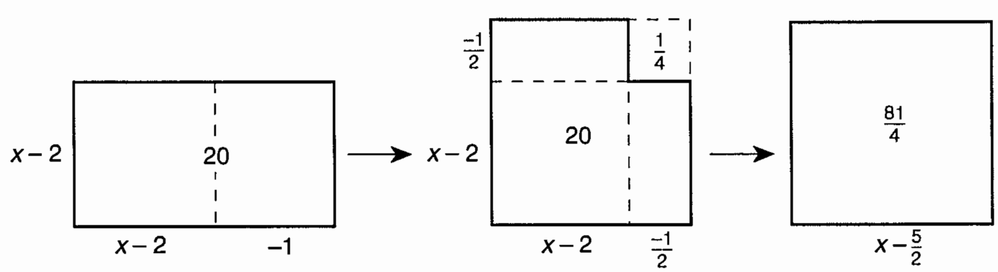


$$12x - 1 = \pm\sqrt{3481} = \pm 59$$
$$12x = 60 \text{ or } -58$$
$$x = 5 \text{ or } -29/6$$

(d) $(x - 2)(x + 3) = 6x + 8$

In the second sketch below, a region is added to eliminate the x term in the value of the original region.





$$x - 5/2 = \pm 9/2$$
$$x = 14/2 \text{ or } -4/2$$
$$x = 7 \text{ or } -2$$

Continued next page.

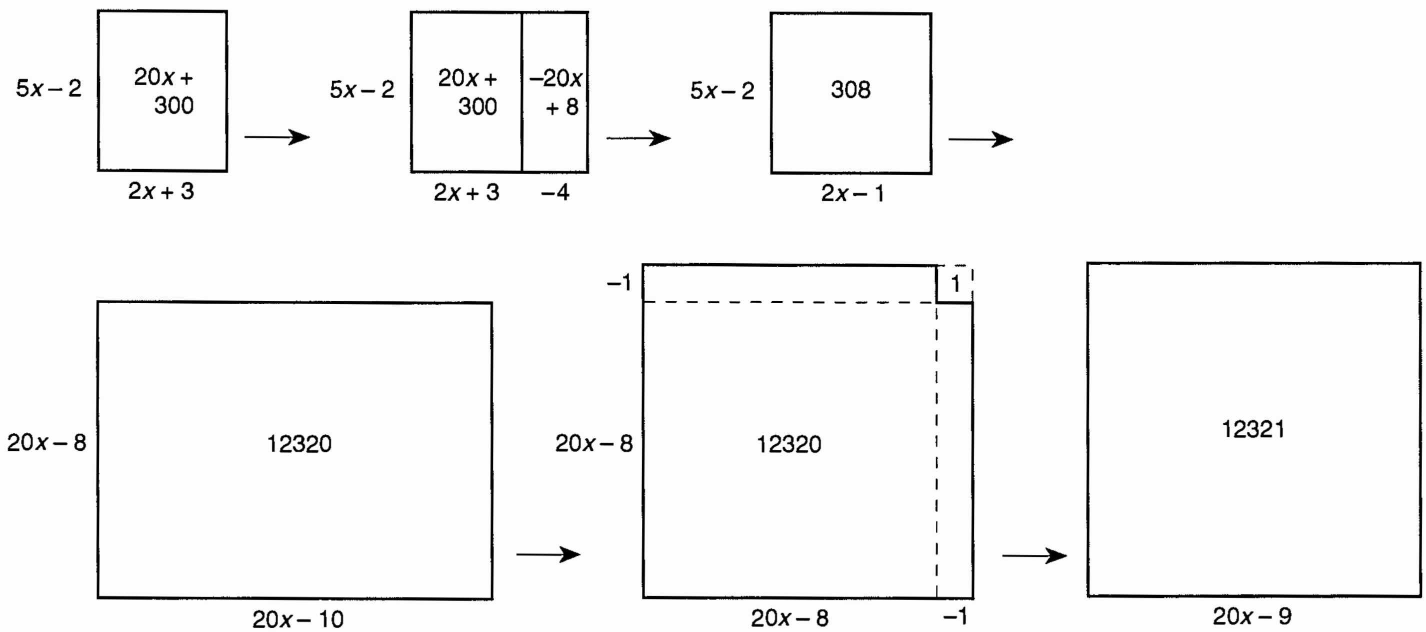
Actions

Comments

2. *Continued.*

(e) **$(5x - 2)(2x + 3) = 20x + 300$**

The solution shown incorporates strategies from the previous examples. (The sketches are not drawn to scale.) The value of the height of the third figure is increased by a factor of 4 and that of its base by a factor of 10 to avoid fractions.



$$20x - 9 = \pm\sqrt{12321} = \pm 111$$
$$20x = 120 \text{ or } -102$$
$$x = 6 \text{ or } -51/10$$

ACTION 2

a $(3x - 2)(3x + 4) = 352$

b $(3x - 2)(x + 4) = 325$

c $(2x + 3)(3x - 5) = 130$

d $(x - 2)(x + 3) = 6x + 8$

e $(5x - 2)(2x + 3) = 20x + 300$

Unit XIII • Activity 4

Equations Involving Rational Expressions

O V E R V I E W

Sketches are used to solve equations involving rational functions.

Prerequisite Activity

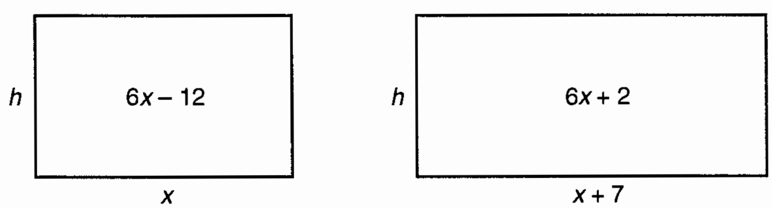
Unit XIII, Activity 3, *Sketching Quadratics, Part II.*

Materials

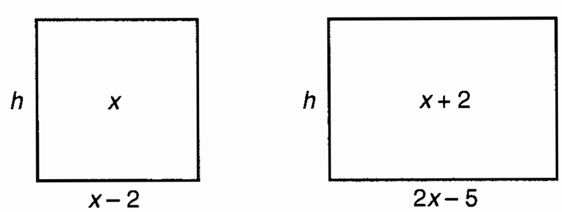
Masters A and B.

Actions

1. (a) Show the students the following two rectangular regions, not necessarily drawn to scale. Tell them the heights of the two regions have the same value h and their areas and bases, expressed in terms of x, have values as shown. Ask the students to find the numerical values of h and x.



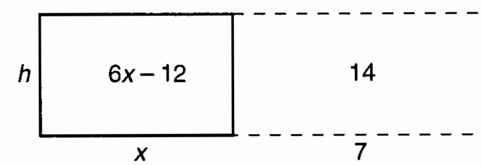
(b) Repeat part (a) for the following pair of rectangular regions.



Comments

1. Actions 1 and 2 are preliminary to Action 3, in which the students are asked to draw sketches to solve equations involving rational expressions.

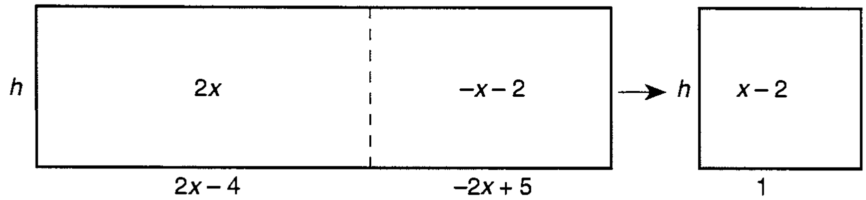
(a) If the base of the first rectangle is increased by 7, as shown below, then the two rectangles have identical dimensions and, hence, equal areas. Thus, the area of the rectangular region added to the first rectangle is 14. Since the base of the added region is 7, its height, h, is 2. Thus, from the first rectangle, $2x = 6x - 12$ and $x = 3$.

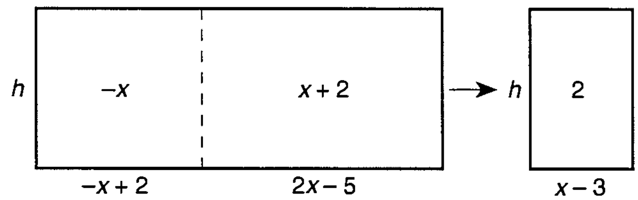


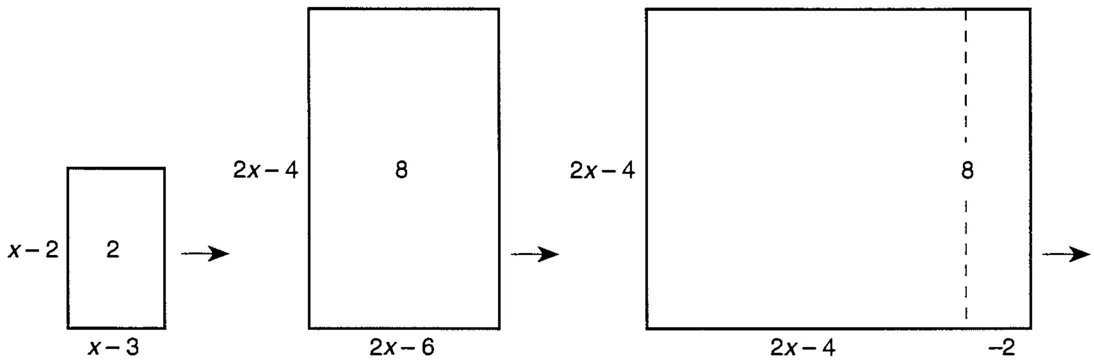
(b) There are a number of ways to determine the values of h and x. One way is to create combinations of the two rectangular regions that result in other rectangular regions which have a constant area or base.

