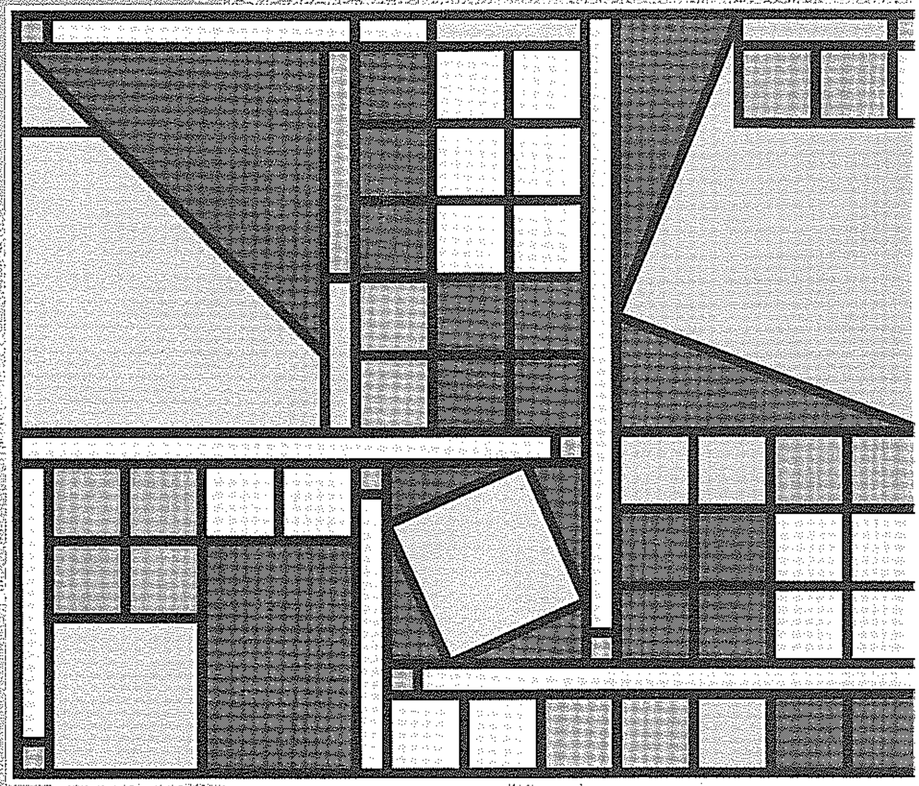
Unit XII / Math and the Mind's Eye Activities



Modeling Real & Complex Numbers

Eugene Maier
& L. Ted Nelson

Modeling Real & Complex Numbers

1 **Heximals & Fractions**
The relationship between heximals and fractions is investigated with the help of base six pieces.

2 **Decimals & Fractions**
The relationship between decimals and fractions is investigated.

3 **Fraction Sums & Differences**
Procedures are developed for finding the sums and differences of algebraic fractions, based on area properties of rectangles.

4 **Fraction Products & Quotients**
Procedures are developed for finding the products and quotients of algebraic fractions, based on area properties of rectangles.

5 **Squares & Square Roots**
Methods of constructing squares of integral area are introduced. Properties of squares and square roots, including the Pythagorean Theorem, are developed.

6 **Complex Numbers**
Green and yellow bicolored counting pieces are used to introduce complex numbers and their arithmetical operations.

Math and the Mind's Eye materials are intended for use in grades 4-9. They are written so teachers can adapt them to fit student backgrounds and grade levels. A single activity can be extended over several days or used in part.

A catalog of Math and the Mind's Eye materials and teaching supplies is available from The Math Learning Center, PO Box 3226, Salem, OR 97302, 503-370-8130. Fax: 503-370-7961.

Math and the Mind's Eye

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Heximals & Fractions

OVERVIEW

The relationship between heximals and fractions is investigated with the help of base six pieces.

Prerequisite Activity

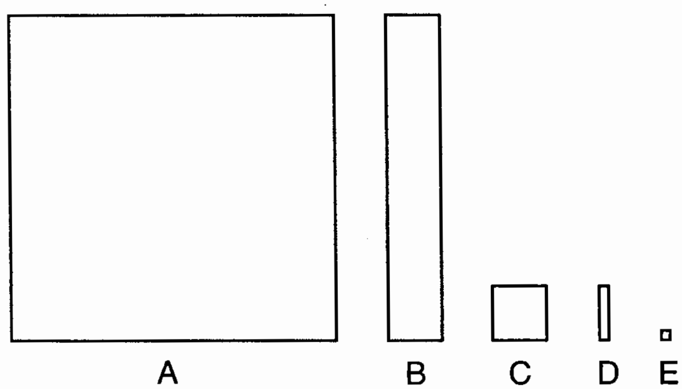
Unit IV, Activity 6, *Introduction to Decimals.*

Materials

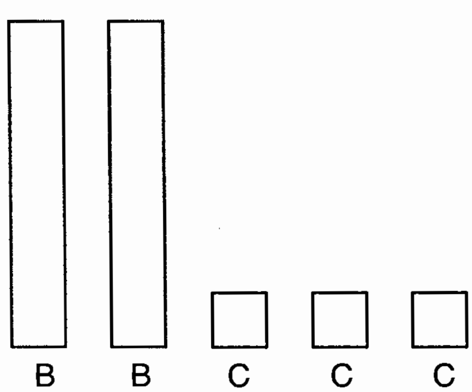
Base six number pieces (see Comment 1).

Actions

1. Distribute base six pieces to each student or group of students. Refer to the pieces, from largest to smallest, by letters A through E as shown below. If piece A has value 1, ask the students to find the values of the other pieces.



2. Have the students form a collection of 2 B and 3 C pieces. Discuss methods for finding the value of this collection. Ask the students to write a base six numeral which represents the value of this collection. Introduce the term *heximal.*



Comments

1. Cardstock base six pieces can be made using attached Masters 1, 2 and 3. One copy of each master is sufficient for each student or group of students.

If the value of piece A is 1, the values of the other pieces are:

B $1/6$

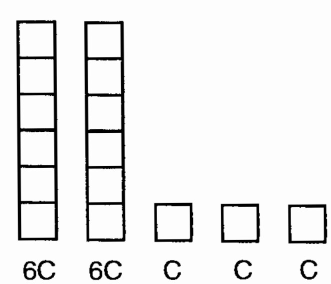
C $1/6^2$ or $1/36$

D $1/6^3$ or $1/216$

E $1/6^4$ or $1/1296$

2. One way to find the value of the collection is to add up the values of the individual pieces. The value of the 2 B pieces is $2/6$ and the value of the 3 C pieces is $3/36$. Hence, the value of the collection is $2/6 + 3/36$.

Another way is to convert the B pieces to C pieces (see below) resulting in a collection of 15 C pieces, each of value $1/36$. Hence, the value of the collection is $15/36$.



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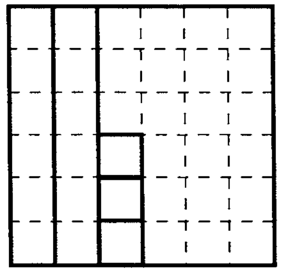
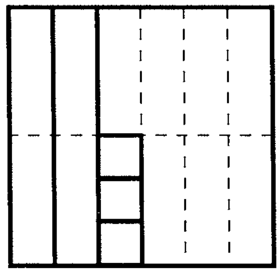
2. *Continued.* Another method for determining the value of the collection is to determine what fraction of a unit piece A the collection covers. If an A piece is subdivided into regions congruent to a C piece, the collection covers 15 of these regions as shown in the figure on the left. Hence, the value of the collection is $^{15}/_{36}$ of the value of an A piece, that is, the collection's value is $^{15}/_{36}$ of 1 or, simply, $^{15}/_{36}$.



A



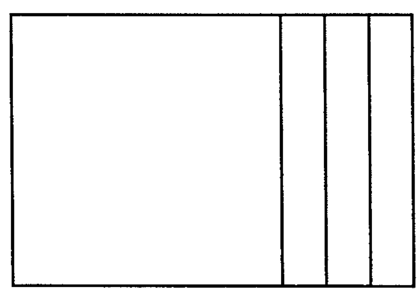
A

Alternatively, one can divide a unit piece A into 12 regions, each congruent to 3 C regions. The collection covers 5 of these regions. Hence, its value is $^5/_{12}$.

As is done in Unit IV, Activity 6, *Introduction to Decimals*, a dot or *point* is used to locate the position of the units place when writing numerals involving fractional base pieces. The point is placed after the number of units. Thus, the numeral for the collection under discussion which consists of 0 unit pieces, 2 one-sixth pieces and 3 one-thirty-sixth pieces is $(0.23)_{six}$. A base six numeral involving a point is called a heximal rather than a decimal. The prefix, *hexi-*, means a sixth part. The prefix, *deci-*, means a tenth part.

3. Ask the students to form a collection whose value is $^3/_2$. Then have them write $^3/_2$ in heximal form. Repeat this action for $^3/_4$. Discuss the methods the students use.

3. A collection consisting of an A piece and 3 B pieces has value $^3/_2$.



A B B B

$$\text{Value} = 1 + 3\left(\tfrac{1}{6}\right) = \tfrac{3}{2}$$

Hence, $^3/_2 = (1.3)_{six}$.

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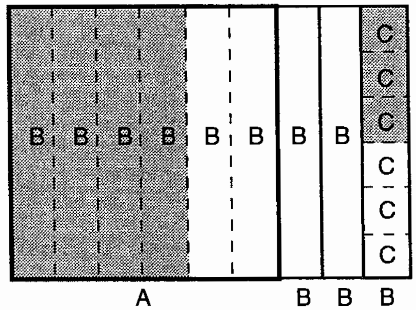
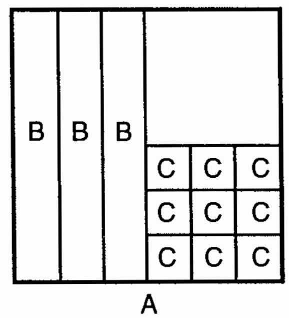
Actions

Comments



3. *Continued.* The students may find the heximal for ¾ in a number of different ways. One way is to note that ¾ is half of $3/2$. By trading pieces, the above collection for $3/2$ can be divided in half to obtain a collection of 4 B pieces and 3 C pieces. Hence, $3/4 = (0.43)_{six}$.

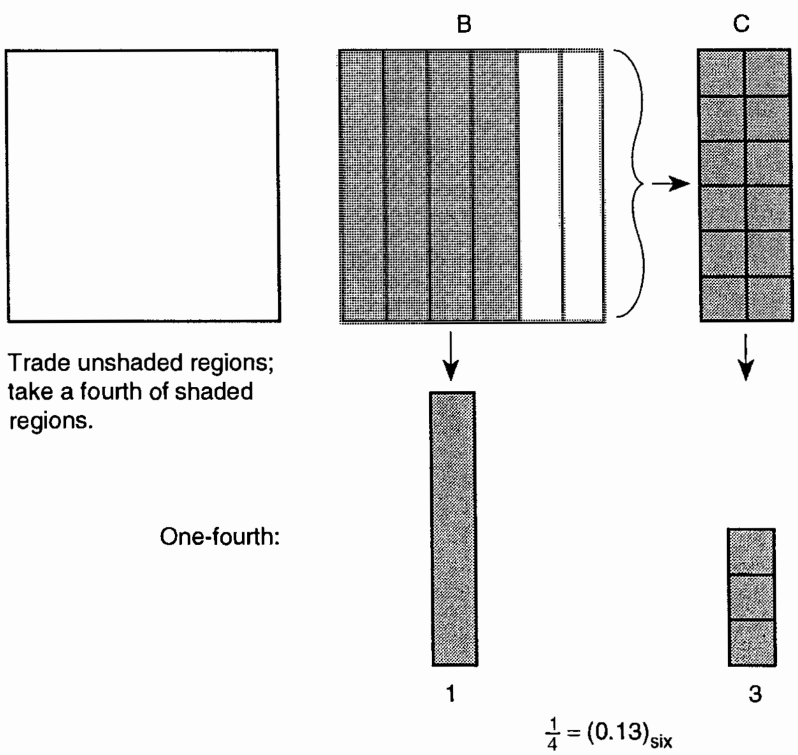


Another method is to observe that a collection of 3 B pieces and 9 C pieces covers three-fourths of a unit piece A. Trading 6 C pieces for 1 B piece results in a minimal collection of 4 B pieces and 3 C pieces.

4. Ask the students to write heximals for the following fractions: ½, ⅓, ¼, ⅕, ⅙, ⅐, ⅛ and ⅑.

4. The value of 3 B pieces is ½ and the value of 2 B pieces is ⅓, hence $1/2 = (0.3)_{six}$ and $1/3 = (0.2)_{six}$.

One-fourth of a unit piece A can be covered by a collection 9 C pieces. Trading pieces provides a minimal collection of 1 B piece and 3 C pieces whose value is ¼. Hence, $1/4 = (0.13)_{six}$.



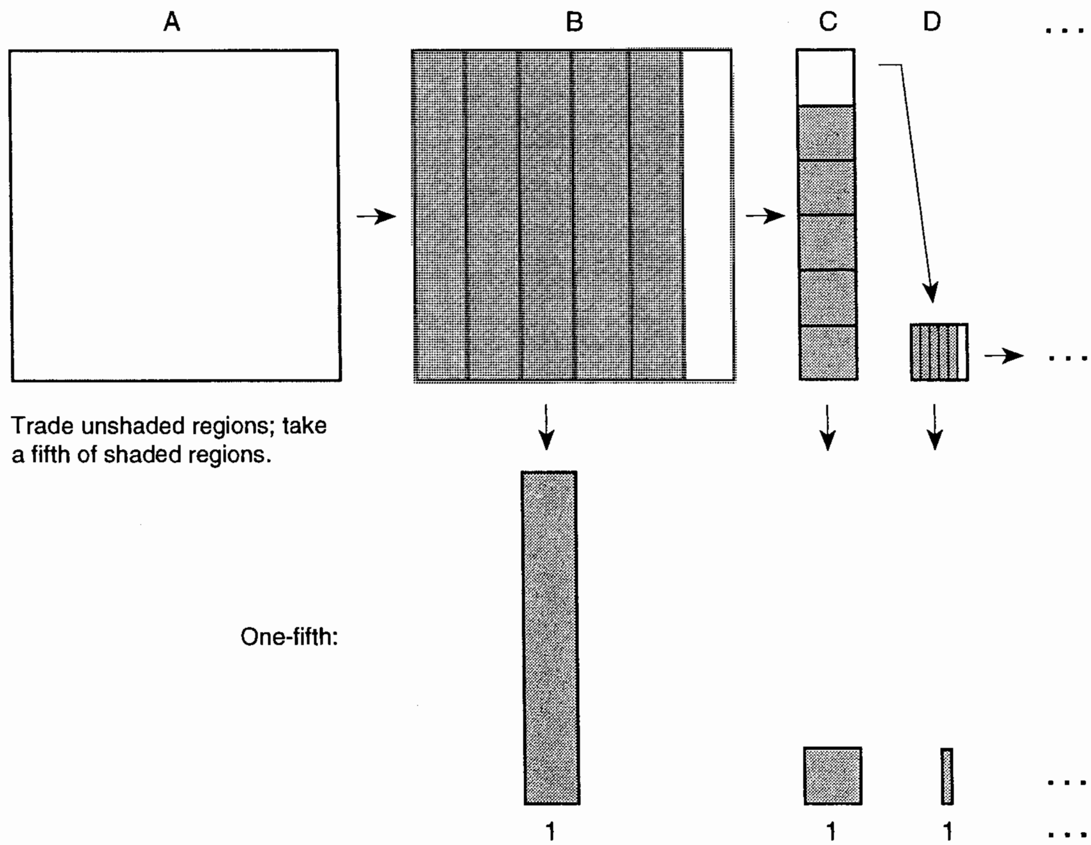
Alternatively, one can obtain a collection which is ¼ of a unit by beginning with an A piece and making trades as necessary: Trade the A piece for 6 B pieces. A fourth of 6 B pieces is 1 B piece with 2 B pieces left over. Trade the 2 B pieces for 12 C pieces. A fourth of 12 C pieces is 3 C pieces. Hence, a fourth of the A piece is equivalent to 1 B piece and 3 C pieces.

Continued next page.

Actions

Comments

4. *Continued.* The process just described, to find a collection whose value is ¼, can be used to find a collection whose value is ⅕. The process, in this case, goes on indefinitely. To start with, a unit piece A is traded for 6 B pieces. A fifth of these pieces is 1 piece with a remainder of 1 piece. This is traded for 6 C pieces and the process repeats. At each stage, a remainder of one piece is left which is traded for 6 of the next smaller piece. A fifth of these pieces is again 1 piece with a remainder of 1 piece.



$\frac{1}{5} = (0.111\ldots)_{six}$

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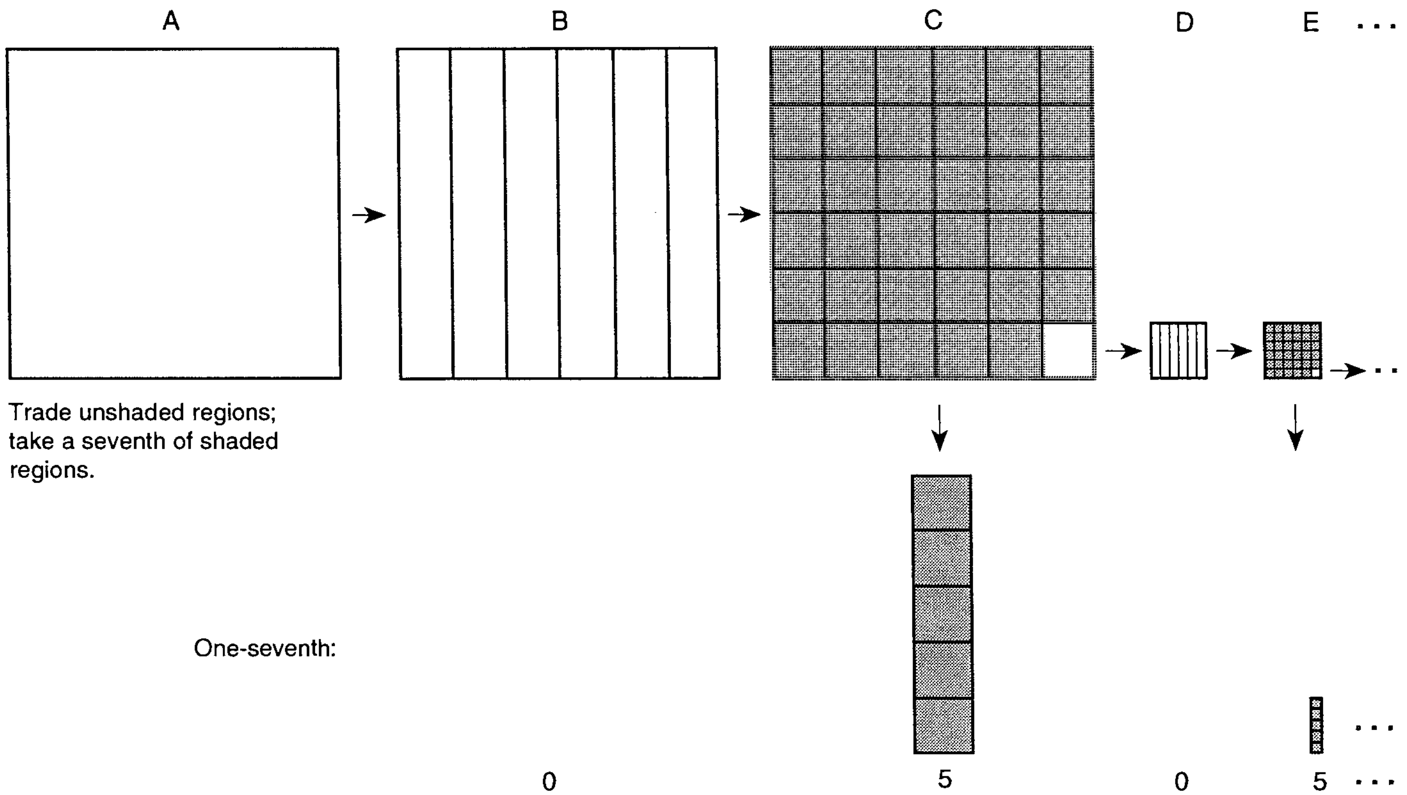
Actions

Comments

4. *Continued.* Thus, a minimal collection for ⅙ consists of 1 B piece, 1 C piece, 1 D piece, and so on indefinitely, that is, ⅕ = $(0.1111\ldots)_{six}$. The set of three dots, called an *ellipsis,* indicates the established pattern continues indefinitely. A heximal which ends with a collection of digits which repeats indefinitely is called a *repeating heximal.*

Since a collection of 1 B piece has value ⅙, ⅙ = $(0.1)_{six}$.

The heximal for ⅐ is repeating: Begin with an A piece and trade it for 6 B pieces. A seventh of these is 0 B pieces with a remainder of 6. Trading these produces 36 C pieces. A seventh of these is 5 C pieces with 1 piece left, and so forth, as indicated in the sketch.