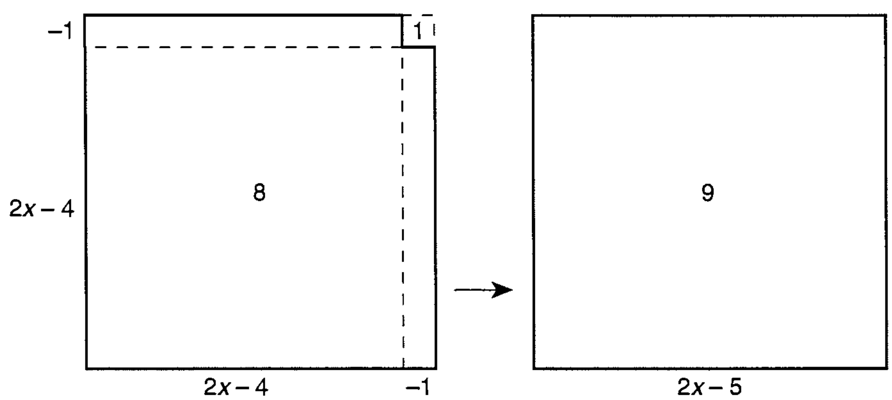
Continued next page.

Actions









Comments

1. (b) *Continued.*

For example, if the value of the base of the first rectangular region (and hence the value of its area) is changed by a factor of 2, and that of the second by a factor of -1, and the resulting regions are combined, the result is a rectangular region of height h, base 1 and area $x - 2$, as shown. Hence, $h = x - 2$.

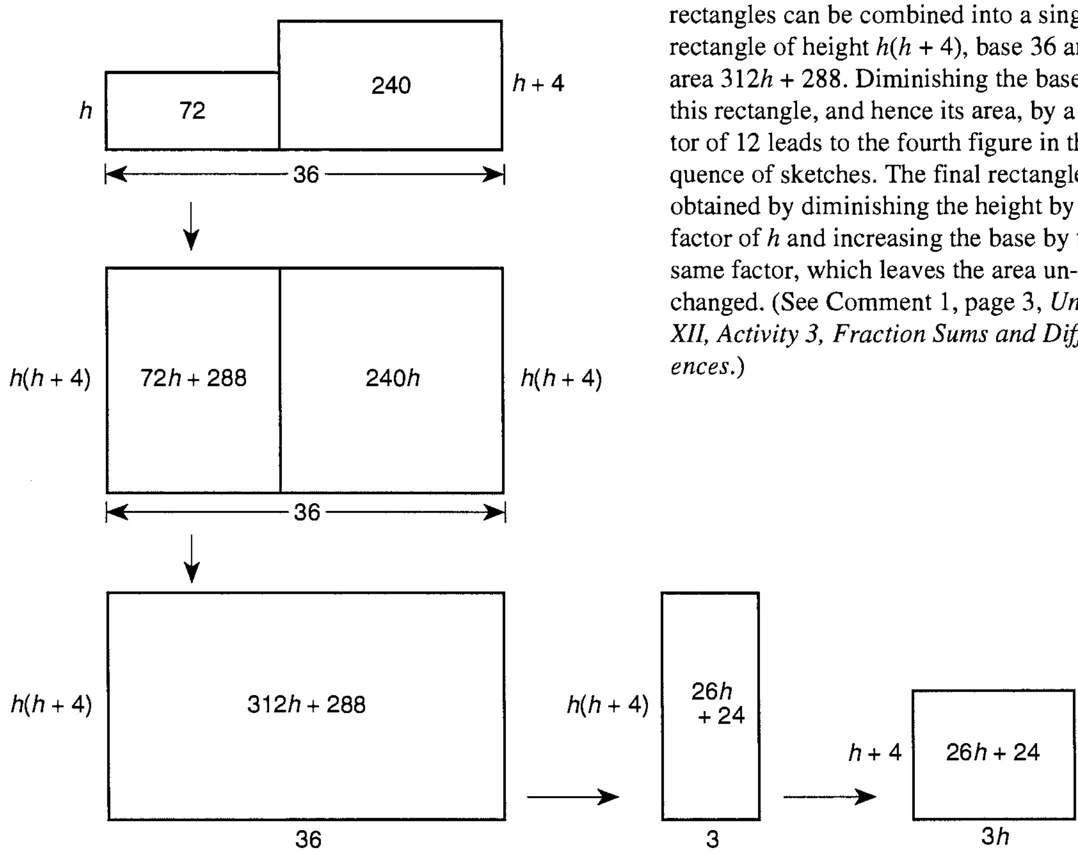
On the other hand, if the value of the base of the first rectanglular region (and hence the value of its area) is changed by a factor of -1 and the result is combined with the second region, the result is a rectangular region of height h, base $x - 3$ and area 2. Since, from above, $h = x - 2$, h can be replaced by $x - 2$ and the square completed on this region, as shown below.

$2x - 5 = \pm 3$,
$2x = 8$ or $2x = 2$,
$x = 4$ or $x = 1$.

When $x = 4$, $h = 2$ and when $x = 1$, $h = -1$.

Actions

2. Tell the students that the areas of two adjacent rectangles are 72 and 240, the sum of their bases is 36 and the difference in their heights is 4. Ask the students to find the dimensions of the rectangles.



Comments

2. In the sketch on the left, the heights of the adjacent rectangles are represented by h and $h + 4$. If the height of the first rectangle (and hence its area) is expanded by a factor of h and the height (and hence the area) of the second by a factor of $h + 4$, the resulting rectangles can be combined into a single rectangle of height $h(h + 4)$, base 36 and area $312h + 288$. Diminishing the base of this rectangle, and hence its area, by a factor of 12 leads to the fourth figure in the sequence of sketches. The final rectangle is obtained by diminishing the height by a factor of h and increasing the base by the same factor, which leaves the area unchanged. (See Comment 1, page 3, *Unit XII, Activity 3, Fraction Sums and Differences.*)

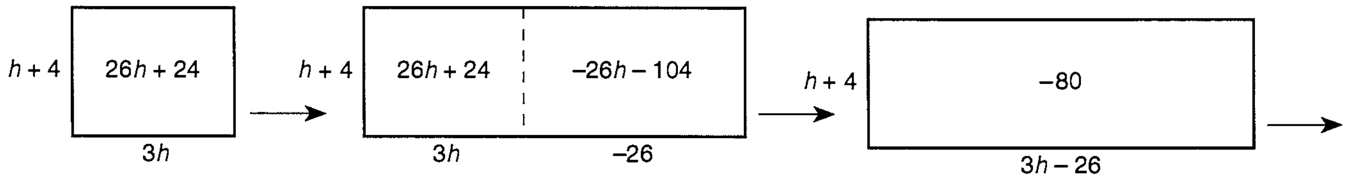
The final rectangle in the sequence above can be converted into a square, as illustrated on the next page, to find that h is 6. Thus, one rectangle is 6×12 and the other is 10×24.

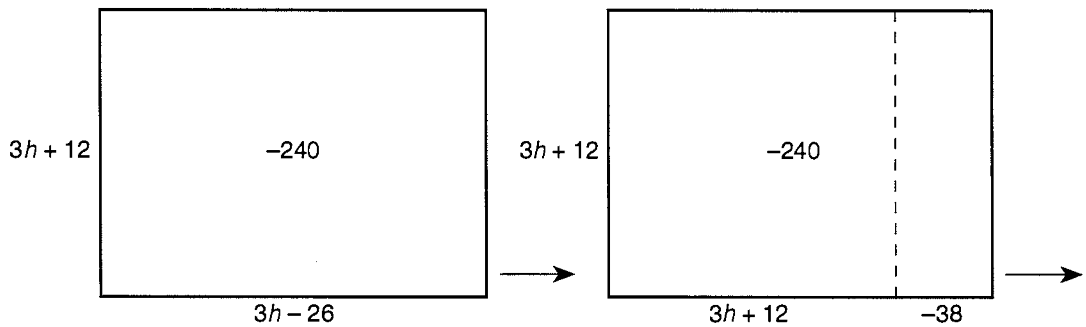
Continued next page.

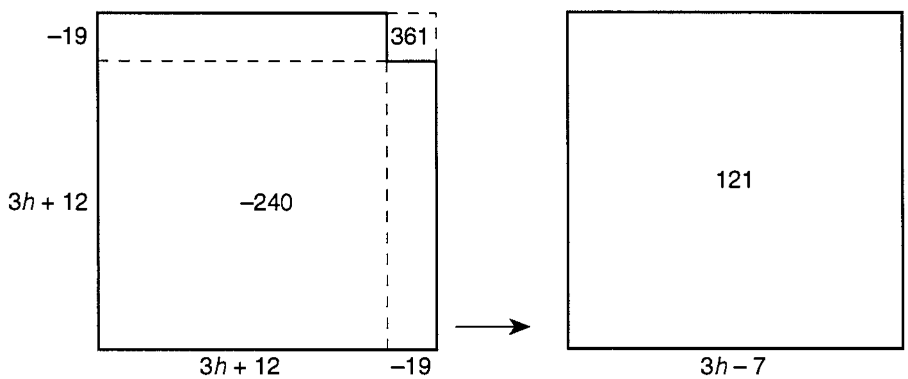
Actions

Comments

2. *Continued.*

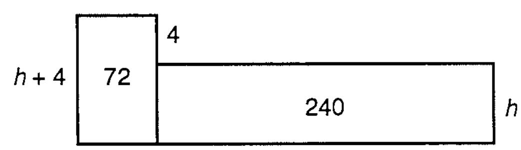






$$3h - 7 = 11,$$
$$h = 6.$$

The heights of the rectangles can be reversed, as shown in the following sketch. In this case, proceeding as above, one finds that h is 8 and the rectangles are 12×6 and 8×30.