$\frac{1}{7} = (0.0505\ldots)_{six}$

Thus, a minimal collection for ⅐ contains 0 B pieces, 5 C pieces, 0 D pieces, 5 E pieces, and so forth. Thus, ⅐ = $(0.050505\ldots)_{six}$.

Continued next page.

Actions

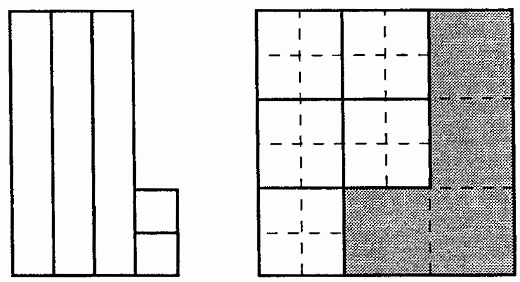
5. Ask the students to find a base ten fraction which has the same value as the heximal $(0.32)_{six}$. Repeat for the following heximals:

(a) $(0.201)_{six}$ (b) $(1.32)_{six}$
(c) $(0.5)_{six}$ (d) $(0.55)_{six}$
(e) $(0.555)_{six}$ (f) $(0.555\ldots)_{six}$

Comments

4. *Continued.* Starting with a unit piece A and trading as necessary, one finds that a minimal collection for $1/8$ is 4 C pieces and 3 D pieces and a minimal collection for $1/9$ is 4 C pieces. Hence, $1/8 = (0.043)_{six}$ and $1/9 = (0.4)_{six}$.

5. The heximal $(0.32)_{six}$ represents a collection of 3 B pieces and 2 C pieces. The value of this collection can be determined in a number of ways (see Comment 2). One way is to convert the collection to 20 C pieces and note that 20 C pieces constitute $5/9$ of a unit piece A, as shown below. Hence $(0.32)_{six} = 5/9$



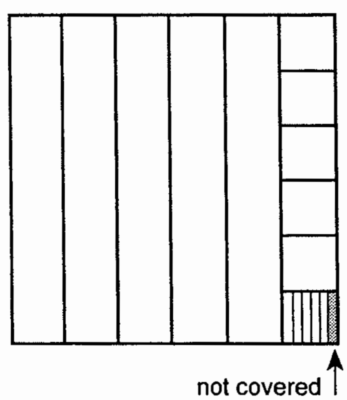
$(0.32)_{six} = \frac{5}{9}$

(a) $(0.201)_{six} = 73/216$

(b) $(1.32)_{six} = 14/9$

(c) $(0.5)_{six} = 5/6$

(d) $(0.55)_{six} = 35/36$



(e) $(0.555)_{six} = 215/216$. One way to find this value is to note a collection of 5 B pieces, 5 C pieces and 5 D pieces covers all but $1/216$ of a unit square.

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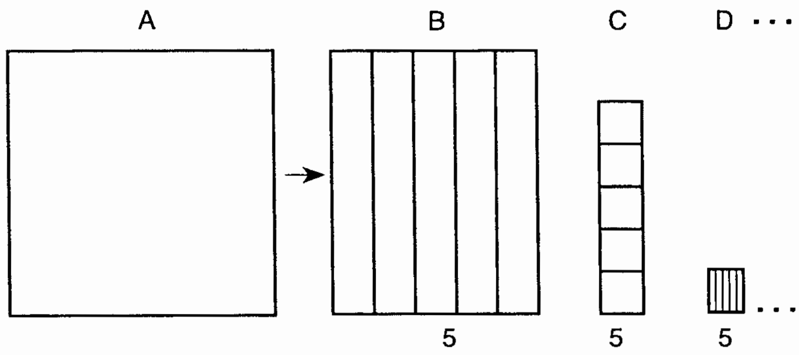
Actions



$1 = (0.555\ldots)_{six}$

6. Have the students form the collection of base pieces whose heximal is $(0.111\ldots)_{six}$. Ask them how many copies of this collection can be obtained from 1 unit piece A. Discuss.

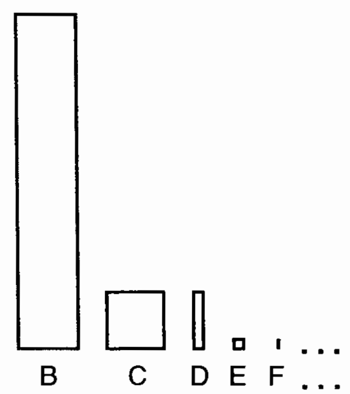
Comments

5. *Continued.*

(f) If a unit piece A is traded for 6 B pieces, one of the B pieces is then traded for 6 C pieces, one of the C pieces is then traded for 6 D pieces, and so on indefinitely; A has been converted to a collection of 5 B pieces, 5 C pieces, 5 D pieces, and so forth. Thus, $1 = (0.555\ldots)_{six}$.

Alternately, one knows from Action 4 that $\frac{1}{5} = (0.111\ldots)_{six}$, that is, the value of a collection which consists of 1 B piece, 1 C piece, 1 D piece, and so on, has value $\frac{1}{5}$. Hence, 5 copies of this collection will have value $5 \times \frac{1}{5}$ or 1.

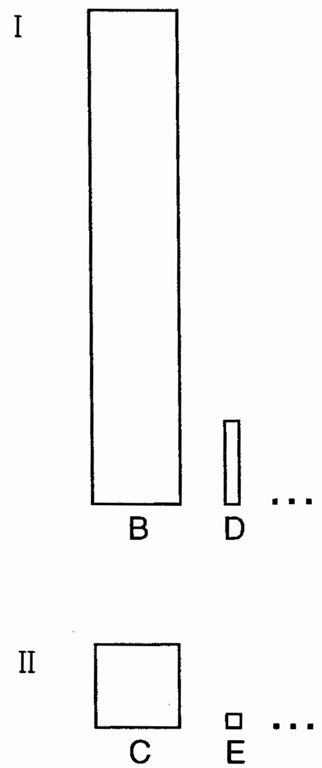
6. A sketch of the collection appears below. The students may recall from Action 4 that $(0.111\ldots)_{six}$ is the heximal for $\frac{1}{5}$. Hence, 5 copies have the value 1, and so are equivalent to 1 A piece.



Alternately, one can convert an A piece to 5 copies of the collection as described in the first paragraph of Comment 5(f). Since 5 copies of the collection have value 1, each copy has value $\frac{1}{5}$, verifying that $(0.111\ldots)_{six} = \frac{1}{5}$.

Actions

7. Ask the students to separate the collection of Action 6 into two collections: Collection I consisting of every other piece starting with piece B and Collection II consisting of every other piece starting with piece C. Ask the students to write the heximals for these two collections and find their values.



Comments

7. The heximals for Collections I and II are $(0.101010\ldots)_{six}$ and $(0.010101\ldots)_{six}$, respectively.

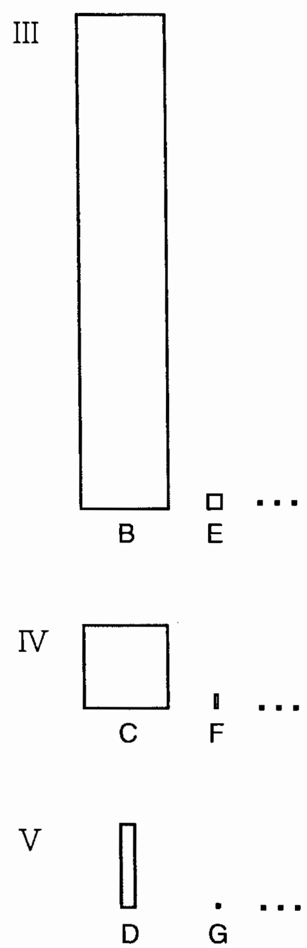
The value of Collection I is 6 times the value of Collection II. The value of the combined collections is $\frac{1}{5}$. Hence, the value of Collection I is $\frac{6}{7} \times \frac{1}{5}$, or $\frac{6}{35}$, and the value of Collection II is $\frac{1}{7} \times \frac{1}{5}$, or $\frac{1}{35}$.

Alternately, an A piece can be converted into 35 copies of Collection II by trading the A piece for 36 C pieces, then trading one of the C pieces for 36 E pieces, then trading one of the E pieces for 36 copies of the next smaller piece in Collection II, and so forth. Hence, the value of Collection II is $\frac{1}{35}$ of the value, 1, of an A piece. The value of Collection I can then be found by either subtracting $\frac{1}{35}$ from $\frac{1}{5}$ or by noting that Collection I is 6 copies of Collection II.

Summarizing, $(0.101010\ldots)_{six} = \frac{6}{35}$ and $(0.010101\ldots)_{six} = \frac{1}{35}$.

Actions

8. Ask the students to separate the collection of Action 6 into three collections: Collection III consisting of every third piece starting with piece B, Collection IV consisting of every third piece starting with piece C, and Collection V consisting of every third piece starting with piece D. Ask the students to write the heximals for these three collections and find their values.



9. Ask the students to find the value of the following heximals.

(a) $(0.030303\ldots)_{six}$ (b) $(0.212121\ldots)_{six}$

(c) $(0.420202\ldots)_{six}$ (d) $(0.412412412\ldots)_{six}$

(e) $(0.0333\ldots)_{six}$ (f) $(0.532122122122\ldots)_{six}$

Comments

8. The heximals for Collections III, IV and V are $(0.100100100\ldots)_{six}$, $(0.010010010\ldots)_{six}$ and $(0.001001001\ldots)_{six}$, respectively.

The value of Collection IV is 6 times the value of Collection V and the value of Collection III is 36 times the value of Collection V, so the combined value of the collections is 43 times the value of Collection V. The value of the combined collections is $\frac{1}{5}$. Thus, the value of Collection V is $\frac{1}{43} \times \frac{1}{5}$ or $\frac{1}{215}$, the value of Collection IV is $\frac{6}{43} \times \frac{1}{5}$ or $\frac{6}{215}$ and the value of Collection III is $\frac{36}{43} \times \frac{1}{5}$ or $\frac{36}{215}$.

Alternately, one can convert an A piece into 215 copies of Collection V by trading the A piece for 216 D pieces, then trading one D piece for 216 copies of the next smaller piece in Collection V, then trading one of these pieces for 216 of the next smaller piece and so forth. Hence, the value of Collection V is $\frac{1}{215}$. The values of the other two collections can then be found by noting that Collection IVis 6 copies of Collection V and Collection III is 36 copies of Collection V.

Summarizing, $(0.100100100\ldots)_{six} = \frac{36}{215}$, $(0.010010010\ldots)_{six} = \frac{6}{215}$ and $(0.001001001\ldots)_{six} = \frac{1}{215}$.

9. (a) Combining 3 copies of Collection II of Action 7 produces a collection whose heximal is $(0.030303\ldots)_{six}$. The value of Collection II is $\frac{1}{35}$. Hence, $(0.030303\ldots)_{six} = 3 \times \frac{1}{35} = \frac{3}{35}$.

Continued next page.

9. *Continued.*

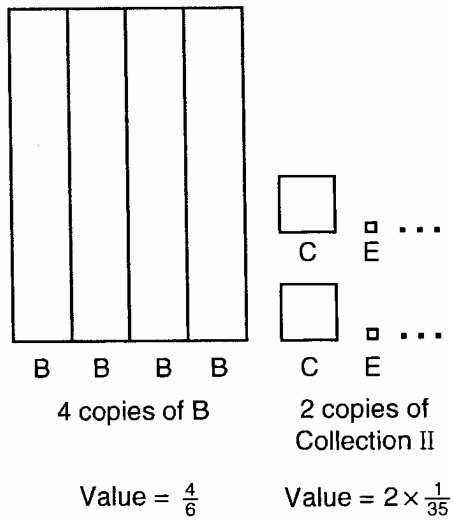
(b) Combining 2 copies of Collection I of Action 7 with 1 copy of Collection II of Action 7 produces a collection whose heximal is $(0.212121\ldots)_{six}$. Hence,

$$(0.212121\ldots)_{six} = (2 \times \tfrac{6}{35}) + \tfrac{1}{35} = \tfrac{13}{35}.$$

(c) Adding 4 B pieces to 2 copies of Collection II, Action 7, produces a collection whose heximal is $(0.420202\ldots)_{six}$. Thus,

$$(0.420202\ldots)_{six} = \tfrac{4}{6} + (2 \times \tfrac{1}{35}) = \tfrac{2}{3} + \tfrac{2}{35} = \tfrac{76}{105}.$$

Collection with heximal $(0.420202\ldots)_{six}$



(d) A collection with heximal $(0.412412412\ldots)_{six}$ is obtained if, in Action 8, 4 copies of Collection III, 1 copy of Collection IV and 2 copies of Collection V are combined. Hence,

$$(0.412412412\ldots)_{six} = (4 \times \tfrac{36}{215}) + \tfrac{6}{215} + (2 \times \tfrac{1}{215}) = \tfrac{152}{215}.$$

(e) Combining 3 copies of the collection of Action 6 and then removing 3 B pieces produces a collection with heximal $(0.0333\ldots)_{six}$. Hence,

$$(0.0333)_{six} = 3 \times \tfrac{1}{5} - 3 \times \tfrac{1}{6} = \tfrac{3}{5} - \tfrac{1}{2} = \tfrac{1}{10}.$$

(f) In Action 8, combining 1 copy of Collection III, 2 copies of Collection IV and 2 copies of Collection V produces a collection with heximal $(0.122122122\ldots)_{six}$. Adding 4 B pieces and 1 C piece to this collection yields a collection with heximal $(0.532122122122\ldots)_{six}$. Hence,

$$(0.532122122122\ldots)_{six}$$
$$= \tfrac{36}{215} + (2 \times \tfrac{6}{215}) + (2 \times \tfrac{1}{215}) + (4 \times \tfrac{1}{6}) + \tfrac{1}{36}$$
$$= \tfrac{36}{215} + \tfrac{12}{215} + \tfrac{2}{215} + \tfrac{2}{3} + \tfrac{1}{36}$$
$$= \tfrac{50}{215} + \tfrac{25}{36} = \tfrac{10}{43} + \tfrac{25}{36} = \tfrac{1435}{1548}.$$

Actions

10. Ask the students to find the heximals for the following fractions: $^{15}/_{32}$, $^{13}/_{14}$, $^{5}/_{11}$.

Comments

10. One way to find the heximal for $^{15}/_{32}$ is to form a collection which is $^{1}/_{32}$nd of a collection of 15 A pieces, making trades as necessary: Trade the 15 A pieces for 90 B pieces; $^{1}/_{32}$nd of 90 B pieces is 2 B pieces with 26 remaining. Trade these for 156 C pieces; $^{1}/_{32}$nd of 156 C pieces is 4 C pieces with 28 remaining. Trade these for 168 D pieces; $^{1}/_{32}$nd of 168 D pieces is 5 D pieces with 8 remaining. Trade these for 48 E pieces; $^{1}/_{32}$nd of 48 E pieces is 1 E piece with 16 remaining. Trade these for 96 F pieces; $^{1}/_{32}$nd of 96 F pieces is 3 F pieces with none remaining. Thus, a collection of 0 A pieces, 2 B pieces, 4 C pieces, 5 D pieces, 1 E piece and 3 F pieces has a value of $^{15}/_{32}$. Hence, $^{15}/_{32} = (0.24513)_{six}$.

The students may find it useful to create a table in which the above information can be recorded as it is obtained. Following is one possibility.

Type of piece	A	B	C	D	E	F
Number of pieces	15	90	156	168	48	96
Total pieces in groups of 32	0	64	128	160	32	96
Remaining pieces	15	26	28	8	16	0
Groups of 32	0	2	4	5	1	3

Here is a table for $^{13}/_{14}$:

Type of piece	A	B	C	D	E	F	...
Number of pieces	13	78	48	36	48	36	...
Total pieces in groups of 14	0	70	42	28	42	28	...
Remaining pieces	13	8	6	8	6	8	...
Groups of 14	0	5	3	2	3	2	...

In the above table, columns C and E are identical. Thus, what comes after E is identical to what comes after C; that is, columns C and D keep repeating. Hence, $^{13}/_{6} = (0.5323232\ldots)_{six} = (0.5\overline{32})_{six}$, where the block of digits under the bar repeat indefinitely.

Continued next page.

10. *Continued.* Here is a table for $\frac{5}{11}$:

Type of piece	A	B	C	D	E	F	G	H	I	J	K	L	...
Number of pieces	5	30	48	24	12	6	36	18	42	54	60	30	...
Total pieces in groups of 11	0	22	44	22	11	0	33	11	33	44	55	22	...
Remaining pieces	5	8	4	2	1	6	3	7	9	10	5	8	...
Groups of 11	0	2	4	2	1	0	3	1	3	4	2	2	...

Since columns B and L are identical, the columns subsequent to B and L will be identical. Thus, columns B through K repeat indefinitely. Thus, $\frac{5}{11} = (0.\overline{2421031342})_{six}$.

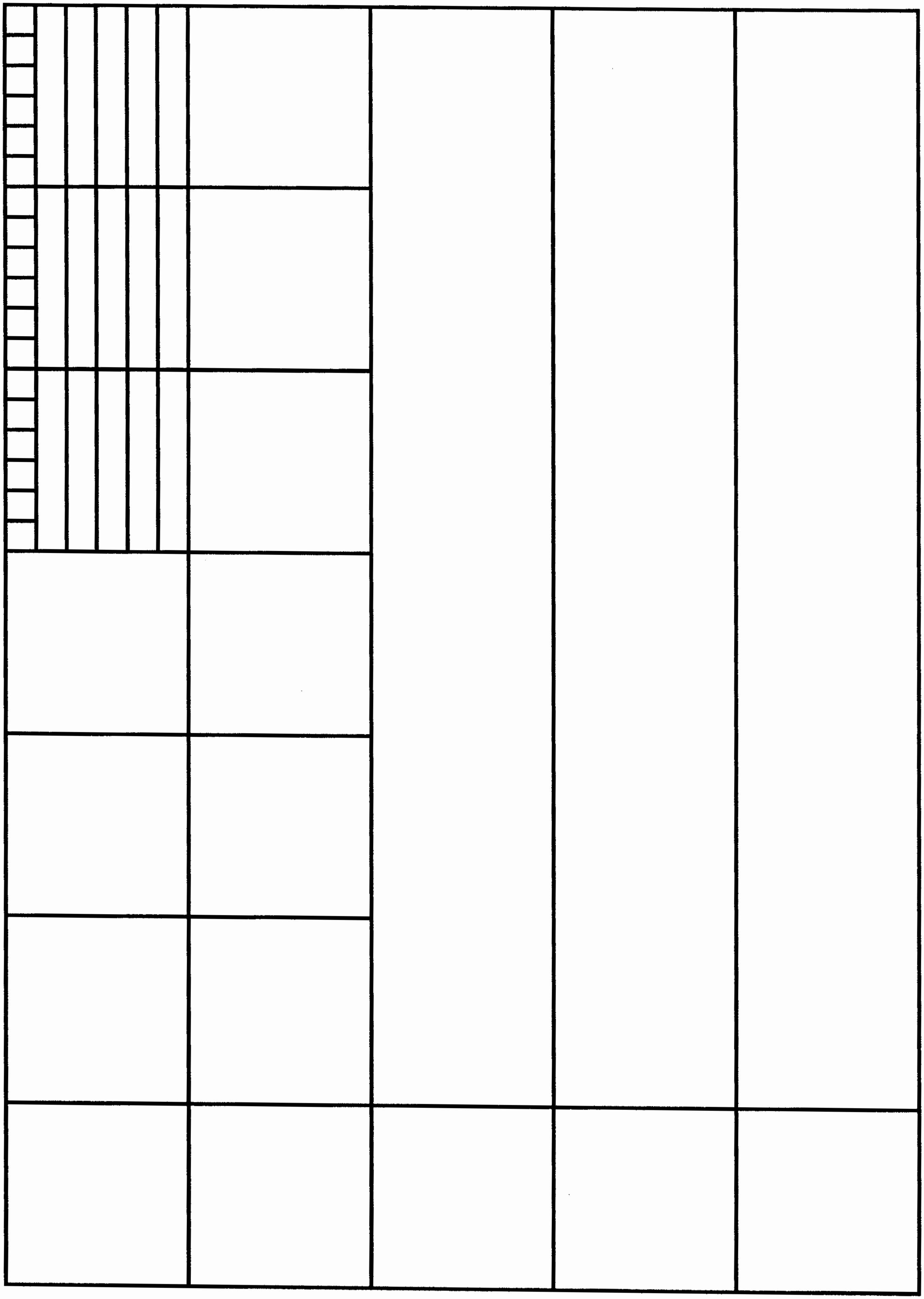
11. Discuss the relationship between fractions and heximals.

11. Using the techniques of Action 10, every fraction can be written as a heximal. If at some point, there are no remaining pieces, the process terminates. Otherwise, at some point, the number of remaining pieces in a column of a table will be the same as the number in some previous column and the subsequent columns will be identical. Hence, the digits in the heximal will begin to repeat. Thus, every fraction can be represented by a heximal which either terminates or eventually repeats.

Conversely, using techniques developed in Actions 5 and 9, every terminating or repeating heximal can be written as a fraction. Hence, the set of all fractions represents the same set of values as the set of all heximals which terminate or repeat.

XII-1 Master 1 Cut across page on above lines. Discard this piece.

XII-1 Master 2 Cut across page on above lines. Discard this piece.



XII-1 Master 3 Cut on lines.

Unit XII • Activity 2

Decimals & Fractions

O V E R V I E W

The relationship between decimals and fractions is investigated.

Prerequisite Activity

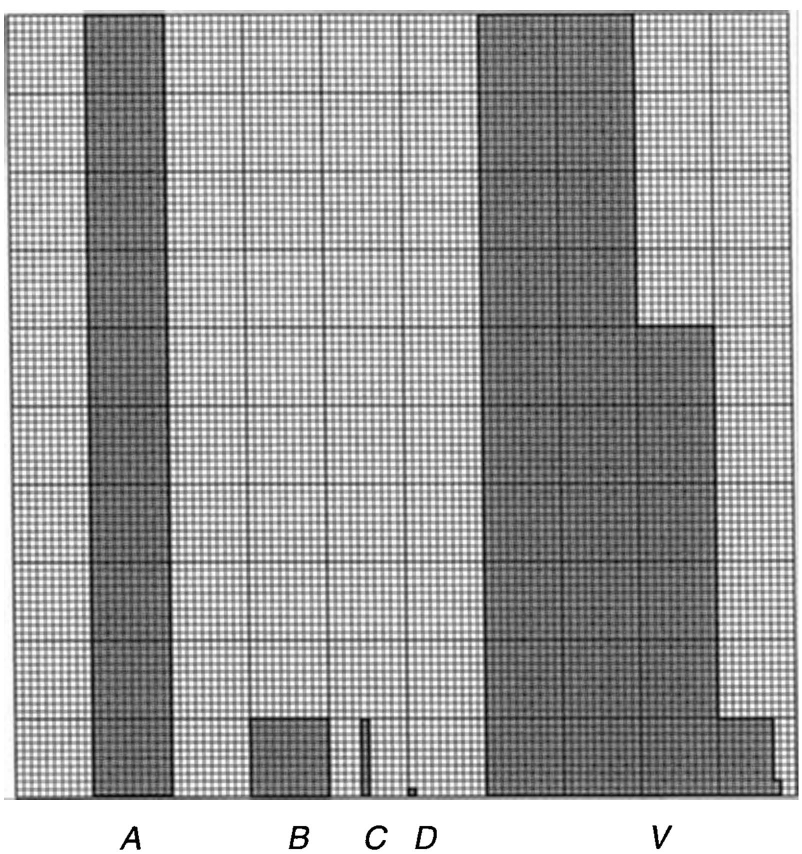
Unit IV, Activity 6, *Introduction to Decimals*; Unit XII, Activity 1, *Heximals and Fractions*.

Materials

Decimal grids (see Action 2).

Actions

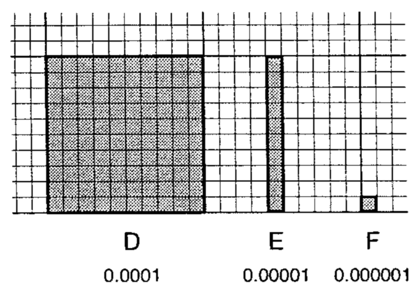
1. Show the students the following figure. Tell them the area of the large square is 1. Ask them to determine the areas of regions *A, B, C, D* and *V*. Discuss how the process of dividing regions into tenths can be continued to obtain areas of 0.00001, 0.00001, 0.000001, etc.



Comments

1. A transparency of the figure can be made from Master 1 which is attached. The areas of the regions are, respectively, 0.1, 0.01, 0.001, 0.0001 and 0.2673.

A region the size of *D* can be divided into tenths—as shown in the magnified version below—to obtain region *E*, whose area is 0.00001. Then, in turn, a region the size of *E* can be divided into tenths to obtain region *F*, whose area is 0.000001, and so forth.



Actions

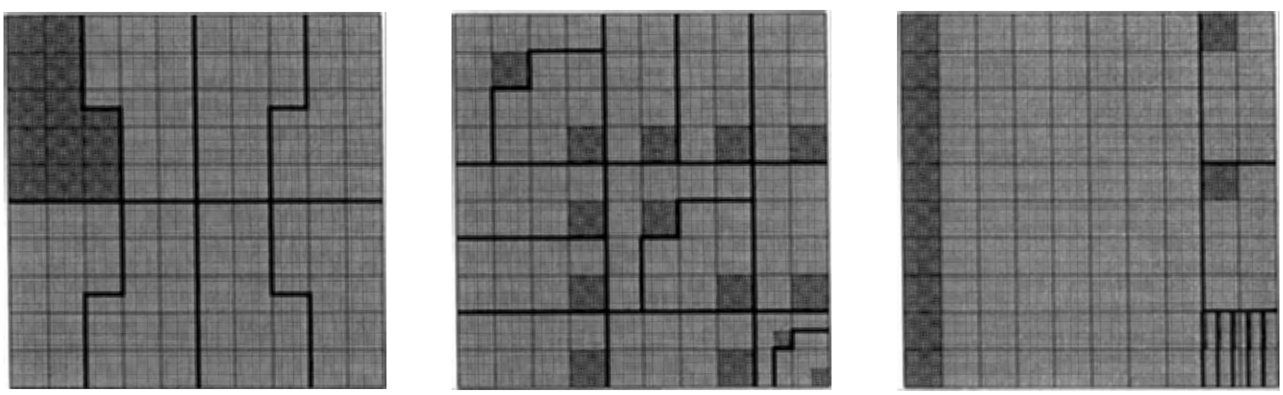
2. Distribute a *decimal grid* to each student. Tell them the area of the large square is 1. Ask the students to shade in parts of the square so that the total amount shaded has area 1⁄8. Then ask them to write the amount of area shaded in decimal form.

Comments

2. A master for decimal grids (Master 2) is attached.

Shown below are some ways of shading in parts of the square so the amount shaded has area 1⁄8. In the last two figures the shaded parts are not connected. In these two figures, the square is divided into regions and 1⁄8 of the area of each of these regions is shaded and, thus, 1⁄8 of the area of the entire square is shaded. Since the area of the square is 1, the total area of the shaded amount is 1⁄8.

In the last figure, the tenths are combined into as many groups of 8 tenths as possible (in this case, only one group is possible) and 1 tenth in each group is shaded. Then the remaining hundredths are combined into as many groups of 8 as possible and 1 hundredth in each group is shaded. Finally, the remaining thousandths are formed into groups of 8 and 1 thousandth in each of these groups is shaded. Since 1 tenth, 2 hundredths and 5 thousandths are shaded, the area of the shaded portion is 0.125.



$\frac{1}{8} = 0.125$

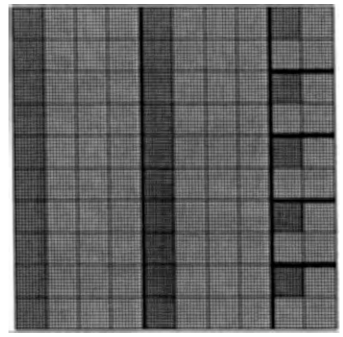
Actions

3. Point out to the students that, in Action 2, they determined that the decimal equivalent of the fraction ⅛ is 0.125. Discuss how decimal grids can be used to illustrate the decimal equivalents of ½, ¼, ⅕ and ⅓.

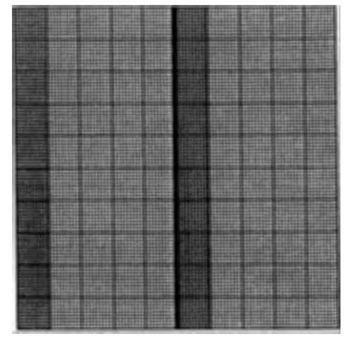
Comments

3. For each fraction, a volunteer can be asked to illustrate its decimal equivalent on a decimal grid. Shown below are decimal grid sketches similar to the last sketch shown for ⅛ in Comment 2.

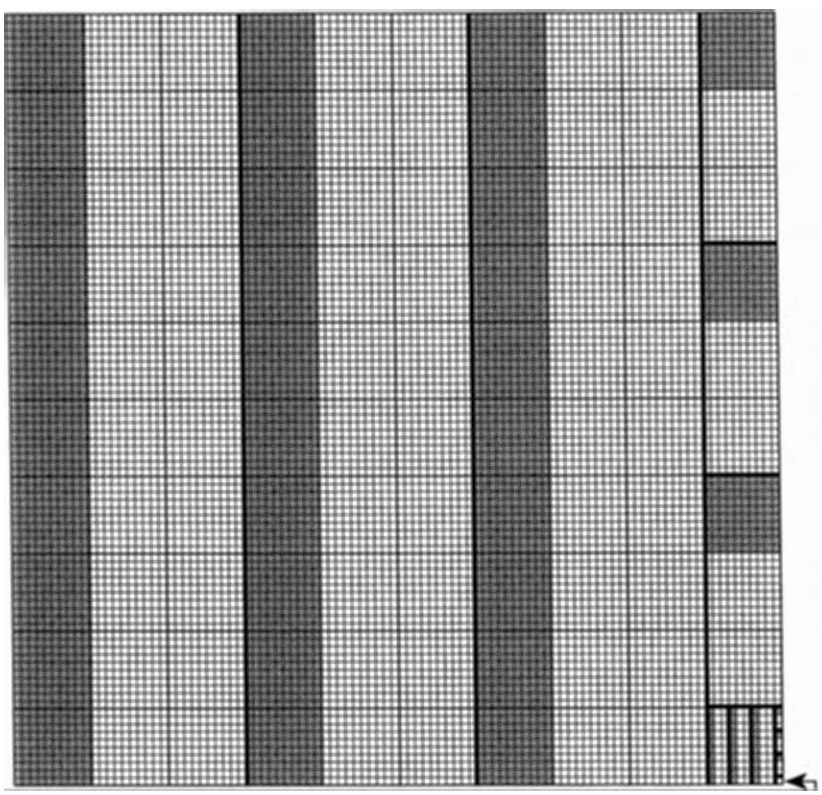
$\frac{1}{2} = 0.5$



$\frac{1}{4} = 0.25$



$\frac{1}{5} = 0.2$



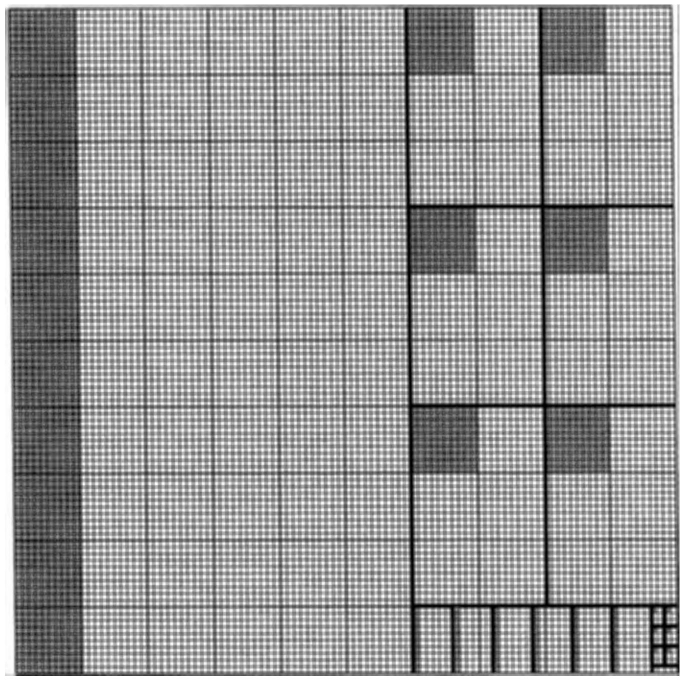
left over

$\frac{1}{3} = 0.33333...$

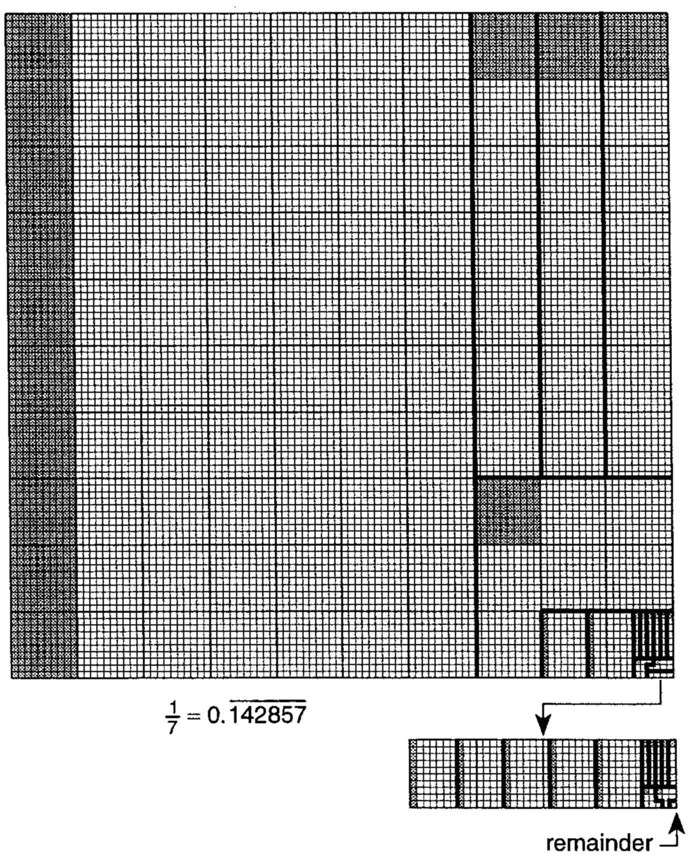
Note that, in the sketch for ⅓, after successively forming tenths, hundredths, thousandths and ten-thousandths into groups of 3, a ten-thousandth is left over. This can be divided into 10 one-hundred-thousandths which can be formed into 3 groups of 3 with 1 remaining. This remaining one-hundred-thousandth can in turn be divided into 10 parts which, again, can be formed into 3 groups of 3 with one remaining, and so on indefinitely. Thus, the decimal equivalent of ⅓ is 0.33333… where the string of 3's never terminates.

Actions

4. Distribute decimal grid paper to the students. Ask them to illustrate the decimal equivalents of 1/6, 1/7, 1/8 and 1/11.



$\frac{1}{6} = 0.1666\ldots$



$\frac{1}{7} = 0.\overline{142857}$

Comments

4. All of the decimal equivalents of these fractions are non-terminating.

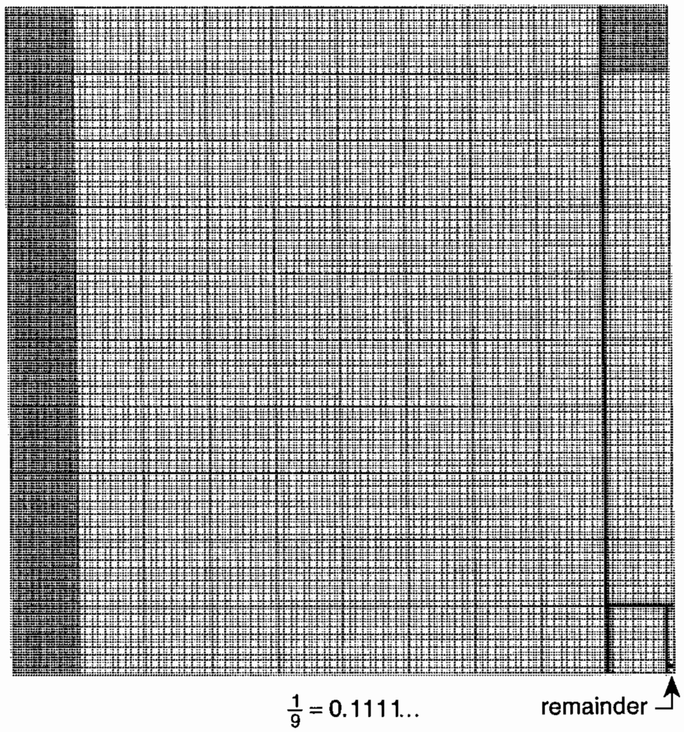
In the illustration of the decimal expansion of 1/6, the leftover region of 4 ten-thousandths, when magnified, is identical to the region of 4 hundredths in the lower right-hand corner of the grid. Thus, as successive subdivisions are made, the pattern of obtaining 6 groups of 6 will continue without termination.

In the sketch for 1/7, 1 group of 7 tenths is formed leaving a remainder group of 3 tenths. This remainder is then divided into 4 groups of 7 hundredths with a remainder group of 2 hundredths, and so forth. This process of dividing remainders into groups of 7 continues until a remainder group is reached which has the same number of elements as a previous remainder group (the original square is considered a remainder group of 1). When this happens, the number of groups of 7 will begin to repeat.

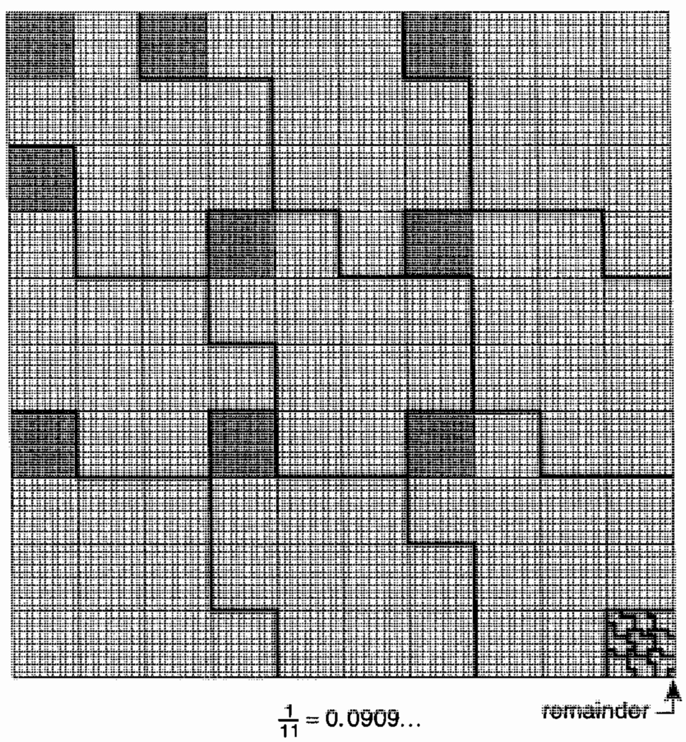
In the sketch for 1/7, one of the remainder groups is enlarged in order to continue the process until repetition occurs. The final remainder group, shown in the sketch, is a single square. When magnified, it is identical to the original square. Thus, the pattern of subdivision of the original square repeats.

Continued next page.

Actions



$\frac{1}{9} = 0.1111...$



$\frac{1}{11} = 0.0909...$

Comments

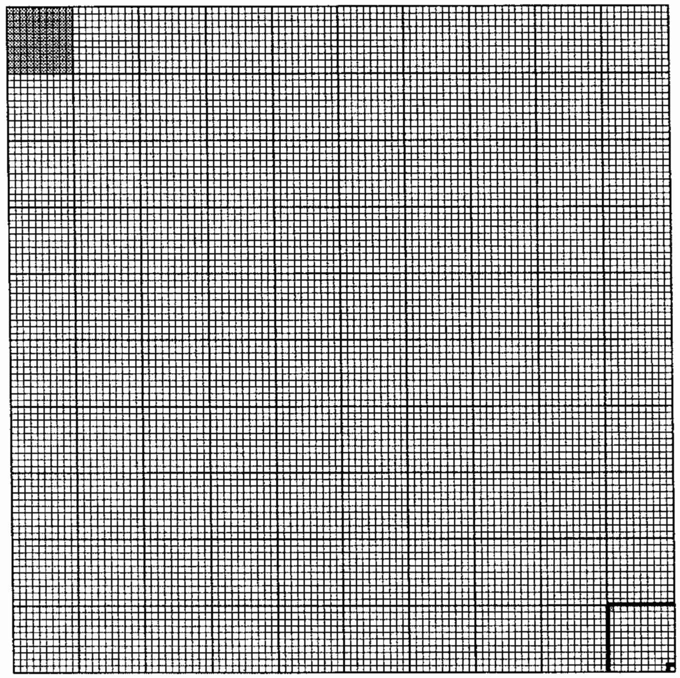
4. *Continued.* Shown here are sketches for 1⁄9 and 1⁄11. A group of 11 tenths is not possible, hence the picture for 1⁄11 begins with groups of 11 hundredths. There are 9 of these with 1 hundredth remaining which, when magnified, is identical to the original square. Thus the pattern of subdivision repeats.

Masters for decimal grid sketches of 1⁄6, 1⁄7, 1⁄9 and 1⁄11 are attached (Masters 3, 4, 5 and 6). The students may be interested in creating other sketches for these fractions.