Actions

3. Ask the students to solve the following equations:

(a) $3/2x = 5/(x-7)$

(b) $4x/(x-1) - (x+3)/x = 4$

(c) $3/(2x+1) + 1/2x = 2/(x+1)$

(d) $3/(x-4) - 2/x = 1/2$

(e) $(x+6)/12 = 3/(5-2x)$

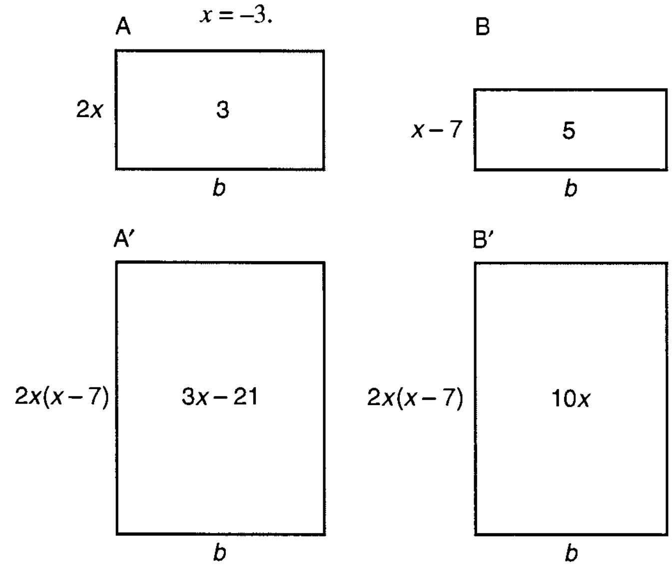
(f) $(3x-3)/(2x+4) = (4x+2)/(x-1)$

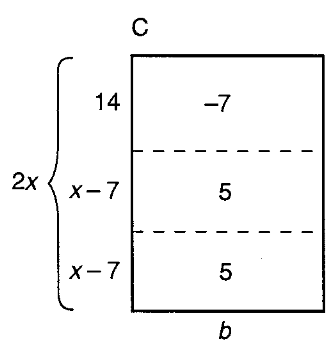
Comments

3. A master of the equations is attached.

(a) $\mathbf{3/2x = 5/(x-7)}$

The values of the bases, b, of rectangular regions A and B, below, are equal. If the value of the height of A is changed by a factor of $x-7$ and that of B by a factor of $2x$ (and the values of the regions changed accordingly), the result is two rectangular regions A' and B' whose bases and heights have equal values. Hence, the regions must have equal values. Thus $3x-21 = 10x$ and $x = -3$.





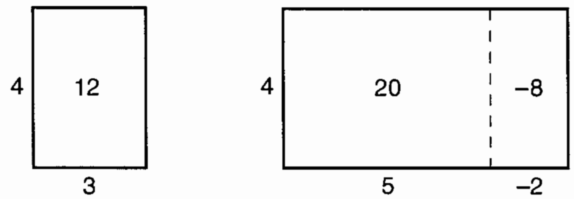
Alternatively, if the value of the height of B is doubled and then increased by 14, the result is region C. Since their edges have the same values, regions A and C have the same values. Thus the value of the top portion of C is -7. So the value of the base b is $^{-7}/_{14}$ or $-1/2$. Thus, from A, $(-1/2)2x = 3$ and $x = -3$.

Continued next page.

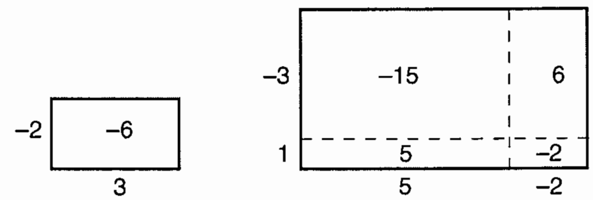
Actions

Comments

3. (a) *Continued.* You may want to discuss with the students why two rectangular regions and their corresponding edges, while differing in appearances, can have equal values. The reason is that edges can have equal amounts of red and black added to them without changing values. Some examples are given below.

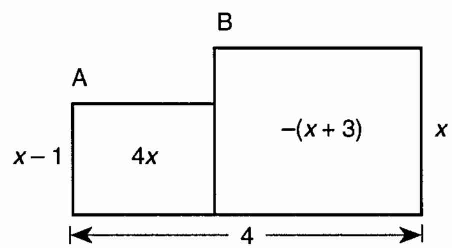


These two rectangular regions and their corresponding edges have equal values.



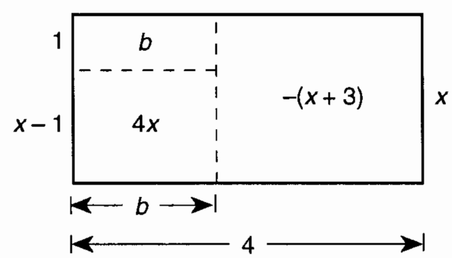
These two rectangular regions and their corresponding edges have equal values.

(b) $4x/(x-1) - (x+3)/x = 4$



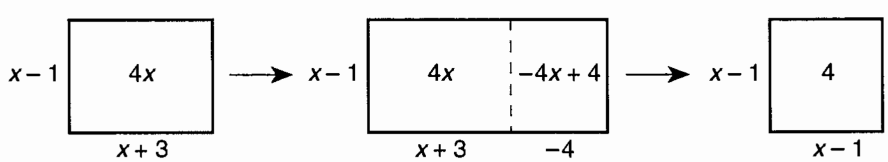
The sum of the values of the bases of the adjacent rectangular regions A and B is 4.

If the height of A is increased by 1, the regions are converted into a single rectangular region whose value is $4x$. If the value of the base of A is b, the value of the added portion is also b since its height is 1. But since the value of the entire region is $4x$, b must be $x + 3$.



Since the value of the entire region is $4x$, $b = x + 3$.

Thus, the base of A has value $x + 3$ and A can be converted to a square as shown below.



$x - 1 = \pm 2$,
$x = 3$ or $x = -1$.

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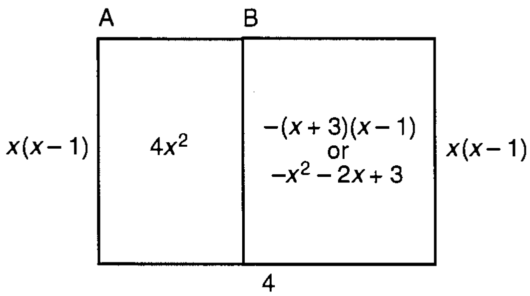
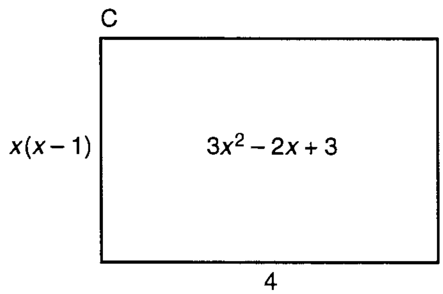
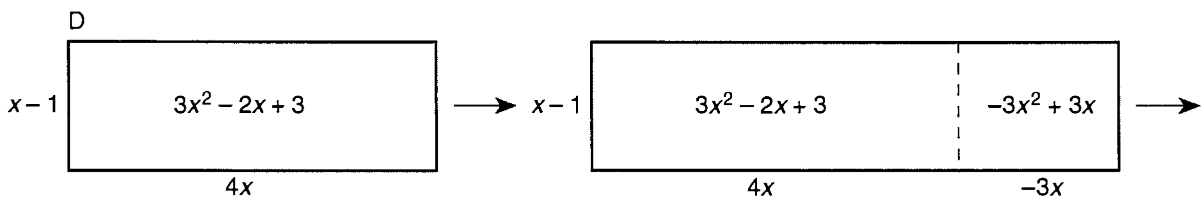
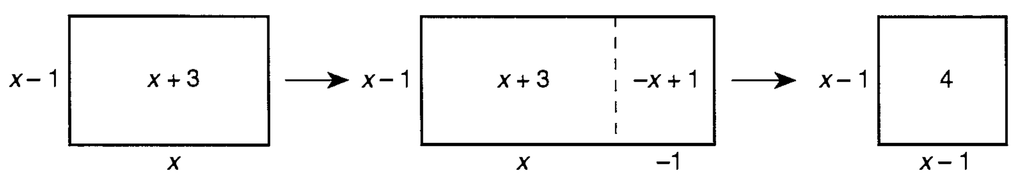
Actions









$x - 1 = \pm 2$
$x = 3$ or
$x = -1$

Comments

3. (b) *Continued.*
Alternatively, if the value of the height of A is increased by a factor of x and that of B by a factor of x, the result is region C. Decreasing the value of the height of C by a factor of x and increasing the value of its base by the same factor, results in region D, which can be converted to a square as shown.