Actions

5. Ask the students to use the information obtained in Actions 3 and 4 to find fractions equivalent to the following decimals:

(a) 0.666... (b) 0.555...
(c) 0.999... (d) 0.010101...
(e) 0.373737... (f) 0.5424242...



$\frac{1}{99} = 0.0101...$

6. Ask the students to determine a fraction equivalent to 0.001001001.... Then have them use this result to find fractions equivalent to $0.\overline{235}$ and $0.27\overline{465}$.

Comments

5. (a). Doubling the shaded area in the picture on page 3 for $\frac{1}{3}$, shows that $0.666... = \frac{2}{3}$. Alternately, shading in 6 of each of the groups of 9 in the earlier picture for $\frac{1}{9}$ shows that $0.666... = \frac{6}{9}$.

(b) $0.555... = \frac{5}{9}$.

(c) Shading in all 9 in each of the groups of 9 in the picture for $\frac{1}{9}$ shades the entire square. Hence, $0.999... = 1$.

(d) 0.010101... is $\frac{1}{9}$ of 0.090909..., that is, $\frac{1}{9}$ of $\frac{1}{11}$ or $\frac{1}{99}$. Alternately, shading in 0.01010... on a decimal grid shades in $\frac{1}{99}$ of the unit square as shown to the left. Note also that 0.010101... is $\frac{1}{99}$ of 0.999...., that is, $\frac{1}{99}$ of 1.

(e) 0.373737... is 37(0.010101...) or $\frac{37}{99}$.

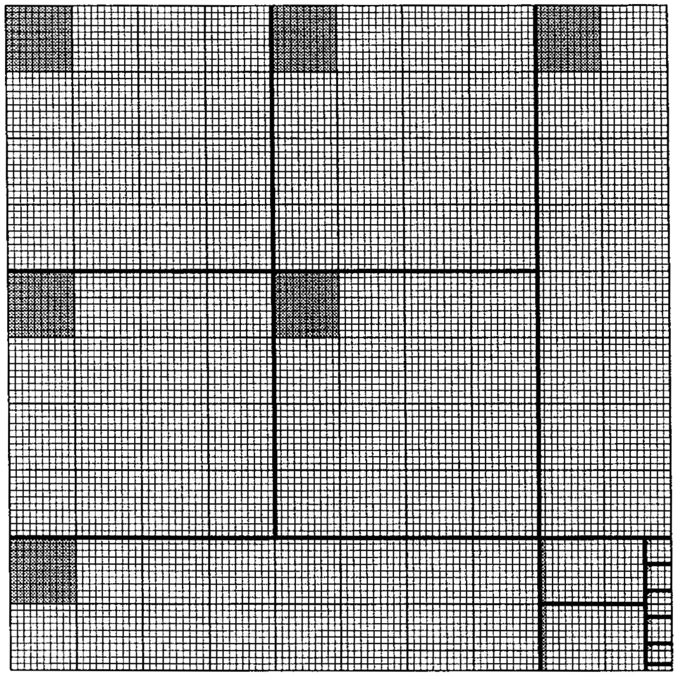
(f) $0.5424242... = 0.3 + 0.242424... = \frac{3}{10} + \frac{24}{99} = \frac{537}{990}$. Alternatively, $0.5424242... = \frac{1}{10}(5.424242...) = \frac{1}{10}(5 + \frac{42}{99}) = \frac{1}{10}(\frac{537}{99})) = \frac{537}{990}$.

6. Since $999(0.0010010001...) = 0.999999999... = 1$, it follows that $0.00100010001... = \frac{1}{999}$.

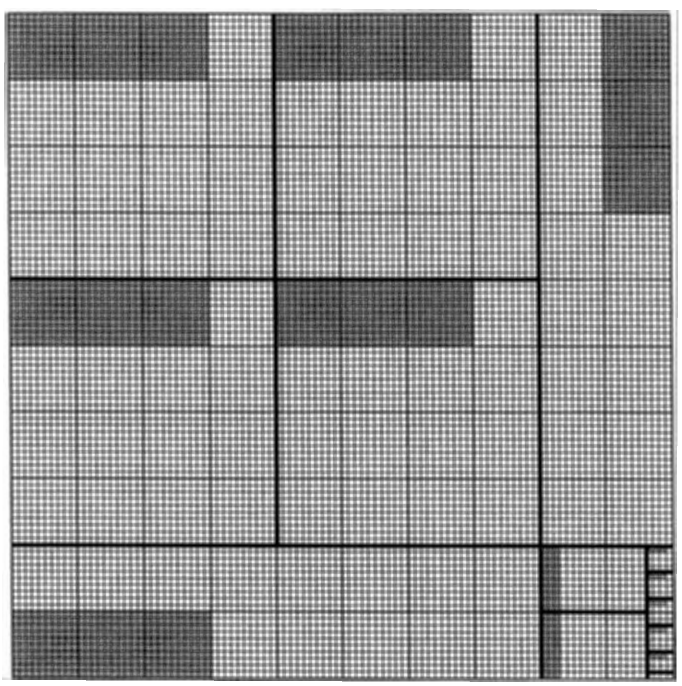
$0.\overline{235} = 235(0.001001001...) = \frac{235}{999}$.
$0.27\overline{465} = 0.27465465465... = 0.654654654... - 0.38 = \frac{654}{999} - \frac{38}{100} = \frac{4573}{16650}$. Alternatively, $0.27\overline{465} = \frac{1}{100}(27.465465465...) = \frac{1}{100}(27 + \frac{465}{999})) = \frac{4573}{16650}$.

Actions

7. Ask the students to find the decimal equivalent of $^3/_{16}$. Discuss.



$\frac{1}{16} = 0.0625$



$\frac{3}{16} = 0.1875$

Comments

7. One way to find the decimal equivalent of $^3/_{16}$ is to multiply the decimal equivalent of $^1/_{16}$ by 3. From a decimal grid sketch, one sees that $^1/_{16} = 0.0625$. Hence, $^3/_{16} = 0.1875$.

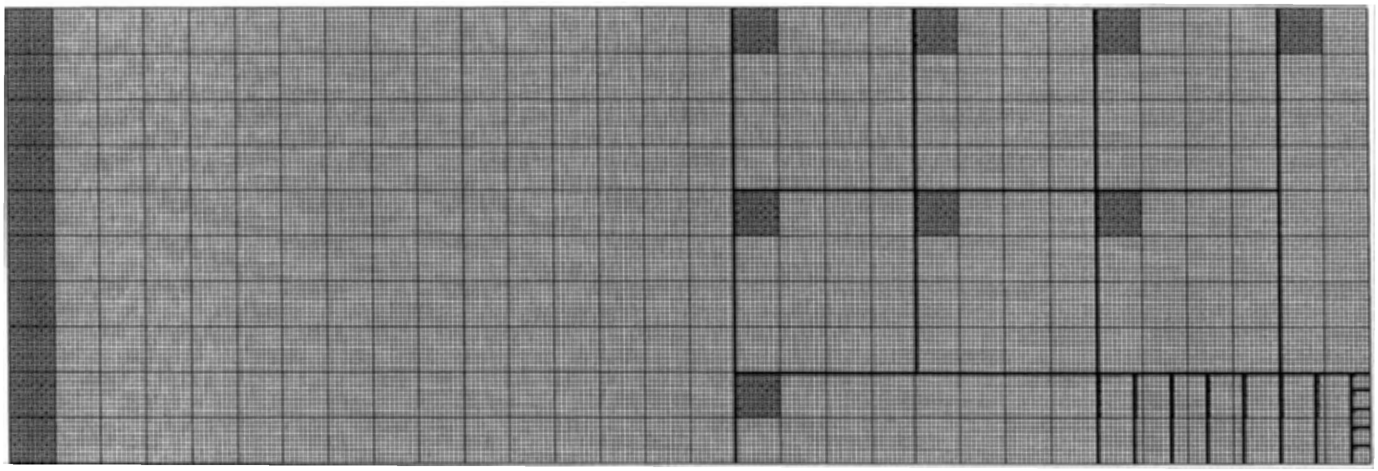
Alternately, a decimal grid sketch of $^3/_{16}$ can be made directly. In the sketch shown here, smaller shaded parts must be combined into larger parts to obtain the decimal equivalent. For example, 10 of the ten-thousandth squares must be combined into a thousandth strip.

Continued next page.

Actions

Comments

7. *Continued.* In this sketch, 3/16 is thought of as 1/16 of 3 unit squares (as opposed to 3/16 of 1 unit square). Notice in this sketch that regrouping is not necessary.



$\frac{3}{16} = 0.1875$

In the second sketch (on page 8), 3/16 is thought of as 1/16 of 3 unit squares (as opposed to 3/16 of 1 unit square). Notice in this sketch that regrouping is not necessary.

The second sketch suggests the following process for finding the decimal equivalent for 3/16: Change 3 into 30 tenths. Form these tenths into as many groups of 16 tenths as possible—in this case, 1 group with 14 tenths left over. Change the leftover tenths into 140 hundredths. Form these hundredths into as many groups of 16 as possible—in this case, 8 groups with 12 hundredths left over. Change the leftover tenths into 120 thousandths. Form these into as many groups of 16 as possible—in this case, 7 groups with 4 thousandths leftover—and continue the process. In the sketch, one element of each group of 16 is shaded. Hence, the number of groups of tenths is the tenths digit of the decimal equivalent, the number of groups of hundredths is the hundredths digit, etc.

Groups of 16		
	30	tenths
1	16	
	140	hundredths
8	128	
	120	thousandths
7	112	
	80	ten-thousandths
5	80	
	0	

This information can be put into tabular form as shown to the left.

The digits of the decimal appear in the left-hand column.

The decimal equivalent of 3/16 can also be obtained by dividing 3 by 16 on a calculator.

Masters for decimal sketches of 1/16 and 3/16 are attached (Masters 7 and 8).

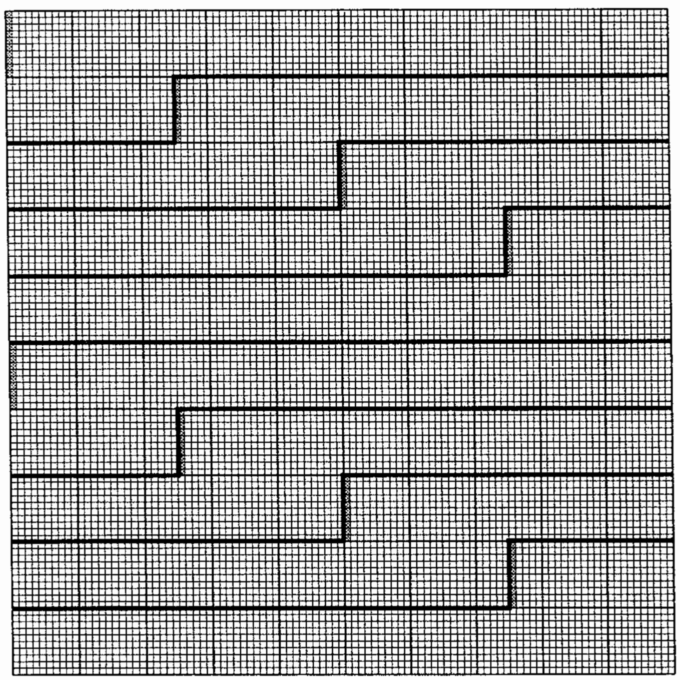
Actions

8. Ask the students to find decimals equivalent to the following fractions:

(a) $7/32$ (b) $64/125$ (c) $17/64$

(d) $19/30$ (e) $37/135$ (f) $29/52$

Groups of 32		
	70	tenths
2	<u>64</u>	
	60	hundredths
1	<u>32</u>	
	280	thousandths
8	<u>256</u>	
	240	ten-thousandths
7	<u>224</u>	
	160	hundred-thousandths
5	<u>160</u>	
	0	



$\frac{1}{125} = 0.008$

Comments

8. (a) There are a variety of ways to determine the decimal equivalent of $7/32$. One way is to find the decimal equivalent of $1/32$ and multiply this by 7. Since $1/32$ is half of $1/16$ and, from Comment 8, $1/16 = 0.0625$, $1/32 = 0.03125$. (This can also be deduced from a decimal grid sketch of $1/32$). Thus, $7/32 = 7(0.03125) = 0.21875$.

One can also carry out the process summarized in tabular form in Action 7. To the left is a table for $7/32$. It begins by converting 7 to tenths.

(b) The students may recognize that $1/125 = 2/250 = 8/1000$. Thus, $1/125 = 0.008$. (An alternative is to draw a decimal grid sketch such as the one shown.). Hence, $64/125 = 64(0.008) = 0.512$.

(c) $17/64 = 0.265625$

Continued next page.

Actions

Table for $\frac{19}{30}$:

Groups of 30		
	190	tenths
6	180	
	100	hundredths
3	90	
	100	thousandths
3	90	
	100	ten-thousandths
3	90	
	100	
⋮	⋮	⋮

Comments

(d) To the left is a table for $^{19}/_{30}$.

As the table continues, the remainder at each step will be the same as that at the previous step. Hence, the column of 3's will continue indefinitely. Thus, $^{19}/_{30} = 0.6333\ldots$.

Table for $\frac{37}{135}$:

Groups of 135	
	370
2	270
	1000
7	945
	550
4	540
	100
0	0
	1000
⋮	⋮

(e) Shown to the left is the beginning of a table for $^{37}/_{135}$. The names of the elements have been omitted. It is understood that the first elements are tenths, the next hundredths, and so forth.

The next step is to determine how many groups of 135 there are in 1000. However, this determination was made earlier in the table and found to be 7. Hence, 7 will be the next entry in the left column and, as before, this will be followed by 4 and then 0, leading to another 7, and so forth. Thus, the cycle 7, 4, 0 will repeat indefinitely. Hence, $^{37}/_{135} = 0.2\overline{740}$.

If 37 is divided by 135 on a standard four function calculator, one gets 0.274074 and it is not clear that the actual decimal representation of $^{37}/_{135}$ is a repeating decimal.

Table for $\frac{29}{52}$:

Groups of 52	
	290
5	260
	300
5	260
	400
7	364
	360
6	312
	480
9	468
	120
2	104
	160
3	156
	40
0	0
	400
⋮	⋮

(f) The beginning of a table for $^{29}/_{52}$ is shown to the left. The next step is determining the number of groups of 52 in 400. This was encountered earlier in the table, so the decimal will repeat from this point on. Thus, $^{29}/_{52} = 0.55\overline{769230}$.

Actions

9. Discuss the relationship between decimals and rational numbers.

Comments

9. A *rational number* is an integer or a *common* fraction, that is a fraction whose numerator and denominator are integers.

Every terminating decimal can be written as a fraction whose denominator is a power of 10 and hence represents a rational number. Also, using techniques similar to those in Actions 5 and 6, every repeating decimal can be written as a common fraction (see the footnote at the end of this comment). Hence every terminating or repeating decimal represents a rational number.

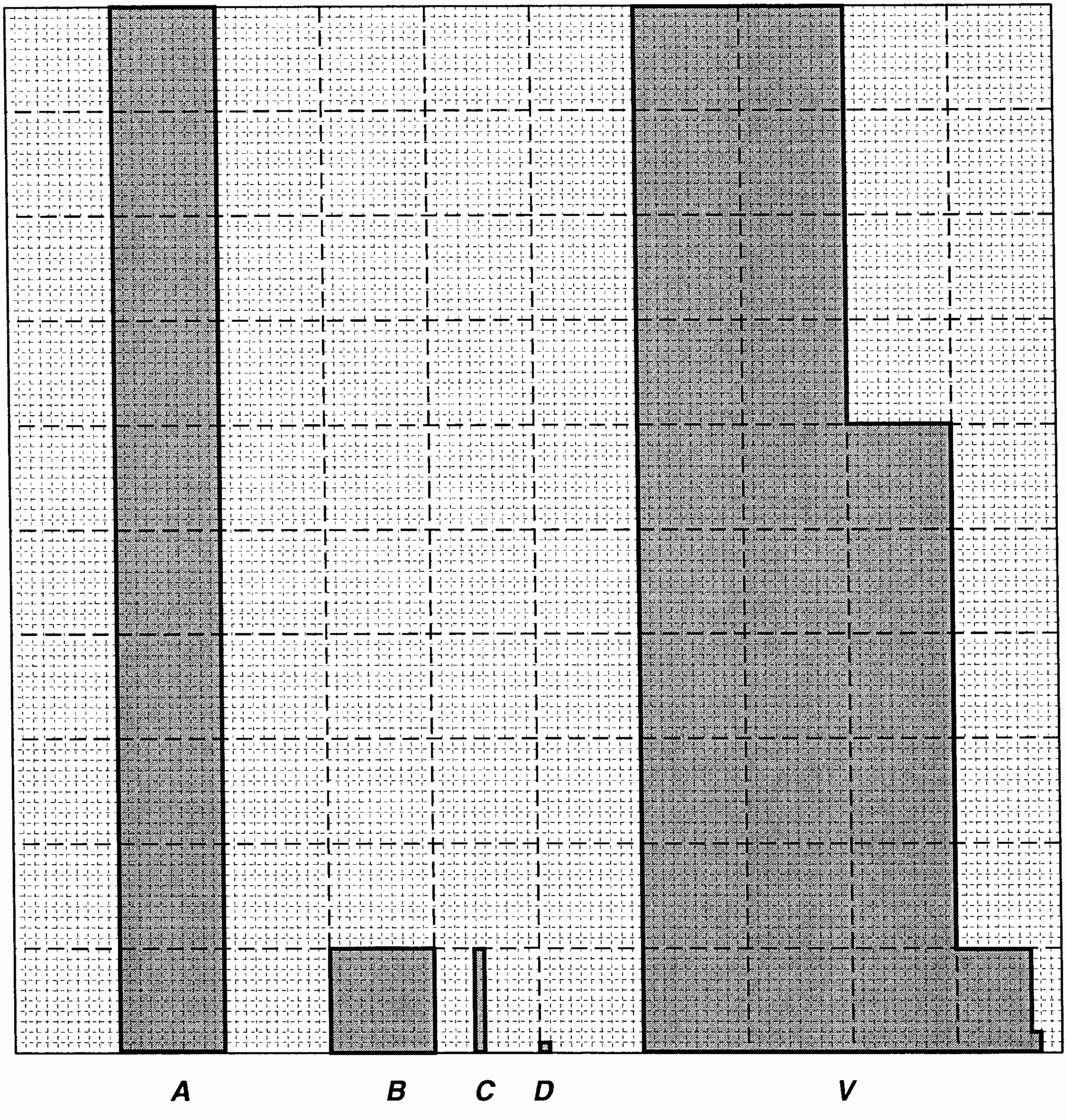
Conversely, using the techniques of Actions 7 and 8, every common fraction can be written as a decimal. For a fraction a/b, this process entails forming elements into as many groups of b as possible. If, at any stage, the number of leftover elements is 0, the decimal terminates. If, at any two stages, the number of leftover elements is the same, the decimal repeats [see Comments 8(e) and 8(f)]. Since the number of leftover elements is one of the b numbers 0, 1, 2, ... $b - 1$, after $b + 1$ stages, the process has either terminated, that is, the number of leftover elements is 0, or there have been two stages for which the number of leftover elements is the same. Hence, the decimal either terminates or repeats.

Thus, the set of rational numbers and the set of numbers whose decimals repeat or terminate are identical. Hence, such numbers as $\sqrt{2}$ and π, which are known to be irrational, have non-repeating, non-terminating decimal expansions.