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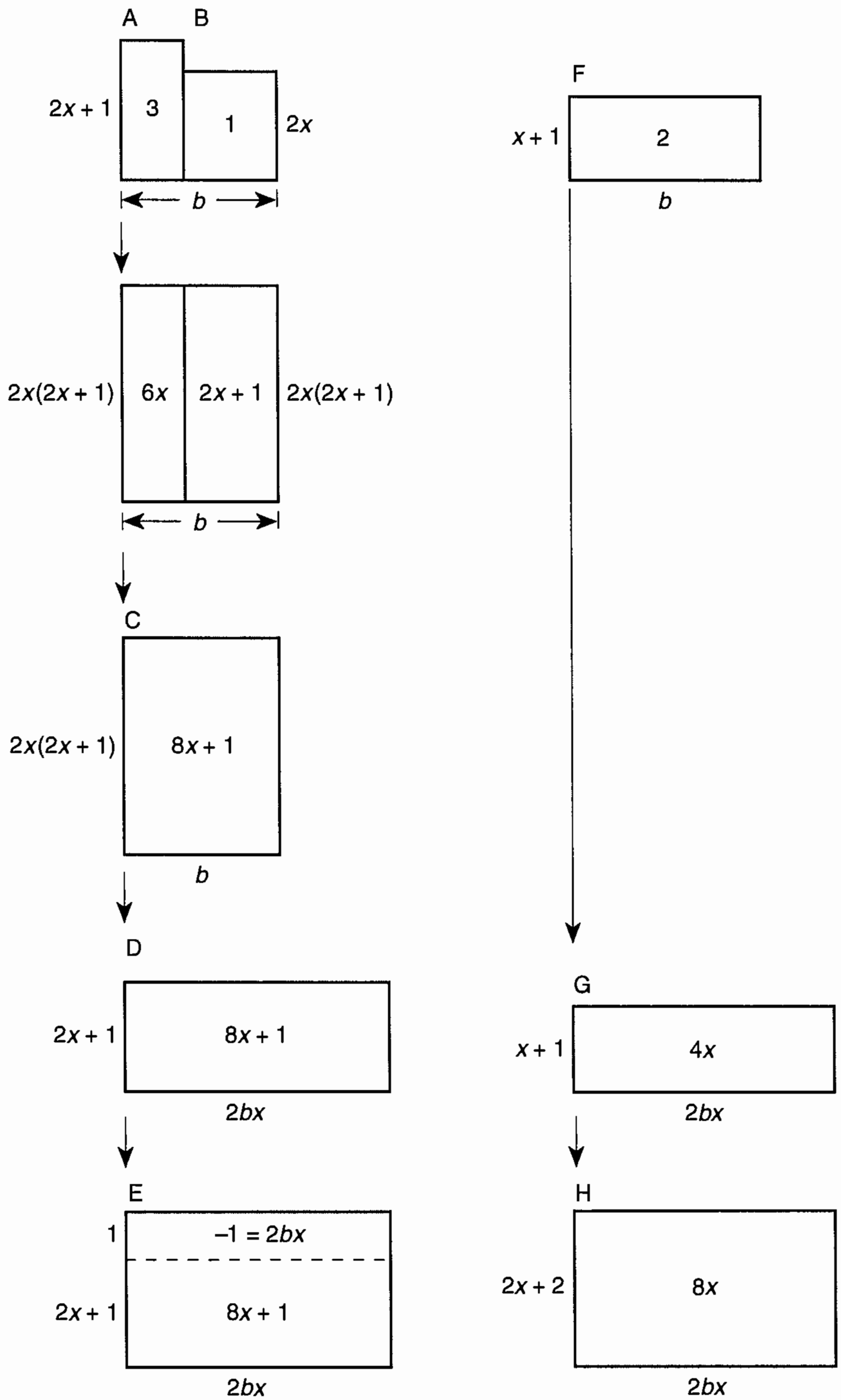
Actions



Comments

3. *Continued.*

(c) $3/(2x + 1) + 1/2x = 2/(x + 1)$

The sum of the bases of adjacent rectangular regions *A* and *B* has the same value as the base of rectangular region *F*.

First, *A* and *B* are combined to form *C*. Then *D* is obtained from *C* by diminishing the value of its height by a factor of $2x$ and increasing the value of its base by the same factor (thus leaving the value of the region unchanged). The value of the base of *D* is the same as that of *G*, which is obtained from *F* by multiplying the value of its base, and hence that of the region, by $2x$.

Now, if the value of the height of *D* is increased by 1 to obtain *E* and the value of the height of *G* is doubled to obtain *H*, then *E* and *H* have dimensions of equal value. Hence, the value of both regions is $8x$, which means the value of the added region in *E* is -1. Since the value of this region is also the product of $2bx$ and 1, it follows that $2bx = -1$.

Thus, the values of the edges of *G* are -1 and $x + 1$. Hence, $-(x + 1) = 4x$ and $x = -1/5$.

Continued next page.

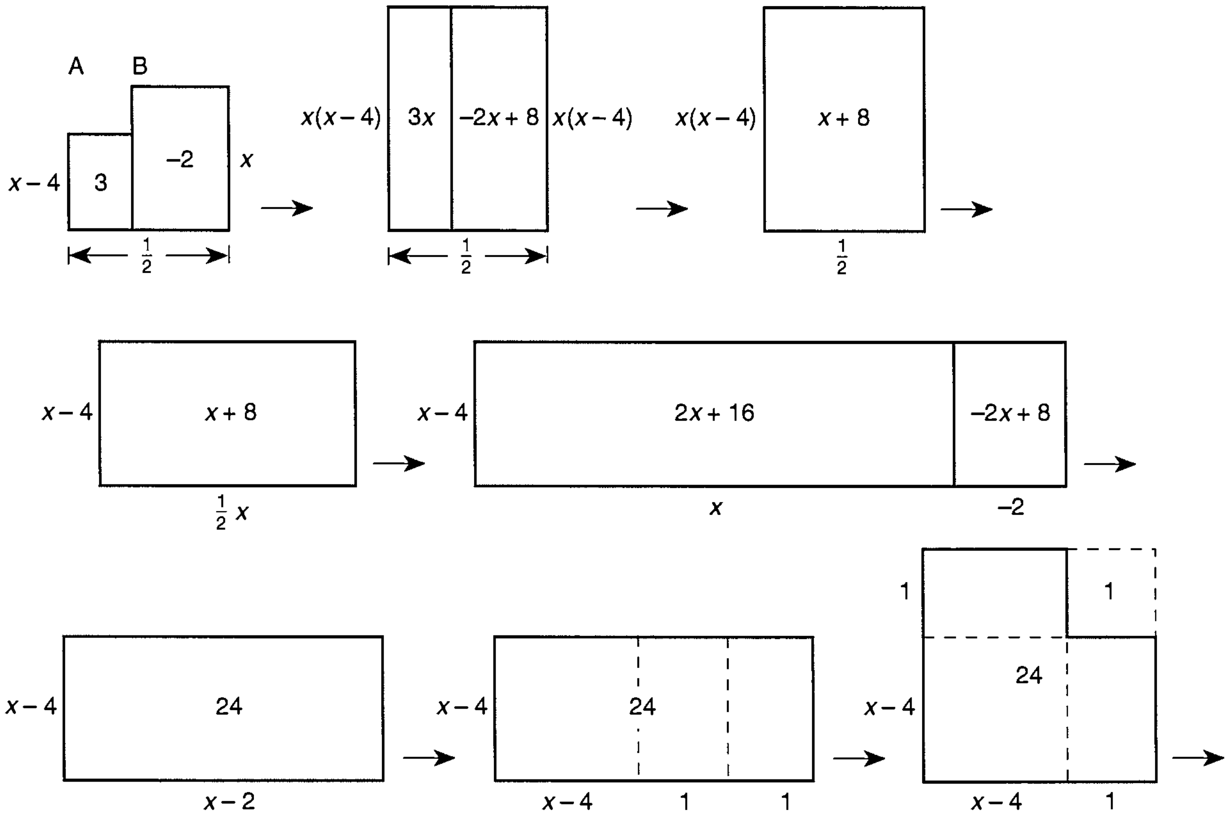
Actions

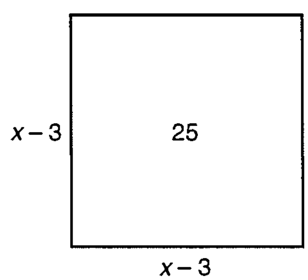
Comments

3. *Continued.*

(d) $3/(x-4) - 2/x = 1/2$

The sum of the values of the bases of regions A and B is ½. These regions can be combined and then adjusted to form a square, as shown.





$x - 3 = \pm 5$,
$x = 8$ or $x = -2$.

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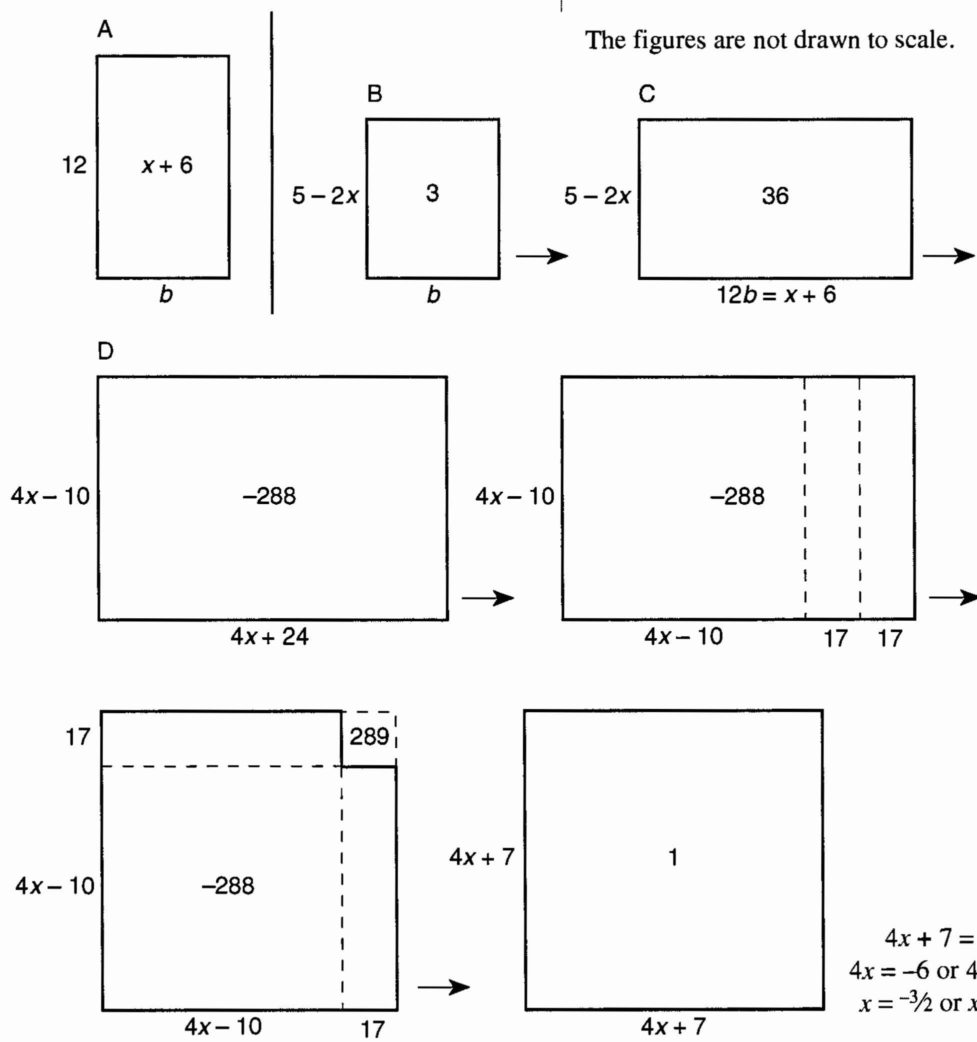
3. *Continued.*

(e) $(x + 6)/12 = 3/(5 - 2x)$

The values of the bases of regions *A* and *B* are equal. Expanding the base of *B* by a factor of 12 produces region *C*. The value of the base of *C* is $12b$, which equals the value, $x + 6$, of the region *A*.