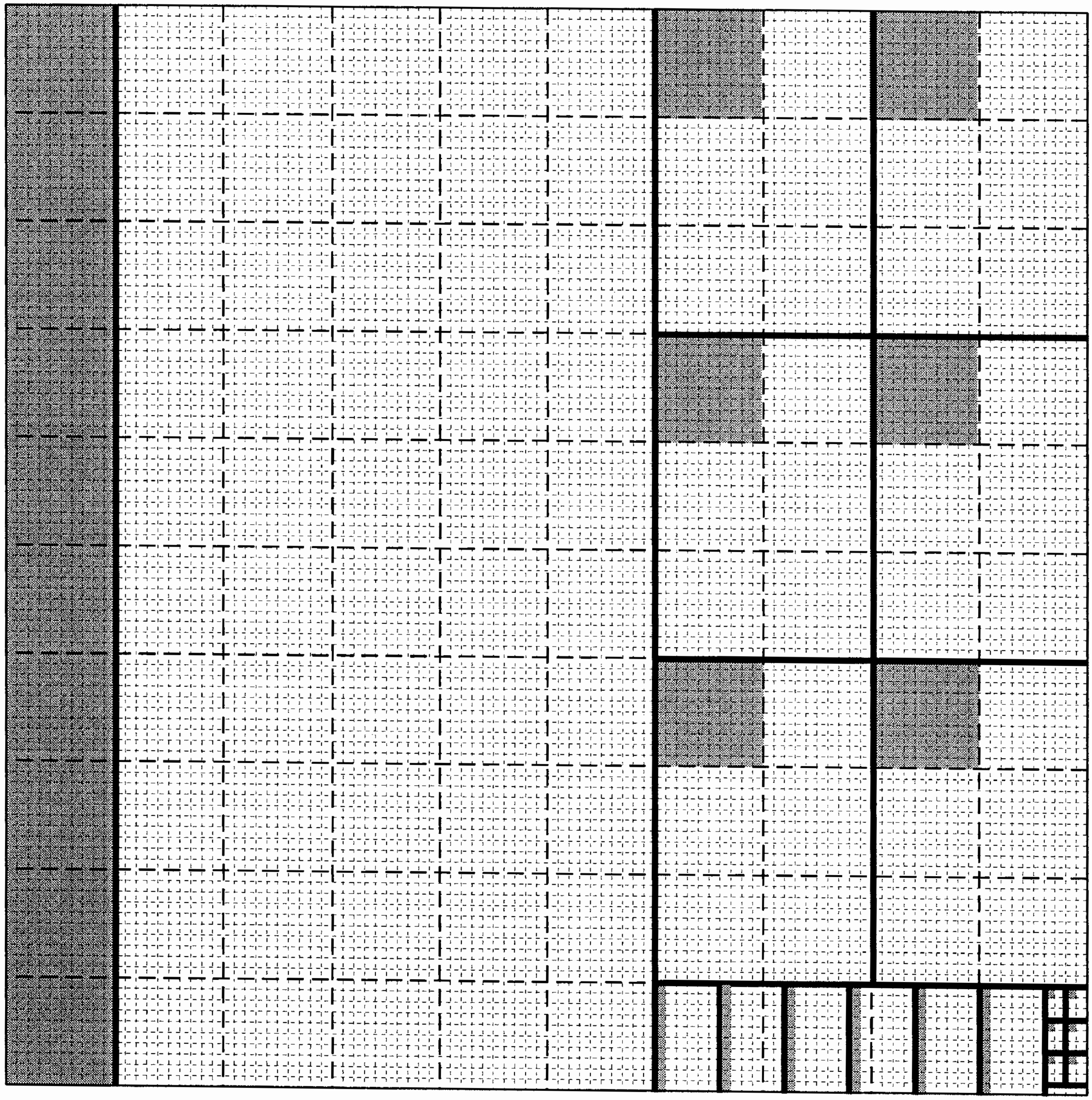
Footnote: Since 0.99999... = 1, one has
0.111... = $1/9$,
0.010101... = $1/99$,
0.001001001... = $1/999$,
0.000100010001... = $1/9999$,
and so forth.

Every repeating decimal can be written in the form $a + bc$ where a and b are rational and c is one of the above repeating decimals and, hence, represents a rational number. For example,

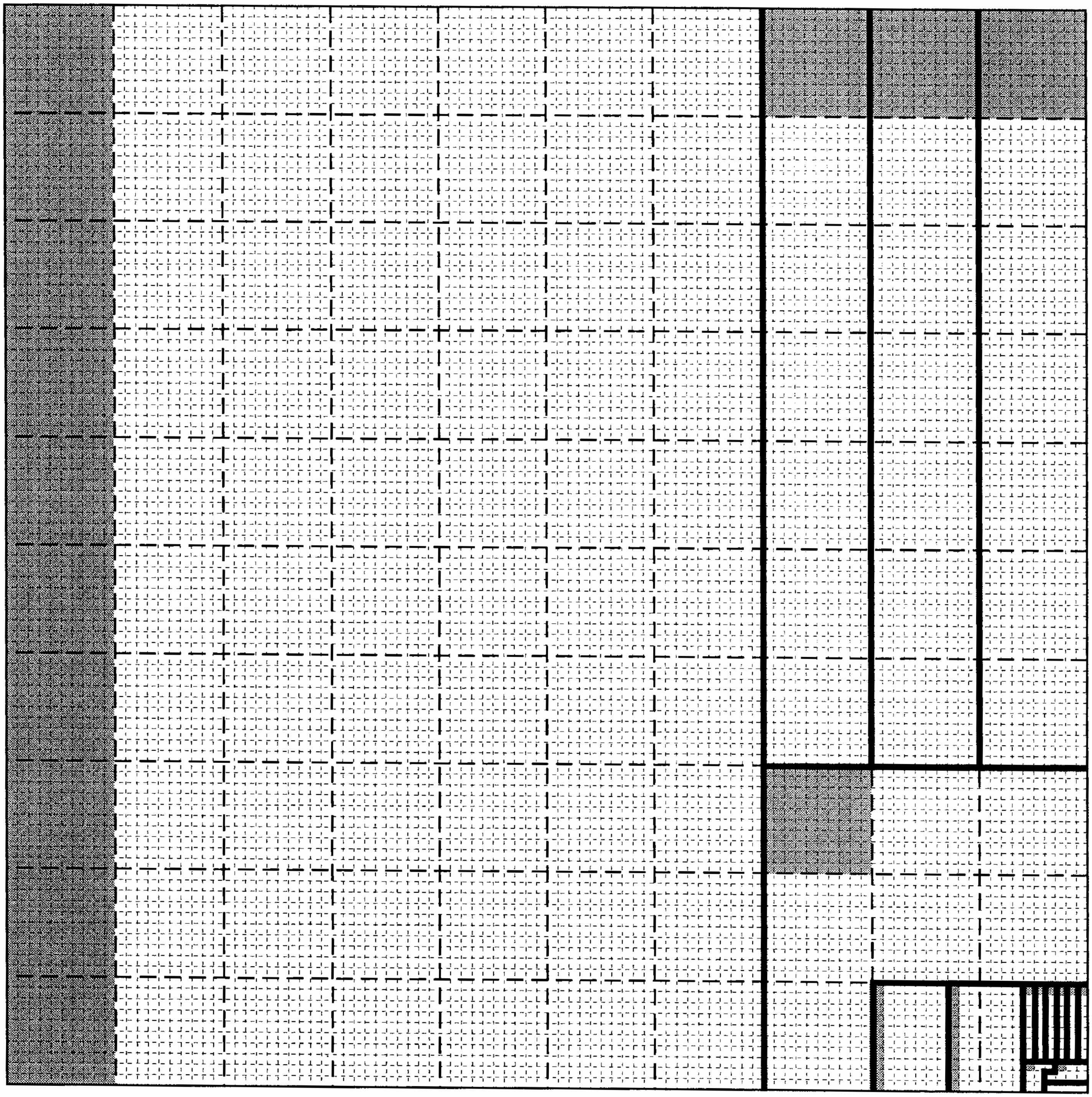
$0.246\overline{3425} =$
$0.246 + 0.000\overline{3425} =$
$0.246 + 0.000342534253425... =$
$0.246 + 1/1000(0.342534253425...) =$
$246/1000 + 3425/1000(0.000100010001...) =$
$246/1000 + 3425/1000(1/9999)$


A
B
C
D
V

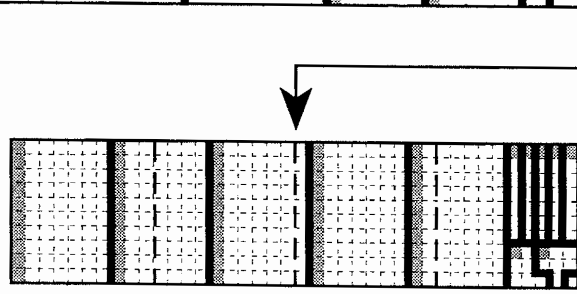
Decimal Grid Paper

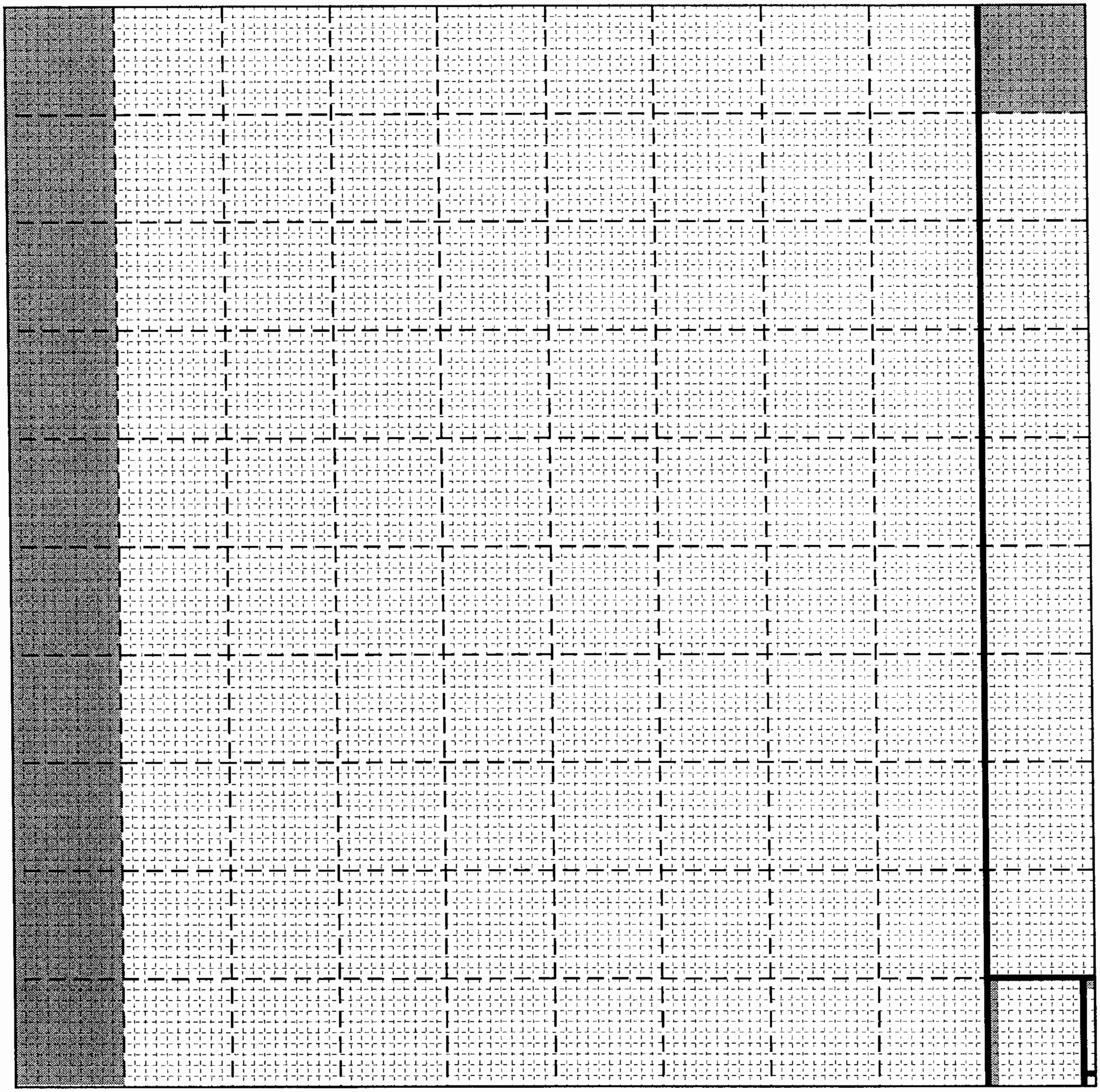


$\frac{1}{6} = 0.1666\ldots$

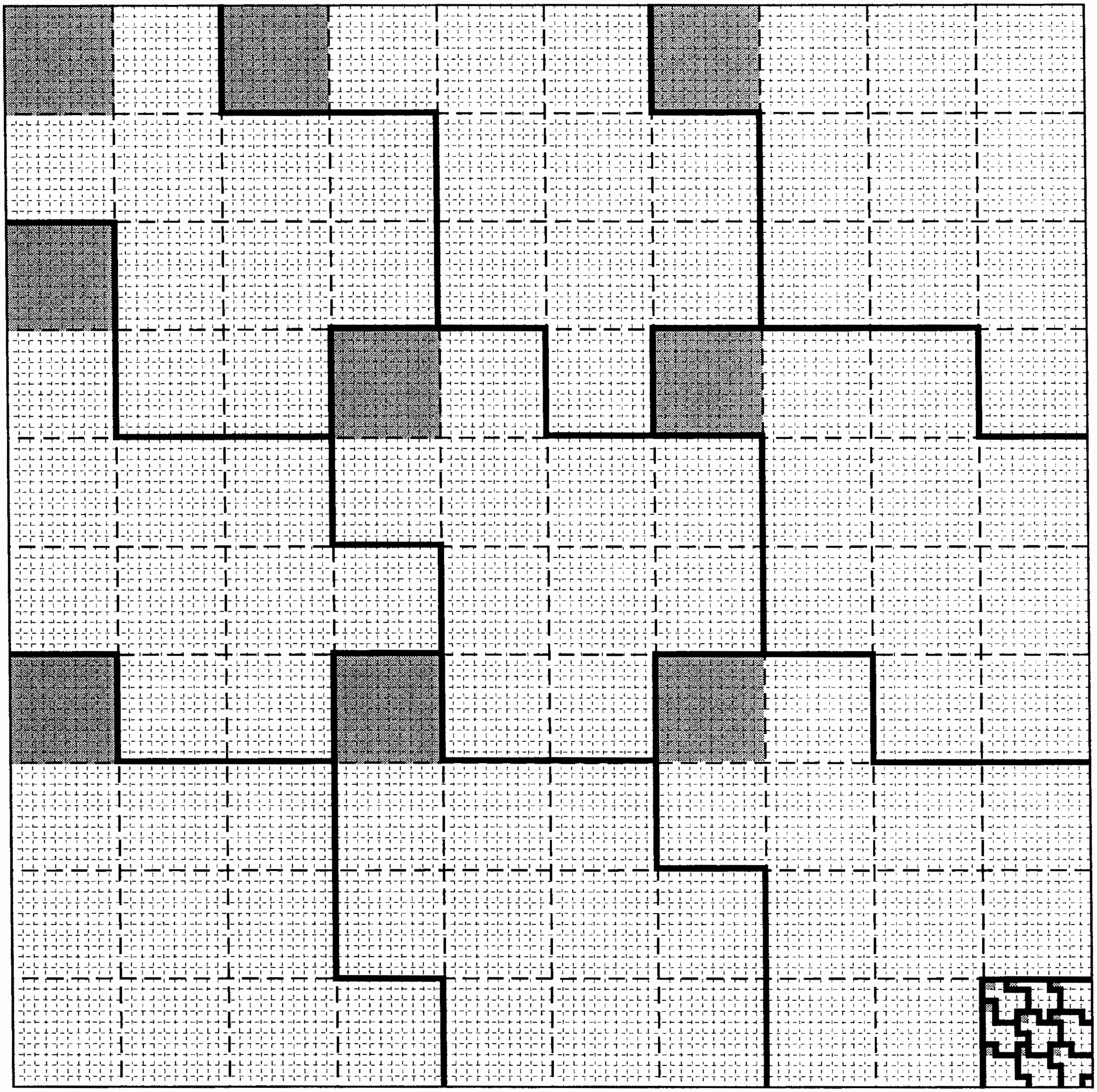


$\frac{1}{7} = 0.\overline{142857}$

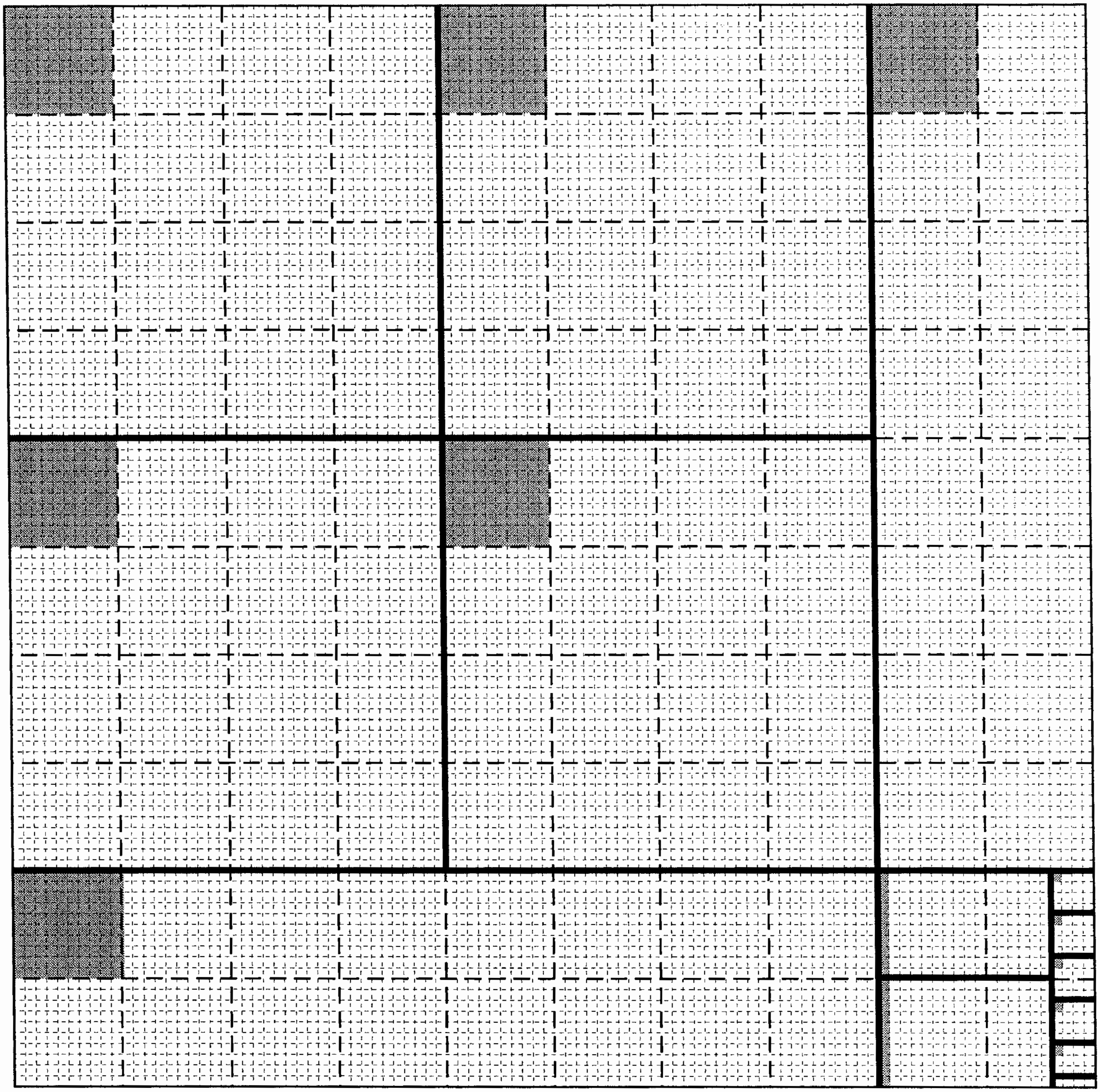




$\frac{1}{9} = 0.1111\ldots$

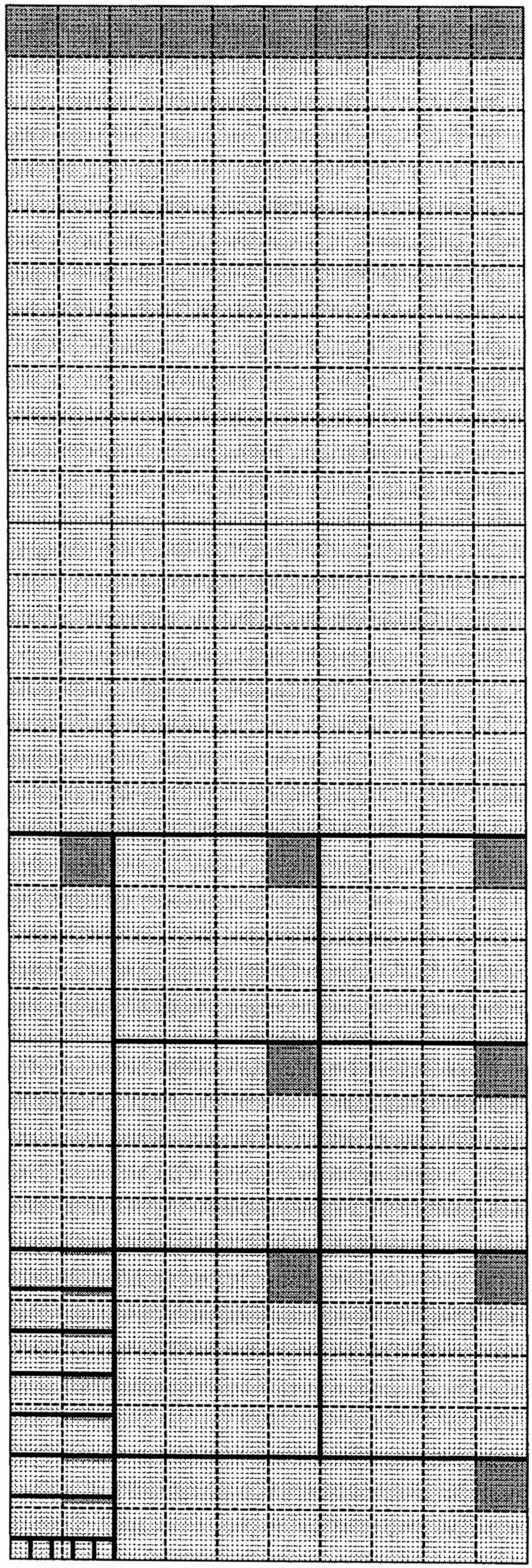


$\frac{1}{11} = 0.0909\ldots$



$$\frac{1}{16} = 0.0625$$

$\frac{3}{16} = 0.1875$



Fraction Sums & Differences

O V E R V I E W

Procedures are developed for finding the sums and differences of algebraic fractions, based on area properties of rectangles.