D is obtained from *C* by multiplying the value of one edge of *D* by –2 and the other by 4, and hence the value of the region by –8. Then *D* can be converted to a square as shown.

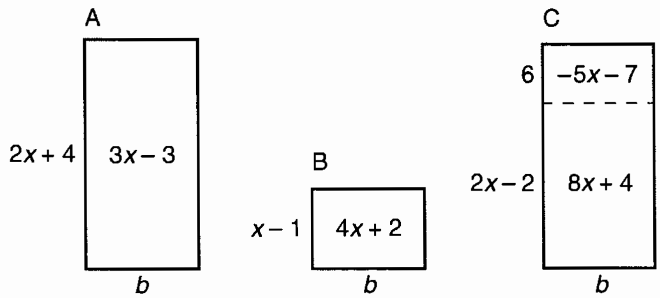
The figures are not drawn to scale.

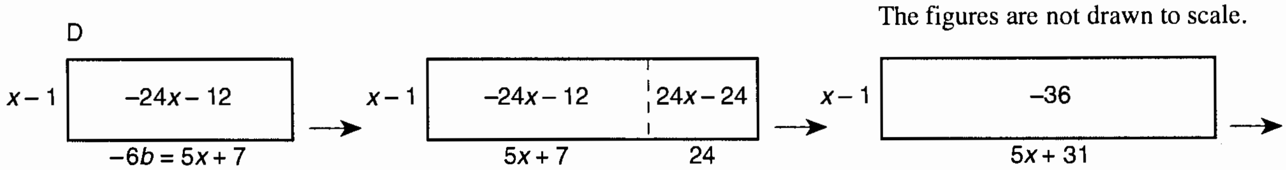


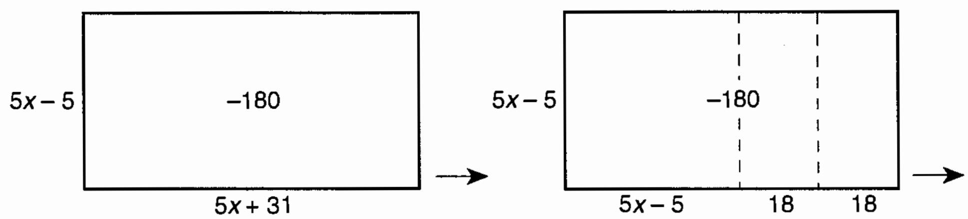
$4x + 7 = \pm 1$,
$4x = -6$ or $4x = -8$,
$x = -3/2$ or $x = -2$.

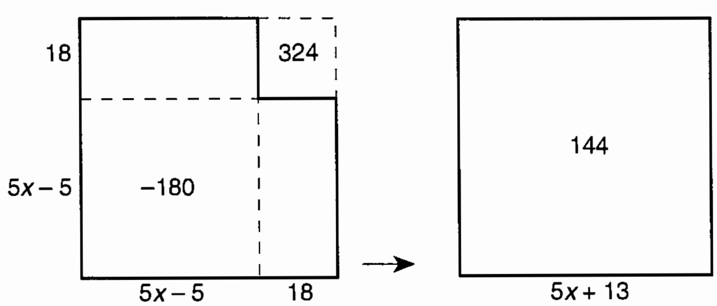
Continued next page.

Actions









$5x + 13 = \pm 12$
$5x = -1$ or $5x = -25$
$x = {}^{-1}\!/_5$ or $x = -5$

Comments

3. *Continued.*

(f) $^{(3x - 3)}\!/_{(2x + 4)} = {}^{(4x + 2)}\!/_{(x - 1)}$

The values of the bases of regions *A* and *B* are equal. If the height of *B* is doubled and then increased by 6, the result is *C*. Since the values of the edges of *A* and *C* are equal, the values of these regions must also be equal. Hence, the value of the region added in *C* is $-5x - 7$ and, since the values of the edges of the added region are 6 and *b*, it follows that $6b = -5x - 7$. Thus, if the base of *B* is multiplied by a factor of -6, the result is region *D* which can be converted to a square as shown.

The figures are not drawn to scale.

Actions

4. Ask the students to use sketches to help them solve the following puzzle problems.

(1) Motorist A travels 8 miles per hour faster than motorist B and covers 448 miles in one hour less time. Find the speed of each motorist.

(2) Twelve pounds of a mixture of nuts contains $30 worth of one kind of nut and $70 worth of a second, more expensive kind of nut. If the difference in the price of the nuts is $4 per pound, how many pounds of each kind of nut are in the mixture?

(3) A motorboat, at open throttle, takes 2 hours to travel 8 miles downstream and 4 miles back on a river which flows at the rate of 2 miles per hour. At what rate does the boat travel, at open throttle, in still water.

(4) The first of three numbers is 9 less than the second and the second is 12 less than the third. The quotient of the first divided by the second is equal to the quotient of the second divided by the third. What are the numbers?

(5) After traveling at a fixed rate of speed for 50 miles, a freight train increases its speed by 10 miles per hour and travels 100 miles farther. If the train took 3 hours to cover the 150 miles, what was its speed for the first 50 miles?

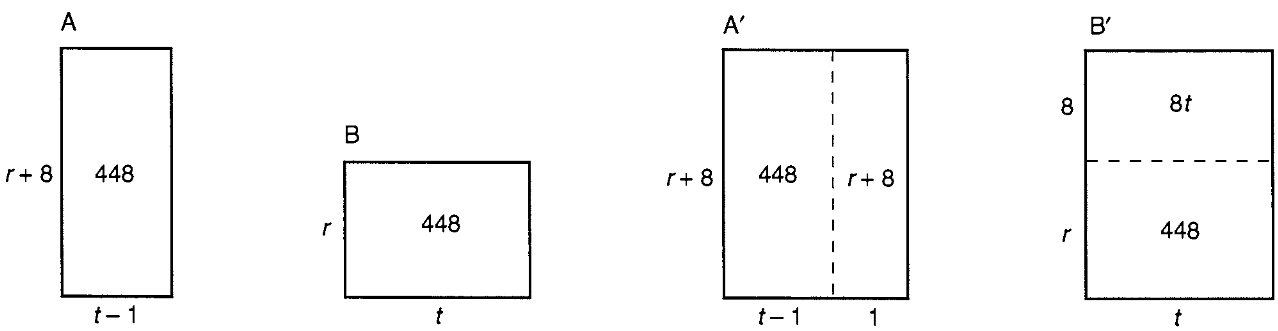
Comments

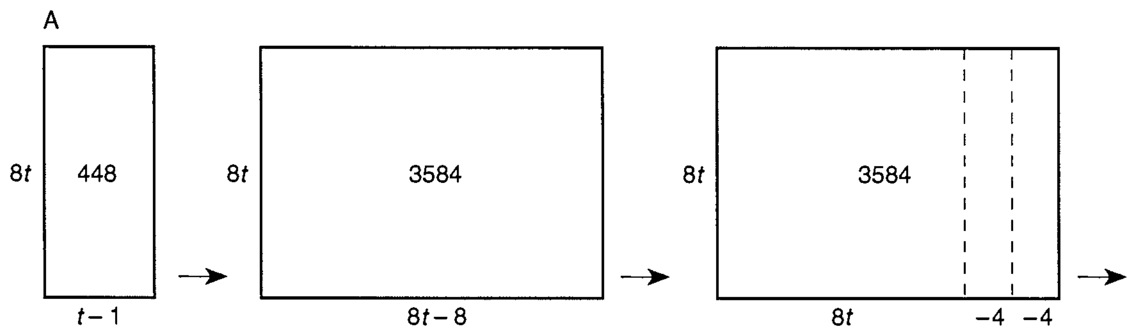
4. A set of sample solutions is attached. Generally, the sketches in the solutions are not drawn to scale.

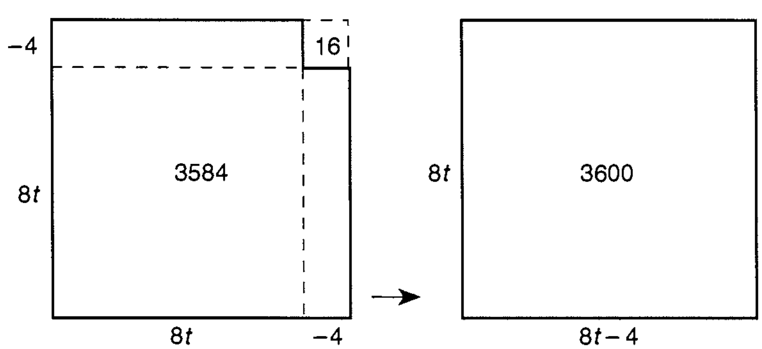
Sample Solutions to Puzzle Problems

1 Motorist A travels 8 miles per hour faster than motorist B and covers 448 miles in one hour less time. Find the speed of each motorist.