Prerequisite Activity

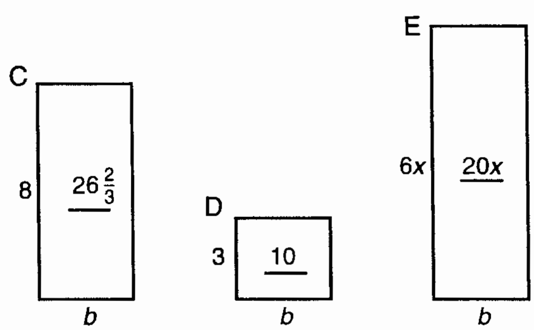
Unit VI, *Modeling Integers*.

Materials

Activity Sheet XII-3 and masters.

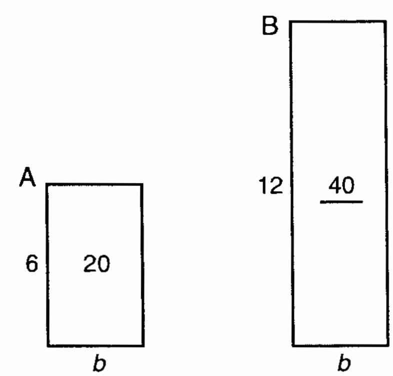
Actions

1. Distribute Activity Sheet XII-3 to the students. Ask them to find the missing values. Discuss the methods the students use and any observations they have. Summarize properties of rectangles illustrated by the sheet.



Comments

1. You may have to emphasize that the entries in the blanks are not to involve *b*. Some students may, for example, write 12*b* as the area of rectangle B. However, one can determine a numerical value for the area of B: since it has the same base as rectangle A and a height twice that of A, the area of B will be twice the area of A, or 40.



Some students may compute the value of *b* and use that to find the requested entries. This can be avoided by comparing the relative sizes of the rectangles. Note, for example, that rectangles C, D and E all have the same base as A. However, their heights are, respectively, $1\frac{1}{3}$, $\frac{1}{2}$ and x times the height of A. Hence, their areas will be, respectively, $1\frac{1}{3}$, $\frac{1}{2}$ and x times the area of A.

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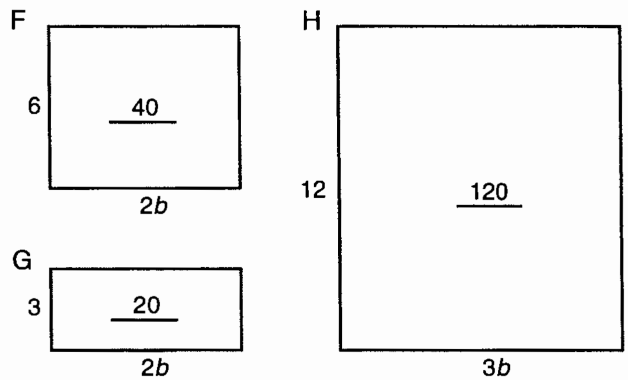
Actions

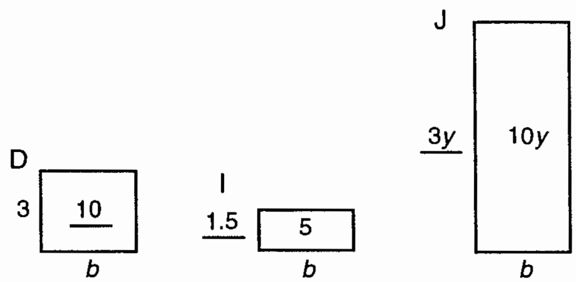
Comments

1. *Continued.* The height of rectangle F is the same as that of A, however its base is twice as long. Thus, its area is twice that of A.

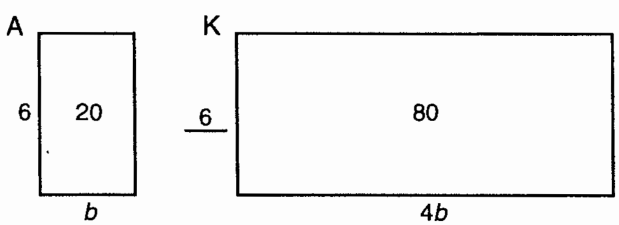


Rectangle G has the same base as F, but half the height, hence it has half the area of F. Alternately, G has half the height and twice the base of A. Since halving one dimension of a rectangle and doubling the other leaves its area unchanged, G has the same area as A.

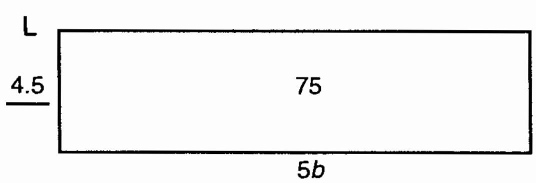
The area of H can be found by comparing it either to B or A: since H has the same height as B but 3 times the base, its area is 3 times that of B; or, noting that its height is 2 times the height of A and its base is 3 times the base of A, its area is 2 × 3, or 6, times the area of A.



Rectangle I has the same base as D but half D's area. Hence, its height is half that of D. Rectangle J has the same base as I but its area is 2*y* times that of I. Thus, its height is 2*y* times that of I.



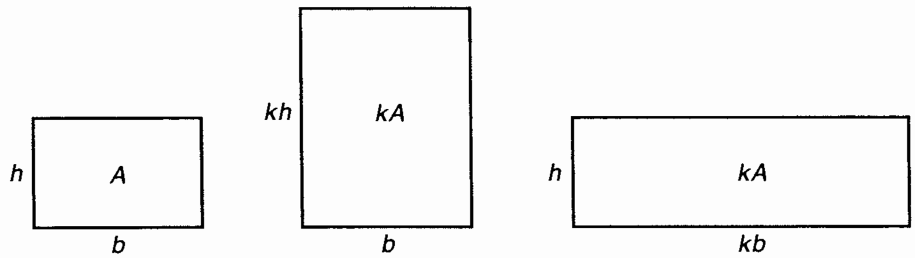
The base of K is 4 times that of A, so is its area. Thus, K has the same height as A.

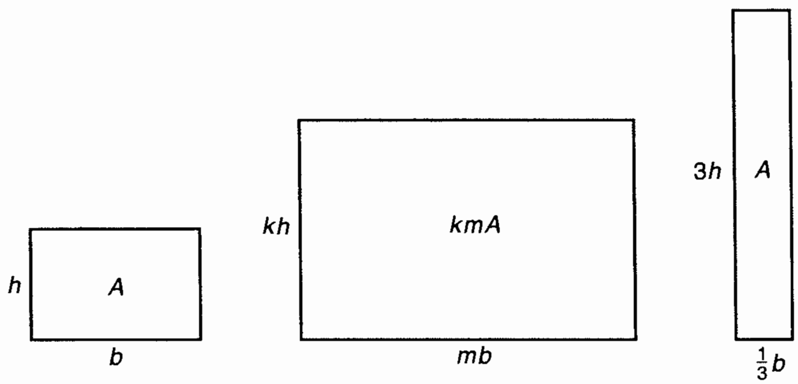


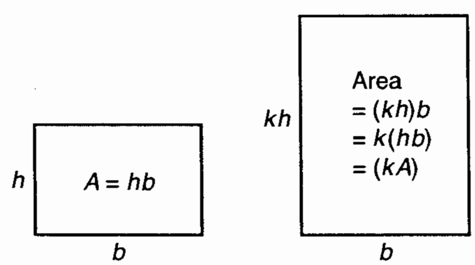
One way to find the height of L is to note that if the base of A were increased by a factor of 5, its area would be 100; however, the area of L is only ¾ of 100, so its height is ¾ that of A. Alternately, if the height of L is kept unchanged and its area is diminished by a factor of 5, the result is a rectangle of base *b* and area 15. This is ¾ the area of A, so the height of the diminished rectangle and, hence, the height of L is ¾ that of A.

Continued next page.

Actions







Comments

1. *Continued.* Some properties of rectangles illustrated by this activity:

- If one dimension of a rectangle is kept fixed and the other dimension is changed by a factor of k, the area is changed by the same factor.

- Similarly, if one dimension of a rectangle is kept fixed and the area is changed by a factor of k, the other dimension is changed by the same factor.

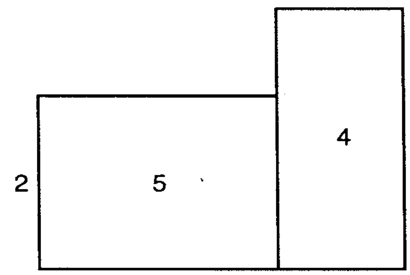
- If one dimension of a rectangle is changed by a factor of k and the other dimension is changed by a factor of m, then the area is changed by the product, km, of these factors. In particular, if one dimension is multiplied by a factor and the other dimension is divided by the same factor, the area is unchanged.

The students are likely to accept these properties without further proof. However, if the students are interested, or it is thought to be instructive, proofs can be constructed based on the proposition that the area of a rectangle is the product of its dimensions. For example, the first property can be established by noting that if a rectangle has dimensions h and b and area A, then $A = hb$. If one of these dimensions, say h, is multiplied by k and a new rectangle is formed whose dimensions are kh and b, this new rectangle has area $(kh)b = k(hb) = kA$. Thus, the new rectangle has an area which is k times the original area.

If the students question the equality of the products $kh(b)$ and $k(hb)$, it may be appropriate to carry out the Action in the appendix, *Arithmetical Properties*, found at the end of this activity.

Actions

2 Show the students the following sketch of *adjacent rectangles*. Ask for a volunteer to record the length of the bases of the rectangles on the sketch.



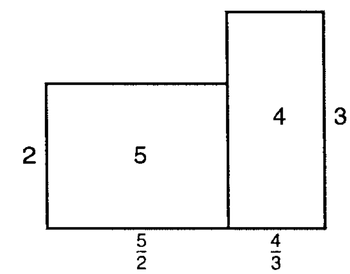
3. Ask the students to sketch the pair of adjacent rectangles obtained by expanding the height of the first rectangle in Action 2 by a factor of 3 and the second by a factor of 2, while leaving the bases unchanged. Note that the resulting pair of rectangles can be combined into a single rectangle. Ask the students to determine the area and dimensions of this rectangle. Discuss the methods they use and any observations they make.

Comments

2. A transparency of the rectangles can be made from Master 1, found at the end of this activity.

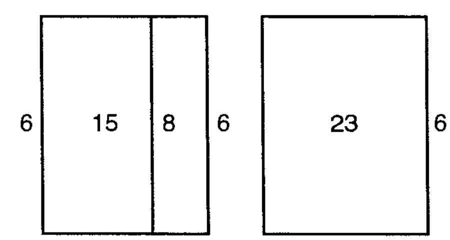
Two rectangles are *adjacent* if their bases are collinear and adjoining. Adjacent rectangles look like side-by-side buildings.

Since the first rectangle has area 5 and one dimension 2, its other dimension is $5 \div 2$ or $5/2$. The length of the base of the second rectangle is $4/3$.



3. It is not critical that the students' sketches are drawn to scale; rough sketches will do.

Since the height of the first rectangle has been tripled, its area has been tripled. Since the height of the second rectangle has been doubled, its area has been doubled. The



area of the combined rectangle is 23. Its height is 6. Hence, the length of its base is $23 \div 6$ or $23/6$. (A transparency of the expanded rectangles can be made from Master 2. The left side of the master shows the expanded rectangles. The right side shows the combined rectangles)

Note that the length of the base of the combined rectangle is the sum of the lengths of the bases of the original rectangles. Thus, $5/2 + 4/3 = 23/6$.

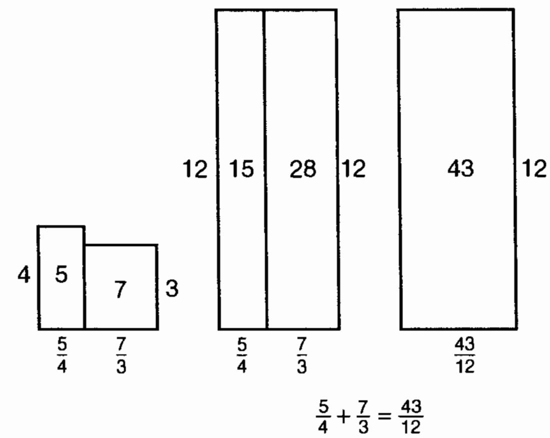
Some students may find the length of the base of the combined rectangle by adding the lengths of the bases of the original rectangles, using previous methods they have learned for adding fractions. Note, however, that the base of the combined rectangle can be found without any knowledge of algorithms for adding fractions.

Actions

4. Ask the students to sketch adjacent rectangles, with integral areas and heights, whose bases have lengths $5/4$ and $7/3$, respectively. Then ask them to expand the heights of these rectangles to form a single rectangle from which they can determine the sum $5/4 + 7/3$.

Comments

4. In the sketches shown here, a rectangle of area 5 and height 4 is drawn adjacent to a rectangle of area 7 and height 3. The height of the first rectangle is expanded by a factor of 3 and the height of the second rectangle is expanded by a factor of 4. Combined, the expanded rectangles form a rectangle of height 12 and area 43. Hence, $5/4 + 7/3 = 43/12$.

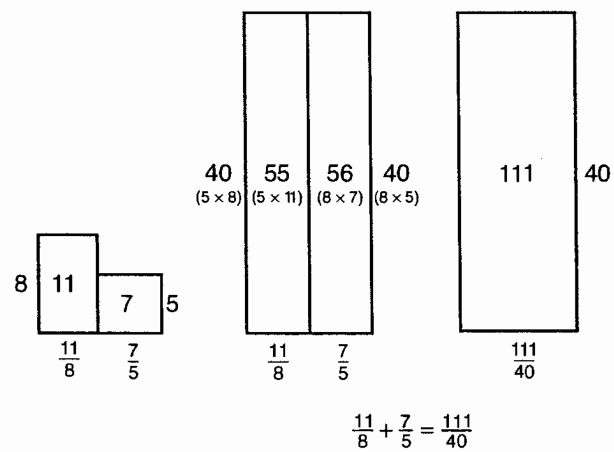


Actions

5. Ask the students to us the method of Action 4 to determine the sum $11/8 + 7/5$. Tell the students that their sketches need not be drawn to scale.

Comments

5. Possible sketches are shown below. The sketches are not drawn to scale.

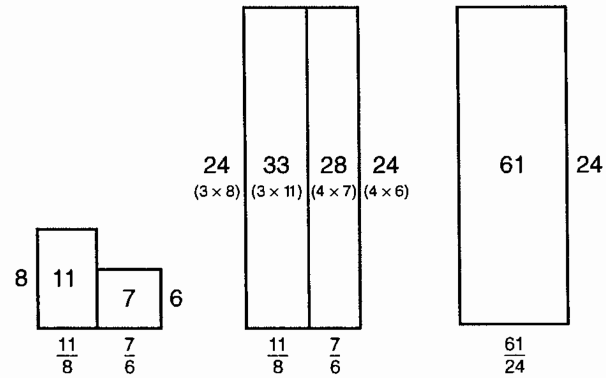


Actions

6. Repeat Action 5 for the sum $^{11}/_{8} + ^{7}/_{6}$.

Comments

6. Shown below, on the left, are adjacent rectangles whose bases have lengths $^{11}/_{8}$ and $^{7}/_{6}$, respectively. Expanding the height of the first rectangle by a factor of 3 and that of the second by a factor of 4 results in adjacent rectangles with a common height of 24. Note that 24 is the least common factor of 8 and 6 or, to put it another way, 24 is the smallest height which is an integral multiple of both 8 and 6.

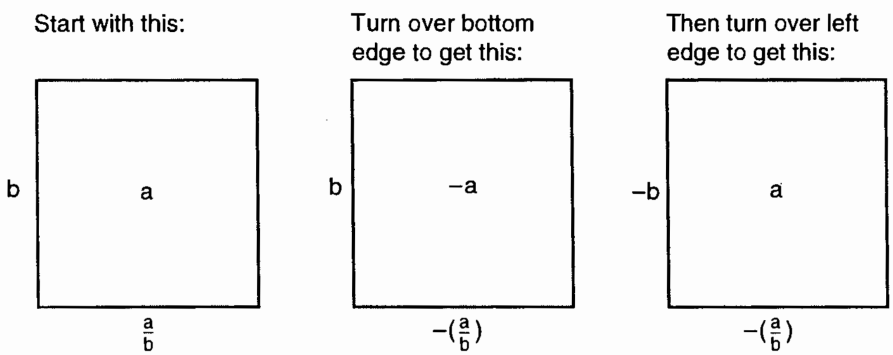


$$\frac{11}{8} + \frac{7}{6} = \frac{61}{24}$$

Some students may obtain adjacent rectangles of common height by expanding the height of the first rectangle by a factor of 6 and the height of the second by a factor of 8. These students are likely to arrive at an equivalent form of the sum, namely, $^{122}/_{48}$.

Actions

7. Discuss with the students how determining the sum of fractions by finding the sum of the bases of adjacent rectangles can be adapted to determine the difference of fractions, such as $^3/_2 - {^1/_3}$.

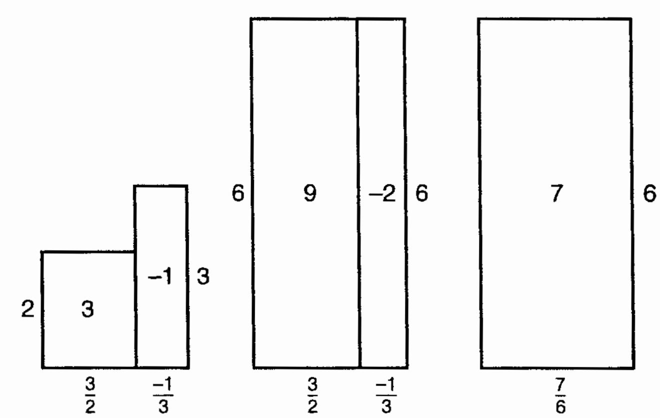


Comments

7. The method can be adapted to subtraction of fractions by considering the values of rectangular regions and their edges, rather than areas and lengths. It is also helpful to know that the opposite of a fraction a/b, namely, $-(a/b)$, is equal to $(-a)/b$, as indicated in the sequence of sketches.

The sketch on the left represents a rectangular region with edges. The region and edges have values as shown. Edges are "turned over" as in the black/red counting piece model. (See Unit VI, *Modeling Integers*.) Recall that "turning over" changes a value to that of its opposite and if an edge is turned over, the region must also be turned over. Since the value of the bottom edge is also the value of the region divided by the value of the left edge, the last two sketches show, respectively, that $-(a/b) = (-a)/b$ and $-(a/b) = a/(-b)$.

Since the difference $x - y$ has the same value as the sum of x and the opposite of y, the value of $3/2 - 1/3$ is the same as the sum of $3/2$ and the opposite of $1/3$. As indicated above, the opposite of $1/3$ is equal to $^-1/3$. Hence, $3/2 - 1/3 = 3/2 + {^-1/3}$. This sum can be determined with the aid of the following sketch. The numerals in the sketch represent values.



$$\frac{3}{2} - \frac{1}{3} = \frac{3}{2} + \frac{-1}{3} = \frac{7}{6}$$

The students may have other methods of adapting sketches of adjacent rectangles to determine the difference of fractions.

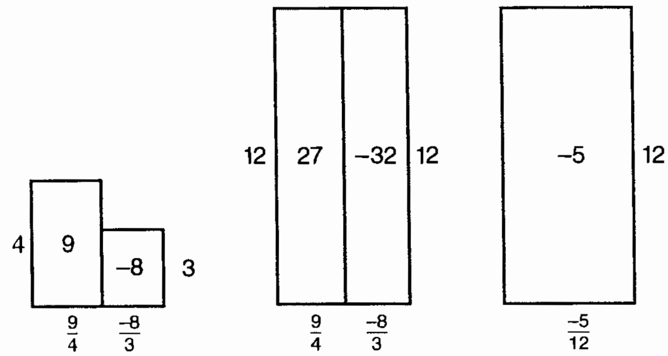
Actions

8. Ask the students to express each of the following as a single fraction:

(a) $9/4 - 8/3$, (b) $x/3 + 1/2$, (c) $2x/5 - 3/4$.