The bases of the rectangles A and B represent time and their heights represent speed. Hence, their areas represent distance traveled. Increasing the base of A by 1 and the height of B by 8 produces rectangles A' and B' which have equal dimensions and, hence, equal areas. Thus, $r + 8 = 8t$. Then, replacing the height, $r + 8$, of A by $8t$, A can be converted into a square as shown.







$8t - 4 = 60$

$8t = 64$

$t = 8$ hrs

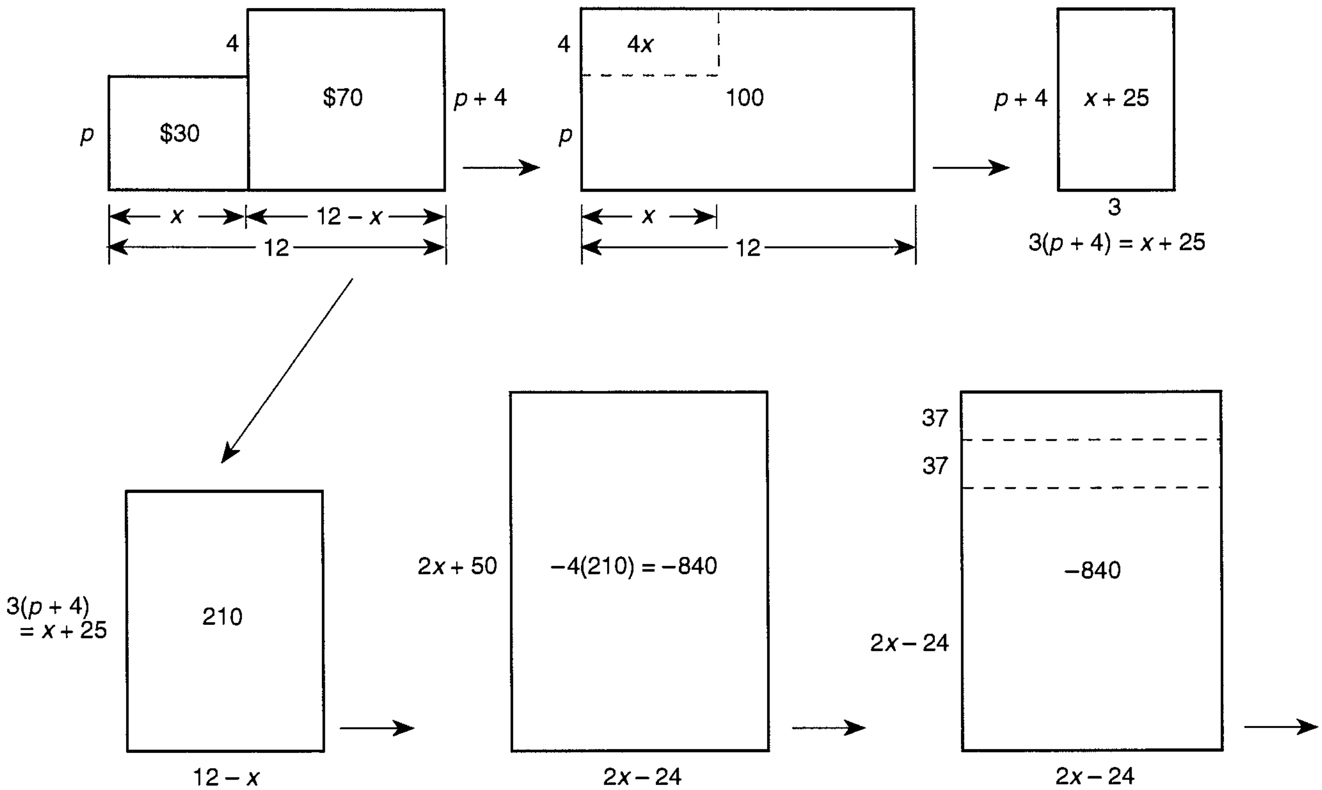
Hence, $r = {}^{448}\!/_{8} = 56$ mph;

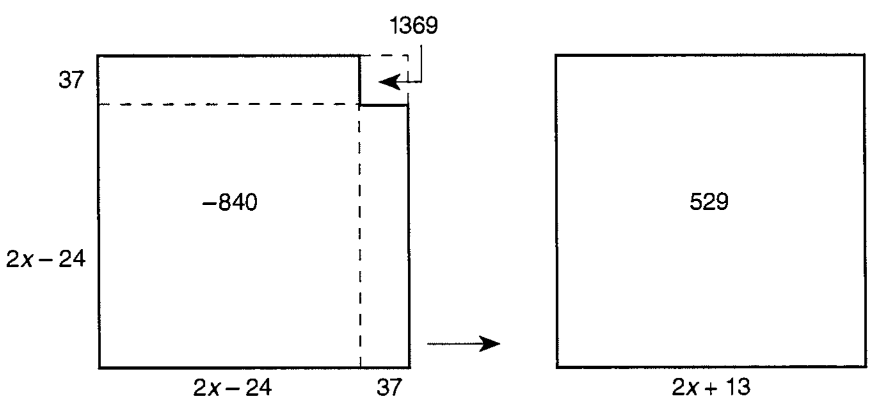
$r + 8 = 64$ mph.

Sample Solutions to Puzzle Problems
Continued

2 Twelve pounds of a mixture of nuts contains \$30 worth of one kind of nut and \$70 worth of a second, more expensive kind of nut. If the difference in the price of the nuts is \$4 per pound, how many pounds of each kind of nut are in the mixture?

The value of the nuts is represented by areas of adjacent rectangles whose heights represent prices per pound and bases represent amounts. The height of the left rectangle is increased to match that of the right, resulting in a rectangle of dimensions $(p + 4) \times 3$ and area $4x + 100$. The base of this rectangle is diminished by a factor of 4 to obtain a rectangle from whence it follows that $3(p + 4) = x + 5$. Then, increasing the height of the larger of the adjacent rectangles by a factor of 3, a rectangle is obtained that can be converted to a square as shown.





$2x + 13 = \sqrt{529} = 23$

$2x = 10$

$x = 5$

$\therefore$ 5 lbs of \$6 mixture
7 lbs of \$10 mixture

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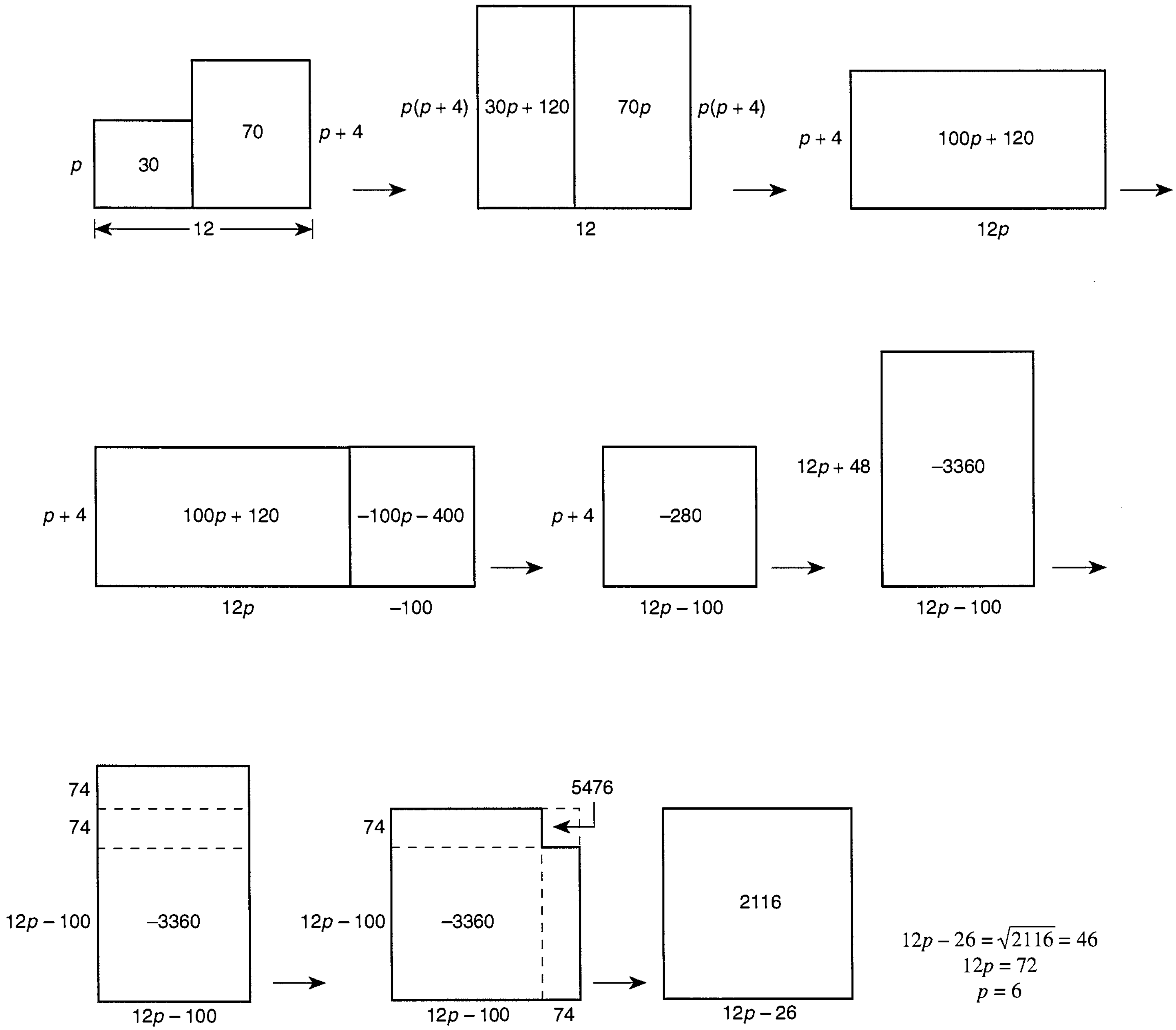
Sample Solutions to Puzzle Problems
Continued

2 (*Continued*) Alternately, the two adjacent rectangles can be converted into a single rectangle which, in turn, can be converted into a square, as shown.

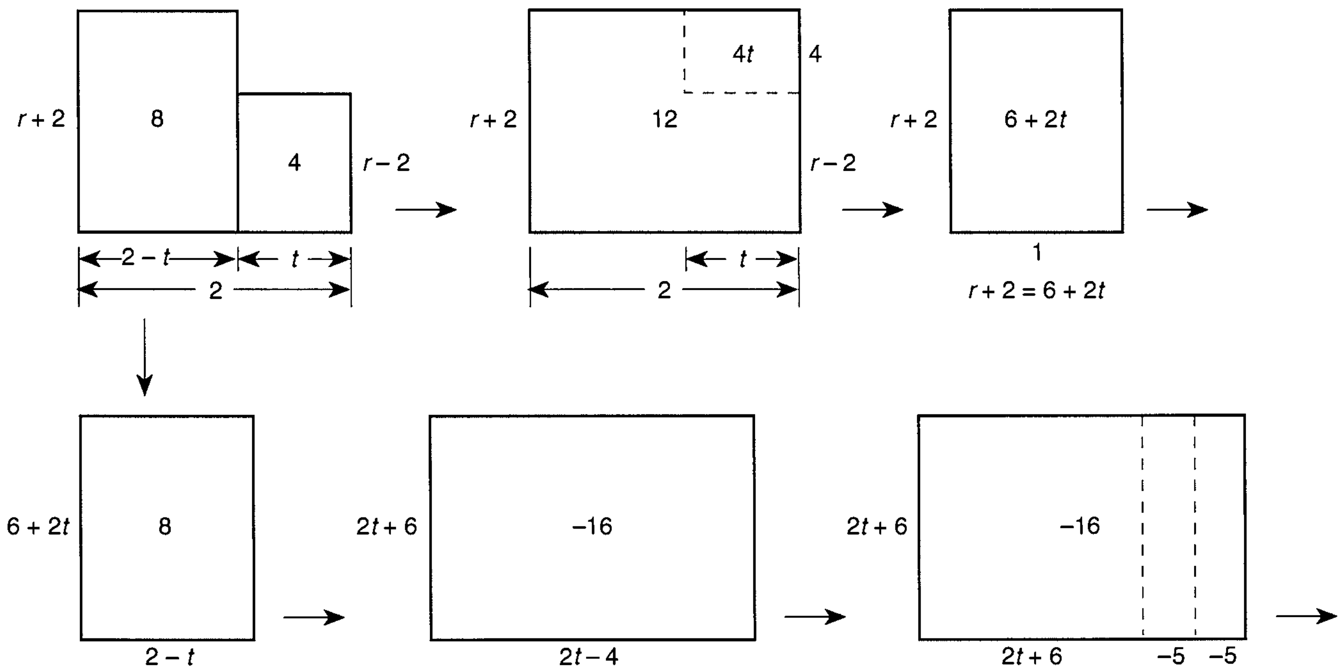


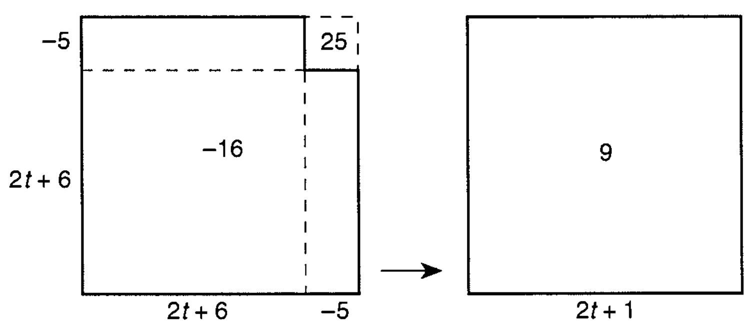
$$12p - 26 = \sqrt{2116} = 46$$
$$12p = 72$$
$$p = 6$$

∴ There are 6 pounds of the first kind and 10 pounds of the second.

3 A motorboat, at open throttle, takes 2 hours to travel 8 miles downstream and 4 miles back on a river which flows at the rate of 2 miles per hour. At what rate does the boat travel, at open throttle, in still water.

The distances traveled downstream are represented by areas of adjacent rectangles whose heights represent rates and bases represent time. (In the sketches, r is the rate of the boat in still water.) By increasing the height of the second rectangle, the adjacent rectangles are converted to a single rectangle from which it is determined that $r + 2 = 6 + 2t$. Knowing this, one can convert the first of the adjacent rectangles into a square to determine that t is 1 hour. So the rate of the boat in still water is 6 mph.

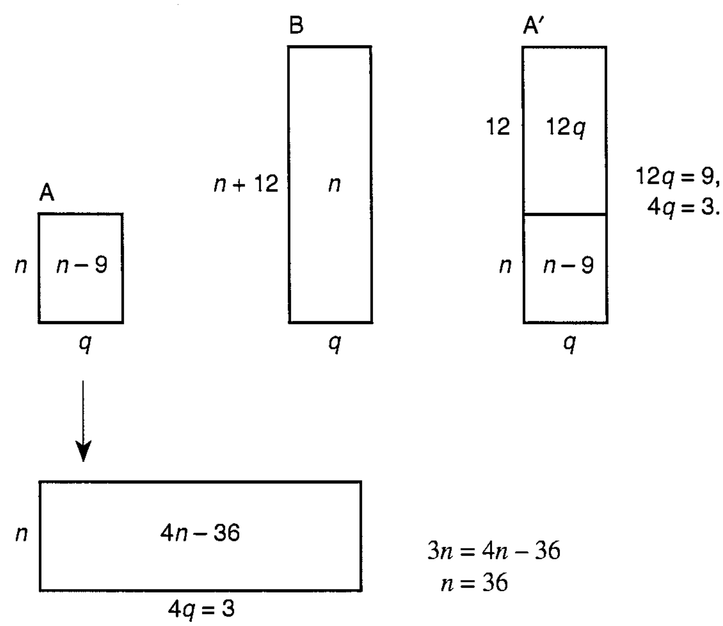




$$\begin{aligned} 2t + 1 &= \sqrt{9} = 3, \\ t &= 1 \text{ hr}; \\ r + 2 &= 8, \\ r &= 6 \text{ mph.} \end{aligned}$$

4 The first of three numbers is 9 less than the second and the second is 12 less than the third. The quotient of the first divided by the second is equal to the quotient of the second divided by the third. What are the numbers?

If the three numbers are $n - 9$, n and $n + 12$, then the bases of rectangles A and B are equal. If A' is the result of increasing the height of A by 12, then A' and B have the same area which means $12q = 9$, that is, $4q = 3$. Then, multiplying the base of A by 4 produces a rectangle whose area is $4n - 36$ and dimensions are n and 3.



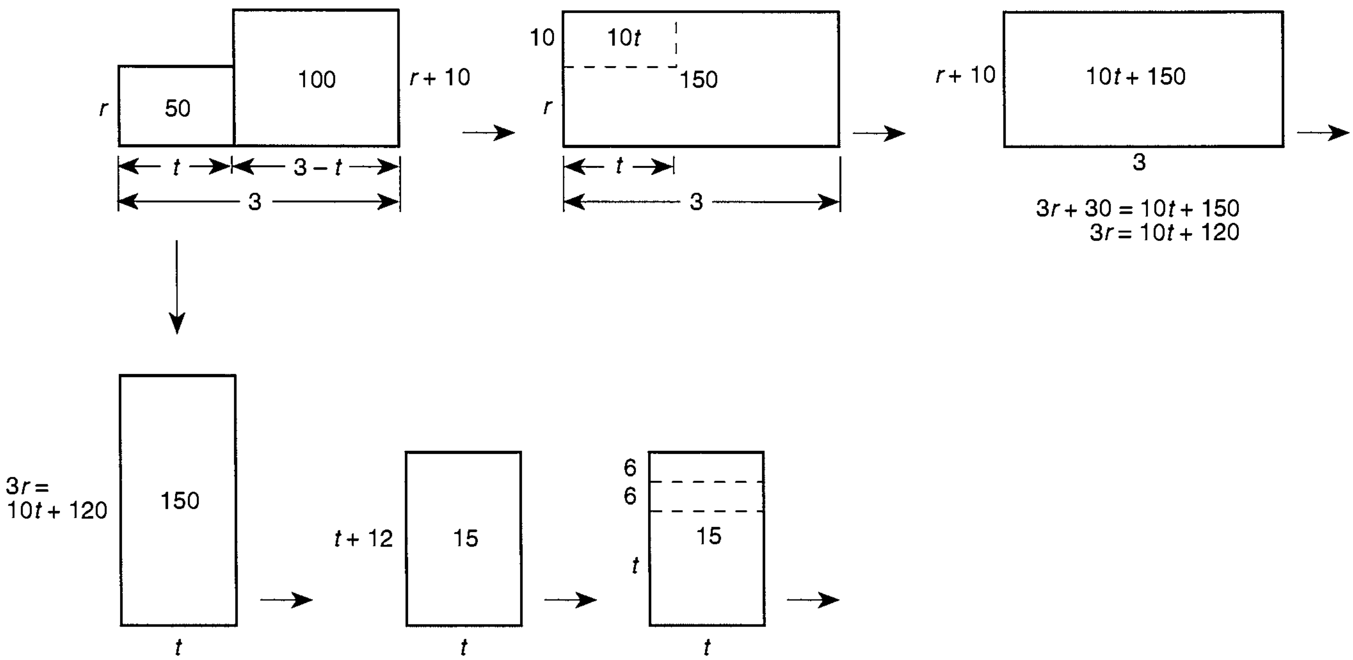
$$3n = 4n - 36$$
$$n = 36$$

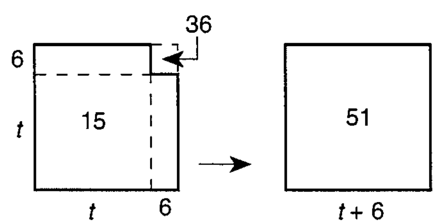
∴ The three numbers are 27, 36 and 48.