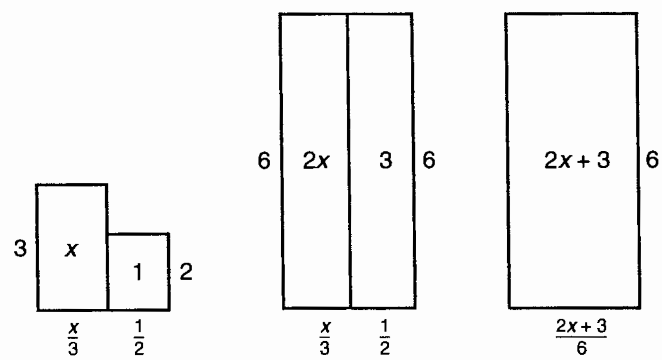
Comments

8. (a) The value of the combined bases of the adjacent rectangles shown on the left below is $9/4 - 8/3$. Expanding the height of the first rectangle by a factor of 3 and that of the second by a factor of 4, results in a pair of adjacent rectangles which combine to form a rectangular region whose value is –5 and has a height whose value is 12. Thus, the value of the combined bases is $-5/12$.



$$\frac{9}{4} - \frac{8}{3} = -\frac{5}{12}$$

(b) Adjacent rectangles can be expanded, as shown below, without changing the values of their bases. The expanded rectangles form a single rectangular region whose value is $2x + 3$ and height is 6.



$$\frac{x}{3} + \frac{1}{2} = \frac{(2x+3)}{6}$$

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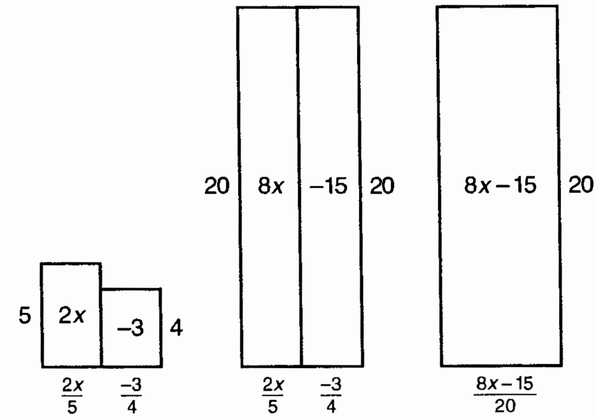
Actions

Comments

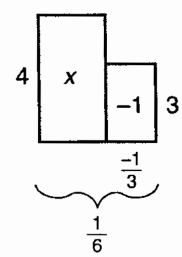
8. *Continued.*
(c) Adjacent rectangles can be expanded and combined into a single rectangular region as shown below.



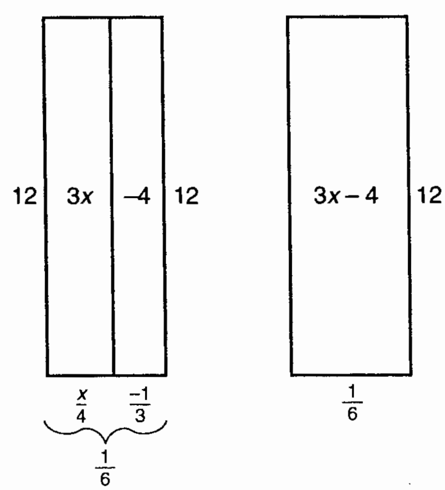
$$\frac{2x}{5} - \frac{3}{4} = \frac{(8x-15)}{20}$$

9. Ask the students to solve the following equations:
(a) $x/4 - 1/3 = 1/6$ (b) $3x/5 + x/3 = 2$ (c) $7/3 + 4/x = 3$

9. (a) The sum of the values of the bases of the adjacent rectangles shown below is $\frac{1}{6}$. Since the value of the base of the rectangle on the right is $-\frac{1}{3}$, or $-\frac{2}{6}$, the value of the base of the rectangle on the left is $\frac{1}{6} + \frac{1}{3}$, or $\frac{1}{2}$. The value, x, of the left rectangle is the product of the values of its edges. Hence, $x = 4(\frac{1}{2})$ or 2.



Alternately, the height of the left rectangle can be expanded by a factor of 3 and that of the right by a factor of 4. The result is two adjacent rectangles that combine to form a single rectangular region whose value is $3x - 4$ and has edges whose values are 12 and $\frac{1}{6}$. Hence, $3x - 4 = 12(\frac{1}{6}) = 2$. In which case $3x$ must be 6 and, thus, x is 2.



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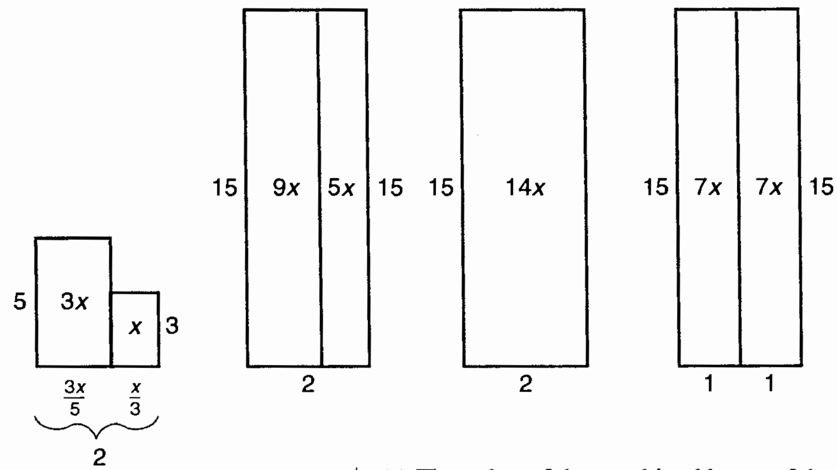
Actions

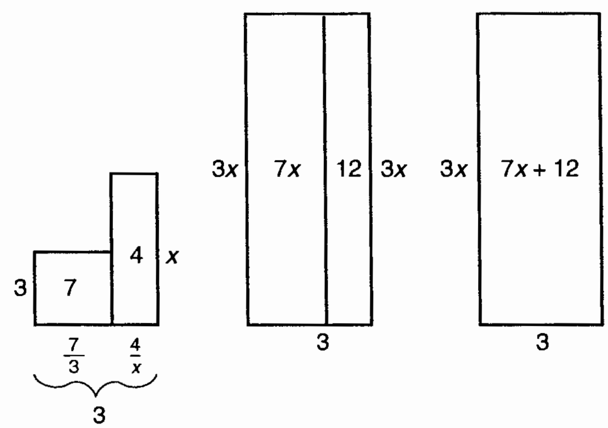
Comments

9. *Continued.*

(b) The heights of the adjacent rectangles shown on the left below can be expanded—the first height by a factor of 3 and the second by a factor of 5—to form a single rectangular region whose value is $14x$ and has edges whose values are 15 and 2. Hence, $14x = 30$ and $x = 30/14 = 15/7$. (Note that the rectangular region can be split into two rectangular regions each of value $7x$ and edges of values 15 and 1.)



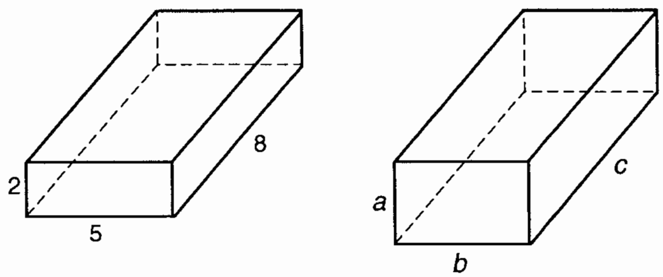
(c) The value of the combined bases of the adjacent rectangles shown on the left below is 3. If the height of the first rectangle is expanded by a factor of x and that of the second by a factor of 3, the resulting rectangles can be combined into a single rectangular region. The value of this region is $7x + 12$. On the other hand, since its edges have values 3 and $3x$, its value is $9x$, or $7x + 2x$. Hence, $12 = 2x$ and $x = 6$.

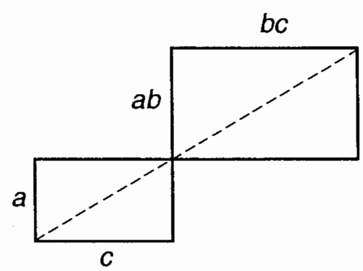


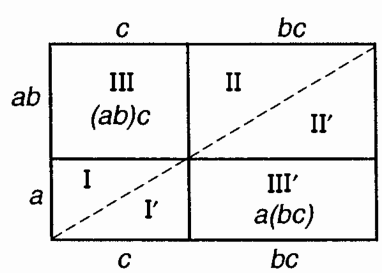
Appendix: Arithmetical Properties

Actions

Show the students a sketch of the solid on the left below. Ask them to compute its volume and describe how they arrived at their answers. Then discuss the ways in which the volume of the solid on the right can be found.







Comments

The volume of the solid on the left is the product of 2, 5 and 8. The order in which these numbers are multiplied is immaterial. For example, the volume can be found by first multiplying 2 and 5 to determine the area of the front face of the solid and then multiplying by 8, that is, by computing $(2 \times 5) \times 8$ where it is understood the product in the parentheses is computed first. Or the volume could be found by multiplying the height, 2, by the area of its base, 5×8; that is, by computing $2 \times (5 \times 8)$. There are other orders in which the multiplications can be made, e.g., $(8 \times 5) \times 2$ and $(2 \times 8) \times 5$.

The volume of the solid on the right can be found by finding the product of a, b and c in any order. Two possible orders are $(a \times b) \times c$ and $a \times (b \times c)$ Thus, $(a \times b) \times c = a \times (b \times c)$. This relationship is called the *associative law for multiplication.*

The associative law for multiplication can also be viewed as follows. Sketch a rectangle whose dimensions are a and c. Change both dimensions by a factor of b to obtain a new rectangle which is placed corner to corner with the original rectangle as shown.

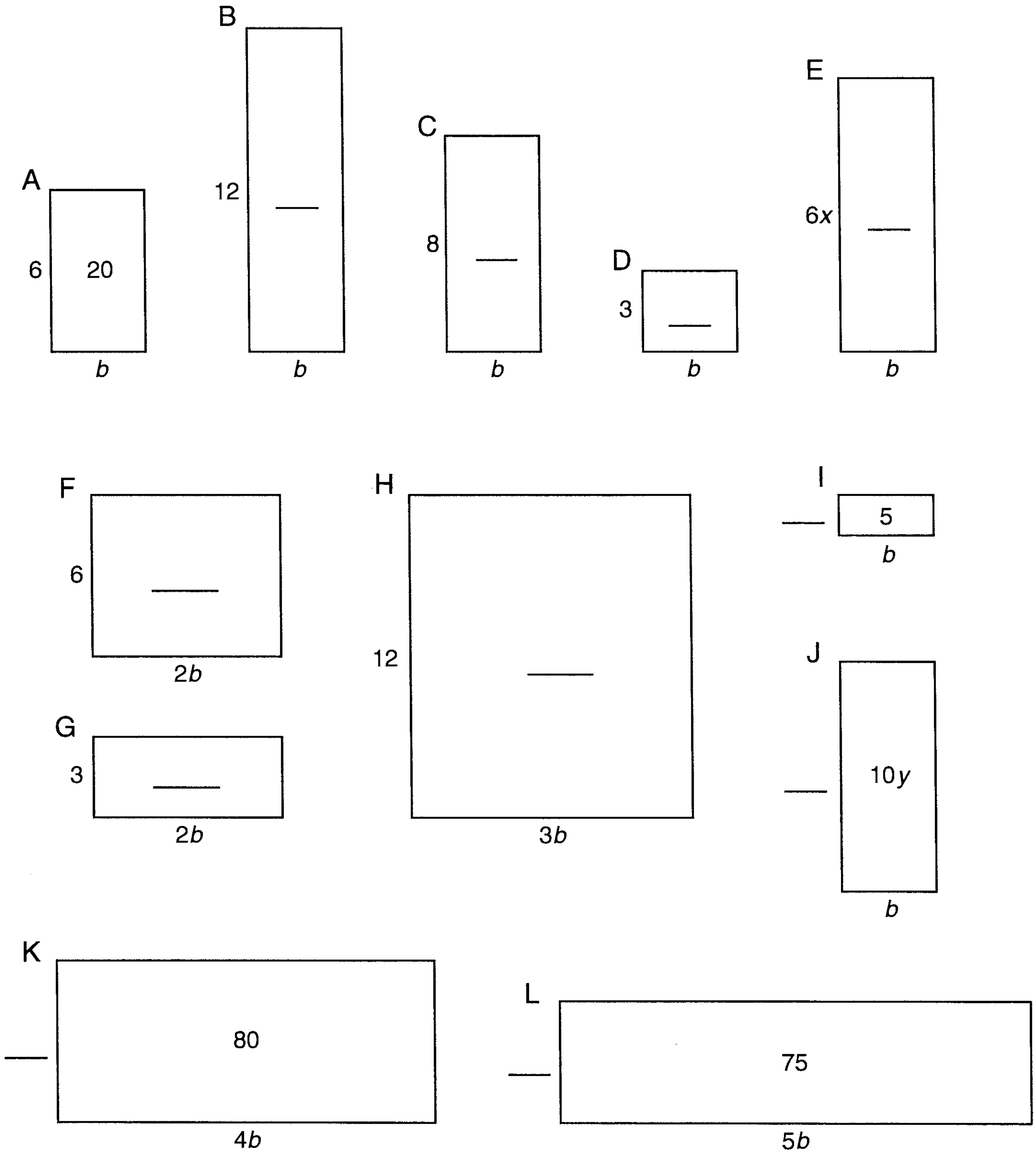
Since one rectangle is an enlargement of the other, their diagonals are collinear. The combined diagonals form the diagonal of a larger rectangle. This diagonal divides the larger rectangle into two triangles of equal area, one composed of regions I, II and III; the other composed of regions I′, II′ and III′. Since regions I and I′ and II and II′ have the same areas, so will regions III and III′. The area of region III is $(ab)c$ and the area of region III′ is $a(bc)$. Thus, $(ab)c = a(bc)$.

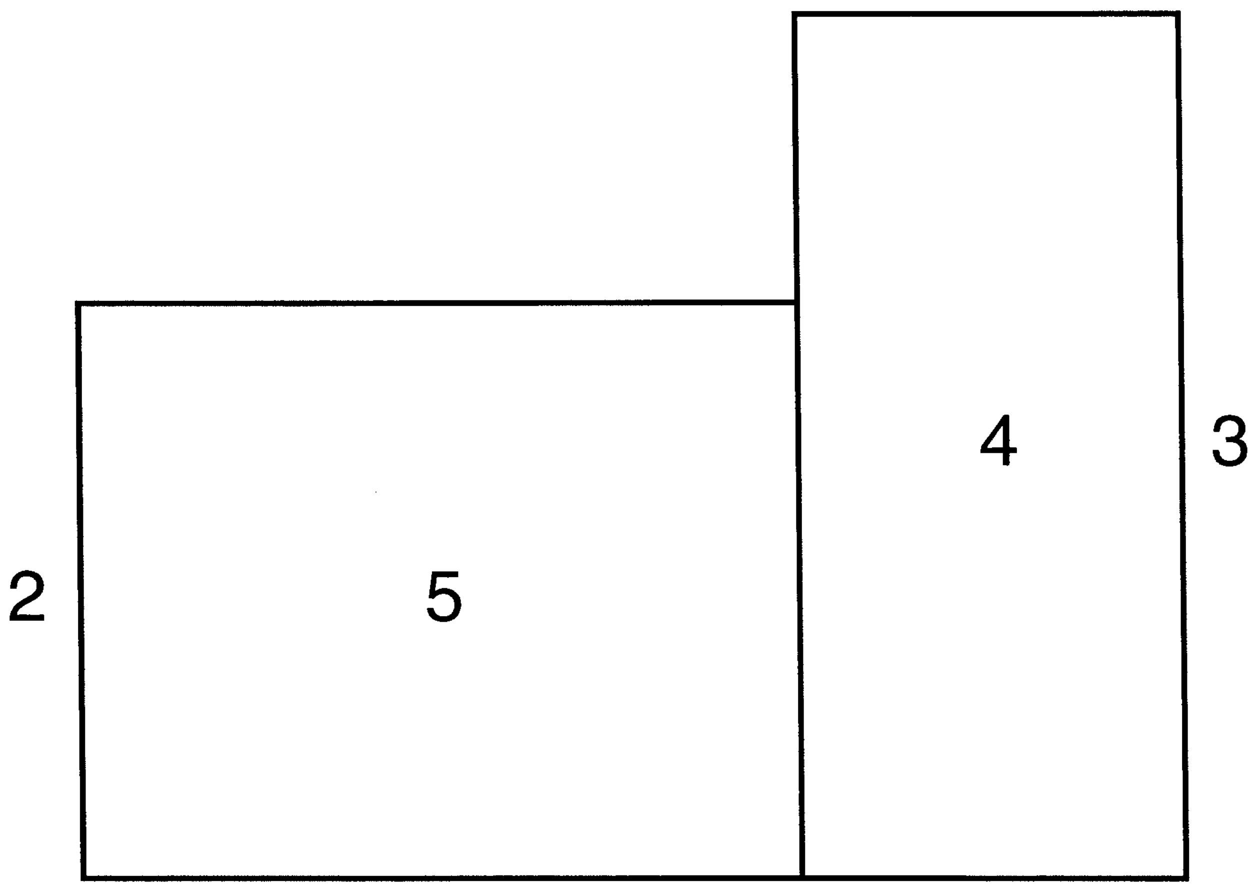
The sheet, *Picturing Properties*, (Master 3) attached to this activity, depicts the common arithmetical properties. On this sheet, $a + b$ is pictured as the length of a line segment composed of a line segment of length a followed by a segment of length b; ab is pictured as the area of a rectangle of height a and base b.

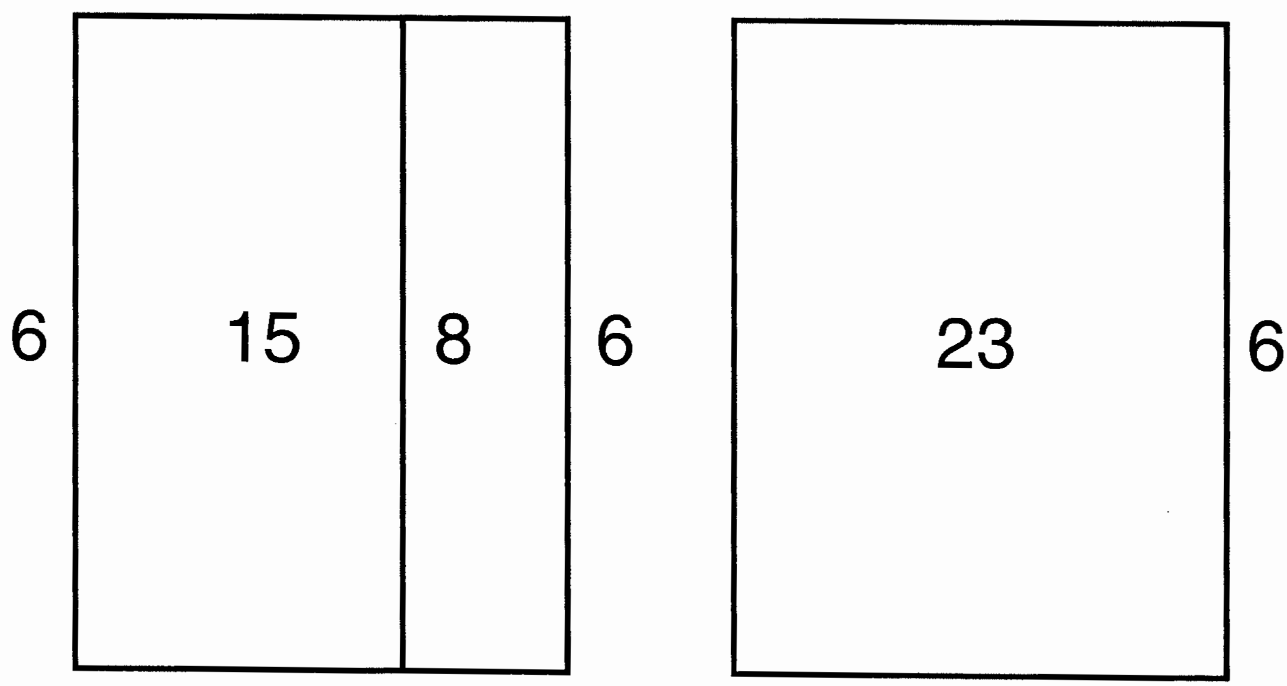
Activity Sheet XII-3

Name ______________________________

Rectangle A has area 20, height 6 and base *b*. For the other rectangles, either the area or the height is missing. Fill in the missing area or height with either a numerical value or an algebraic expression that does not involve *b*.