Sample Solutions to Puzzle Problems
Continued

5 After traveling at a fixed rate of speed for 50 miles, a freight train increases its speed by 10 miles per hour and travels 100 miles farther. If the train took 3 hours to cover the 150 miles, what was its speed for the first 50 miles?

The distances traveled at the two different speeds are represented by areas of adjacent rectangles whose heights represent rates and bases represent times. Converting the adjacent rectangles into a single rectangle, as indicated, shows that $3r = 10t + 120$. Using this fact, the first of the adjacent rectangles, after its height is increased by a factor of 3, can be converted into a square.





$$t + 6 = \sqrt{51},$$
$$t = \sqrt{51} - 6;$$

$$\therefore\ 3r = 10t + 120$$
$$= 10\sqrt{51} - 60 + 120;$$
$$r = {}^{10\sqrt{51}}\!/_{3} + 20,$$
$$= 43.8 \text{ mph.}$$

Action 3

a $\frac{3}{2x} = \frac{5}{(x-7)}$

b $\frac{4x}{(x-1)} - \frac{(x+3)}{x} = 4$

c $\frac{3}{(2x+1)} + \frac{1}{2x} = \frac{2}{(x+1)}$

d $\frac{3}{(x-4)} - \frac{2}{x} = \frac{1}{2}$

e $\frac{(x+6)}{12} = \frac{3}{(5-2x)}$

f $\frac{(3x-3)}{(2x+4)} = \frac{(4x+2)}{(x-1)}$

Action 4

1 Motorist A travels 8 miles per hour faster than motorist B and covers 448 miles in one hour less time. Find the speed of each motorist.

2 Twelve pounds of a mixture of nuts contains $30 worth of one kind of nut and $70 worth of a second, more expensive kind of nut. If the difference in the price of the nuts is $4 per pound, how many pounds of each kind of nut are in the mixture?

3 A motorboat, at open throttle, takes 2 hours to travel 8 miles downstream and 4 miles back on a river which flows at the rate of 2 miles per hour. At what rate does the boat travel, at open throttle, in still water.

4 The first of three numbers is 9 less than the second and the second is 12 less than the third. The quotient of the first divided by the second is equal to the quotient of the second divided by the third. What are the numbers?

5 After traveling at a fixed rate of speed for 50 miles, a freight train increases its speed by 10 miles per hour and travels 100 miles farther. If the train took 3 hours to cover the 150 miles, what was its speed for the first 50 miles?

Irrational Roots

OVERVIEW

The irrationality of $\sqrt{3}$ is established. The method is extended to other roots.

Prerequisite Activity

Unit XII, Activity III, *Fraction Sums and Differences,* especially Action 1.

Materials

None.

Actions

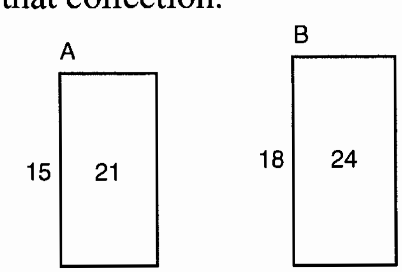
1. Tell the students a *type R* rectangle is a rectangle whose height and area are positive integers. Ask the students to sketch two examples of rectangles that have a base of $5/2$ and (a) are type R, (b) are not type R.

2. Sketch for the students the two type R rectangles A and B shown below. Tell the students that two type R rectangles are *equivalent* if their bases are equal. Then have them consider the following two collections of rectangles:

Collection 1. All type R rectangles equivalent to A.

Collection 2. All type R rectangles whose heights are between 40 and 50 and are equivalent to B.

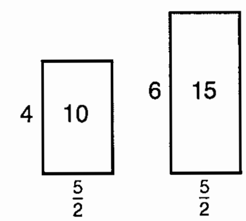
For each collection, ask the students to list the heights of all the rectangles in that collection.



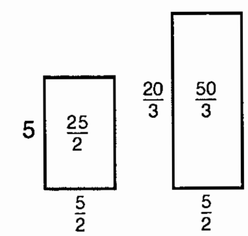
Comments

1. Here are some examples.

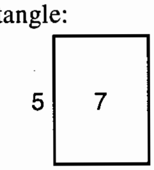
These rectangles are type R:



These rectangles are not:

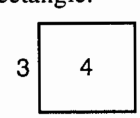


2. Since $21/15 = 7/5$, A is equivalent to the following rectangle:



Other rectangles in Collection 1 have heights which are multiples of the height of this rectangle. Thus the list of heights is 5, 10, 15,....

Rectangles in Collection 2 are equivalent to the following rectangle:



Only three have heights between 40 and 50, namely those of height 42, 45 and 48.

Actions

Comments

3. Point out to the students that each of the collections in Action 2 contains a *minimal* rectangle, i.e., a rectangle whose height is less than that of any other rectangle in the collection, while only Collection 2 contains a *maximal* rectangle, i.e., a rectangle whose height is greater then that of any other rectangle in the collection. Discuss with the students why—though it is not the case for a collection of rectangles in general—every collection of type R rectangles must contain a minimal rectangle.

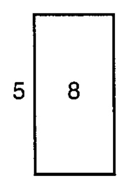
3. The heights of the rectangles in a collection of type R is a set of positive integers. Since every set of positive integers contains a smallest one, the rectangle whose height is this smallest integer will be the minimal rectangle for the collection.

In general, not every collection of rectangles will contain a rectangle of minimal height. For example, the collection of all rectangles whose base is 10 and height is greater than 4 does not contain a rectangle of smallest height—for if A is a rectangle in this collection and its height is h, and t is any number between h and 4 (e. g., the average of h and 4), then the rectangle whose base is 1 and height is t is in the collection and has a smaller height than A. So for any rectangle in the collection, there is one of smaller height. Hence, there is none whose height is smallest.

4. Discuss the relationship between type R rectangles and rational numbers.

4. The base of a type R rectangle is a rational number, i.e., the quotient of two integers, and conversely, every positive rational number is the base of a type R rectangle (actually, of many type R rectangles).

If r is a positive rational number, the minimal rectangle in the collection of all type R rectangles with base r is the rectangle whose height is the smallest denominator that occurs when r is written as the quotient of positive integers. Thus, for example, if r is $8/5$, the minimal rectangle for the collection of all type R rectangles whose base is $8/5$ is the one shown.

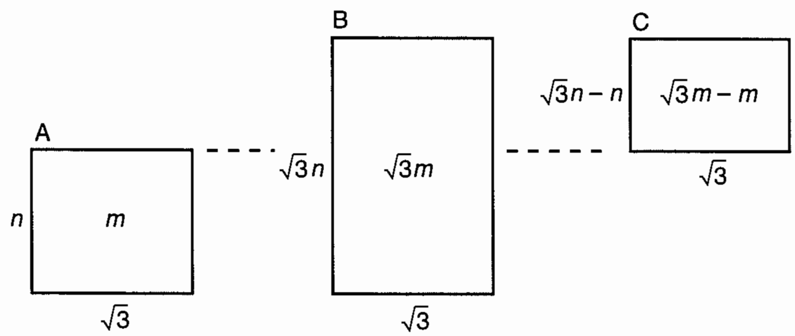


Minimal type R rectangle whose base is $8/5$.

If a number t is rational, then the collection of all type R rectangles whose base is t contains a minimal rectangle. Thus, one way to show that a number t is irrational—and the method used in Actions 5 and 6 to show $\sqrt{3}$ is irrational—is to show that if it were rational, there would be no minimal rectangle in the collection of type R rectangles whose base is t, which is an impossible situation. The only alternative is that t is irrational. Briefly, the form of the argument is this: Either t is rational or t is irrational. The former alternative leads to an impossible situation, hence the latter alternative must be the case.

Actions

5. Show the students the following rectangles.



Tell them that rectangle B is obtained from A by expanding its height by a factor of $\sqrt{3}$ and that rectangle C is obtained by cutting off from B a rectangle congruent to A. Ask the students to show that if A is type R, so is C and, further, the height of C is less than the height of A.

6. Discuss with the students why Action 5 implies that $\sqrt{3}$ is irrational.

7. Ask the students to adapt Actions 5 and 6 to show $\sqrt{2}$ is irrational.

Comments

5. From A, $\sqrt{3}n = m$. Hence, the height of $C = \sqrt{3}n - n = m - n$ and the area of C $= \sqrt{3}m - m = \sqrt{3}(\sqrt{3}n) - m = 3n - m$. If the height, n, and area, m, of A are positive integers then the height, $m - n$, and the area, $3n - m$, of C are integers. Furthermore, they are positive since C has positive height and area. Thus, if A is type R, so is C.

Because $\sqrt{3}$ is between 1 and 2, $\sqrt{3} - 1 < 1$. Hence, the height of C is less than that of A since $\sqrt{3}n - n = (\sqrt{3} - 1)n < n$.

6. If $\sqrt{3}$ were rational, then there would exist type R rectangles with base $\sqrt{3}$ and the collection of all such would contain a minimal rectangle. However, Action 5 shows this is impossible; that is, Action 5 shows that no type R rectangles with base $\sqrt{3}$ could be minimal—if one existed, there would always be another of shorter height. Hence, the assumption that $\sqrt{3}$ is rational leads to an impossible situation. The only alternative is that $\sqrt{3}$ is not rational, i.e., it is irrational.

Note that this implies that the decimal representation of $\sqrt{3}$ is non-repeating and non-terminating. (See Comment 9, Unit XII, Activity 2, *Decimals & Fractions*.)

7. In Actions 5 and 6, replace 3 by 2.