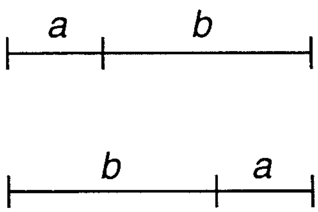



2
5
4
3

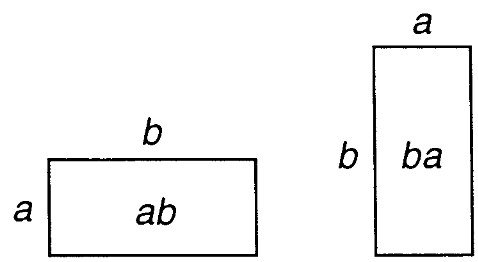

6
15
8
6
23
6

Picturing Properties

I. Commutative Properties

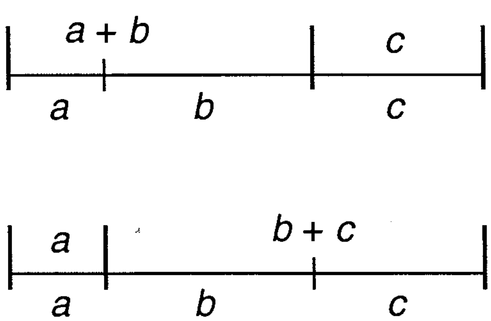


$a + b = b + a$

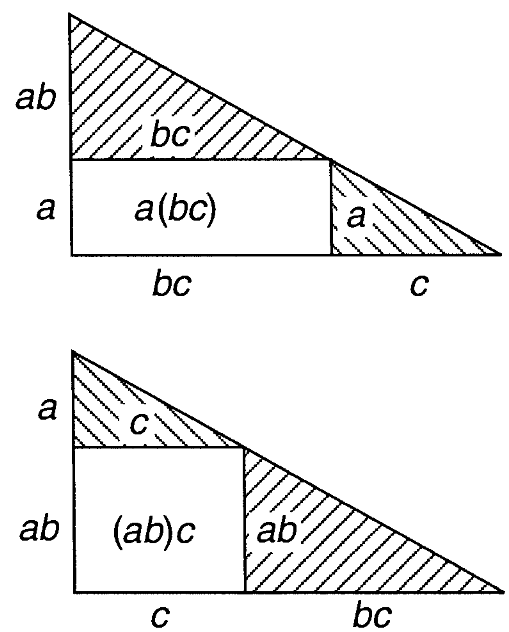


$ab = ba$

II. Associative Properties

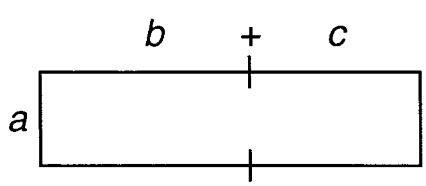


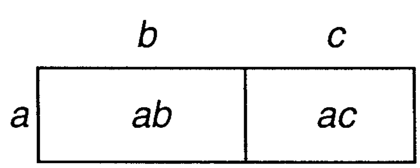
$(a + b) + c = a + (b + c)$



$a(bc) = (ab)c$

III. Distributive Properties





$a(b + c) = ab + ac$

Fraction Products & Quotients

O V E R V I E W

Procedures are developed for finding the products and quotients of algebraic fractions, based on area properties of rectangles.

Prerequisite Activity

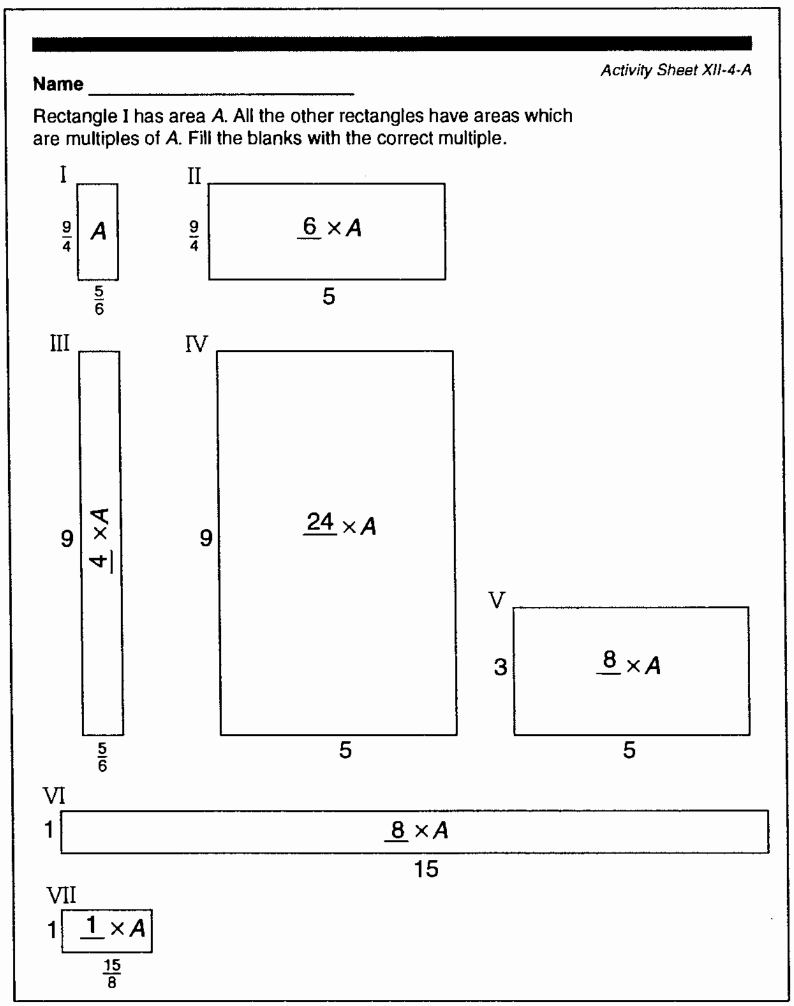
Unit XII, Activity 3, *Fraction Sums and Differences.*

Materials

Activity sheets as noted.

Actions

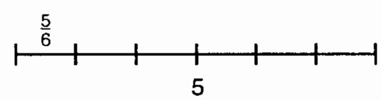
1. Distribute copies of Activity Sheet XII-4-A to the students. Ask them to complete the sheet. Ask for volunteers to describe the methods they used Then ask the students to determine the value of A. Discuss.



Comments

1. A master of the activity sheet is attached.

The completed activity sheet is shown on the left. The multiples of A can be determined by using the properties of rectangles discussed in Comment 1 of Unit VII, Activity 3, *Fraction Sums and Differences.* For example, the height of rectangle II is the same as that of rectangle I while its base is 6 times that of I, hence its area is $6 \times A$. (This argument requires that one recognizes 5 is 6 times $\frac{5}{6}$. One way to see this is to think of $\frac{5}{6}$ as 5 divided into 6 parts. Then 6 of these parts is 5.)



$$6 \times \frac{5}{6} = 5$$

The multiples of A for the remaining rectangles can be obtained by comparing its dimensions with those of a preceding rectangle. For example, the height of V is $\frac{1}{3}$ that of IV and its base is the same, hence its area is $\frac{1}{3}$ that of IV; the height of VI is $\frac{1}{3}$ that of V and its base is 3 times that of V, hence its area is the same as that of V.

Since the area of rectangle VII is A and its dimensions are 1 and $\frac{15}{8}$, it follows that A is $\frac{15}{8}$. The value of A can also be determined in other ways. For example, rectangle IV shows that $24 \times A = 45$; hence, $A = \frac{45}{24}$. Since from rectangle I, $A = \frac{9}{4} \times \frac{5}{6}$, it follows that $\frac{9}{4} \times \frac{5}{6} = \frac{15}{8} = \frac{45}{24}$.

Students who know how to multiply fractions may have found the value of A directly from rectangle I.

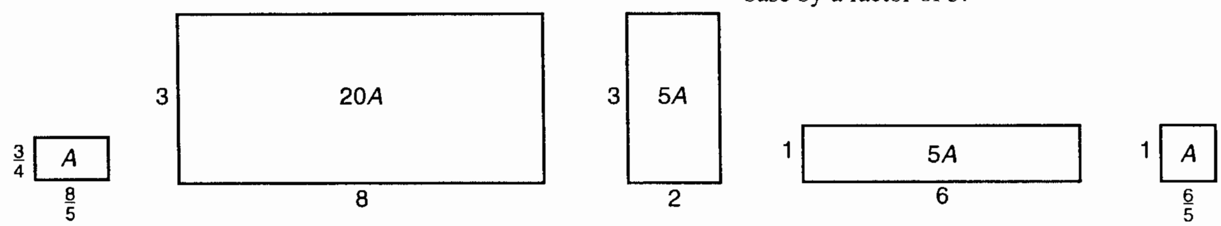
Actions

2. Ask the students to sketch a rectangle whose area is A and dimensions are $\frac{3}{4}$ and $\frac{8}{5}$. Then ask them to sketch other rectangles, similar to those illustrated on the activity sheet, from which they can easily determine the value of A. Discuss.

Comments

2. The sketches need not be drawn to scale.

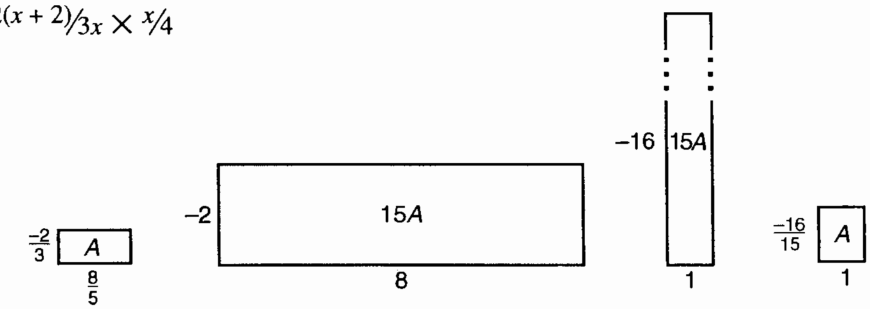
From the last sketch in the sequence of sketches shown below, one sees that $A = \frac{6}{5}$. In the sequence, the second rectangle was obtained from the first by simultaneously increasing its height by a factor of 4 and its base by a factor of 5.



3. Ask the students to draw sketches, similar to those in Action 2, from which they can determine the following products:

(a) $^{-2}/_{3} \times {}^{8}/_{5}$ (b) $^{4}/_{x} \times {}^{(x^2)}/_{6}$ (c) $^{2(x+2)}/_{3x} \times {}^{x}/_{4}$

3. (a) From the first and last sketch in the sequence below, $^{-2}/_{3} \times {}^{8}/_{5} = A = {}^{-16}/_{15}$.



$$A = \frac{-2}{3} \times \frac{8}{5} = \frac{-16}{15} \times 1 = \frac{-16}{15}$$

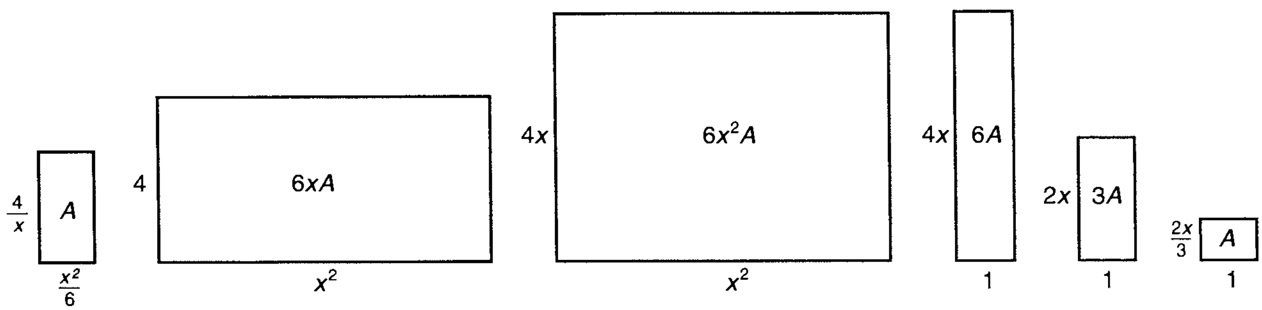
Continued next page.

Actions

Comments

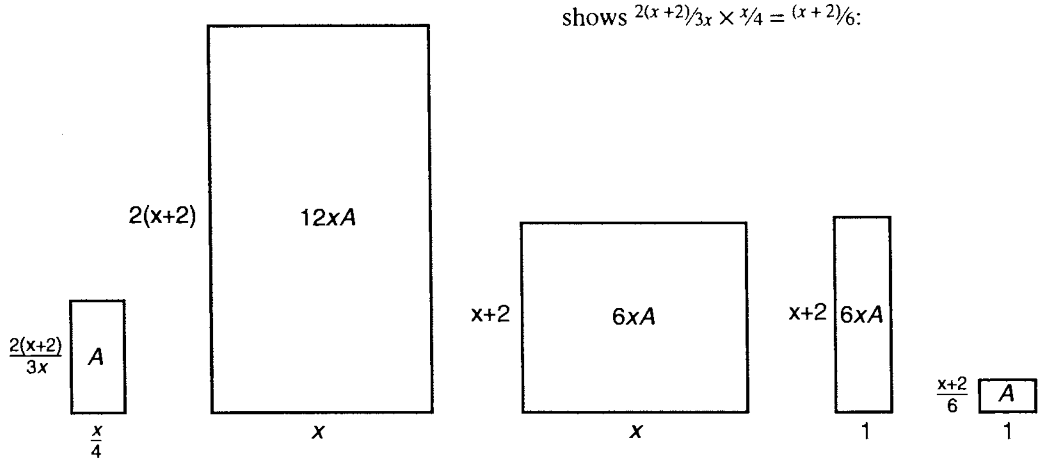
3. *Continued.*

(b) As illustrated below, $4/x \times (x^2)/6 = 2x/3$. In this sequence of sketches, the second rectangle is obtained from the first by multiplying its base by 6 and its height by x and, hence, its area by $6x$. This eliminates fractional dimensions. The third rectangle is obtained from the second by multiplying its height and area by x, providing a rectangle whose base and area can be divided by x^2 to obtain a rectangle—the fourth—which has a dimension of 1. The height and area of this rectangle is divided by 2 to get the fifth rectangle. Finally, the area and height of the fifth rectangle is divided by 3 to get a final rectangle whose area A is equal to its height. Other sequences of sketches are possible.



$$A = \frac{4}{x} \times \frac{x^2}{6} = \frac{2x}{3}$$

(c) The following sequence of sketches shows $2(x+2)/3x \times x/4 = (x+2)/6$:



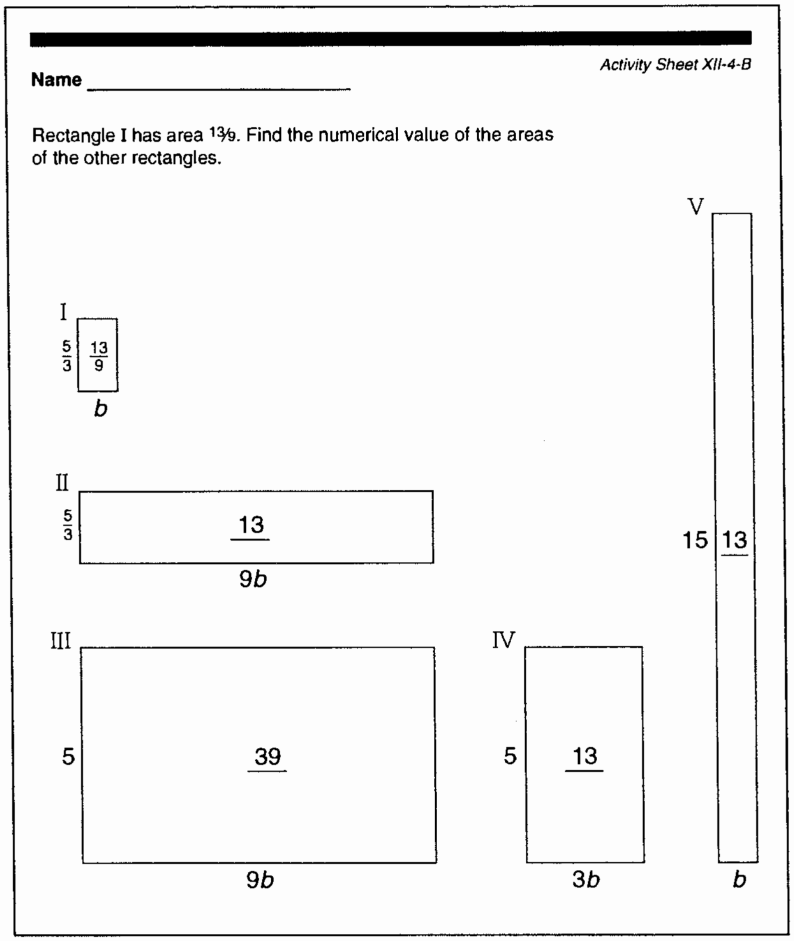
$$A = \frac{2(x+2)}{3x} \times \frac{x}{4} = \frac{x+2}{6}$$

Actions

4. Distribute copies of Activity Sheet XII-4-B to the students. Ask them to complete the sheet. Ask for volunteers to describe the methods they used. Then ask the students to determine the value of b. Discuss.

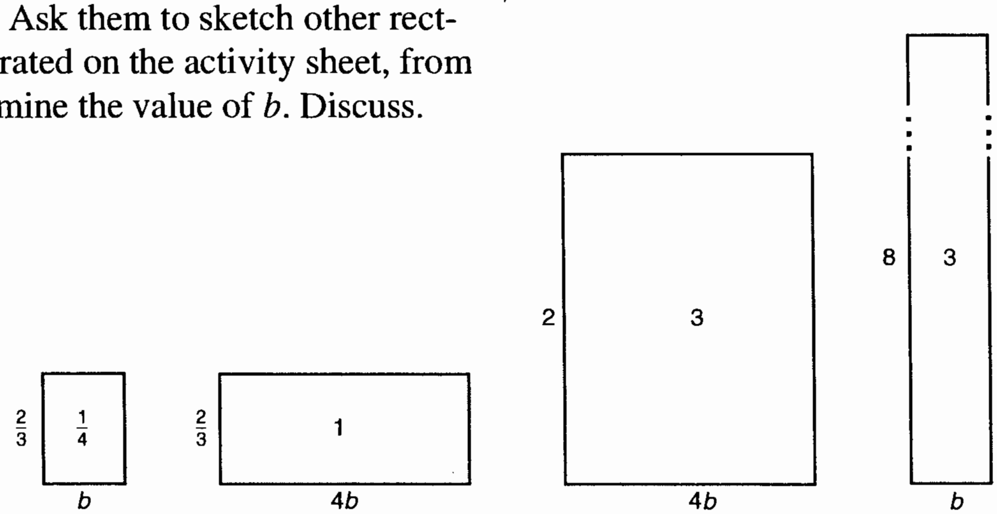
Comments

4. The completed activity sheet is shown below. From rectangle V, one sees that $b = 13/15$. Since, from rectangle I, $b = 13/9 \div 5/3$, it follows that $13/9 \div 5/3 = 13/15$.



5. Ask the students to sketch a rectangle whose area is ¼ and dimensions are ⅔ and b. Ask them to sketch other rectangles, similar to those illustrated on the activity sheet, from which they can readily determine the value of b. Discuss.

5. The sketches need not be drawn to scale.



$$b = \frac{1}{4} \div \frac{2}{3} = \frac{3}{8}$$

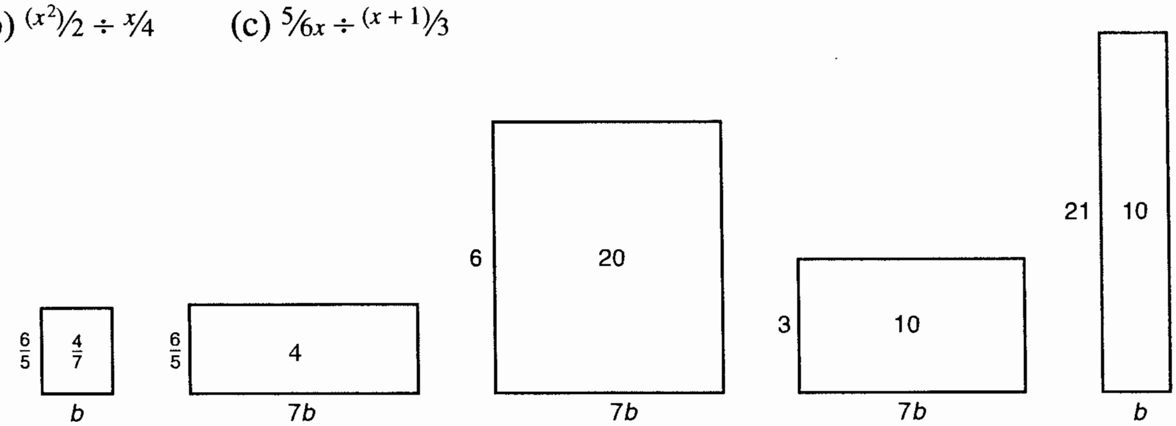
Actions

6. Ask the students to draw sketches, similar to those in Action 5, from which they can determine the following quotients:

(a) $\frac{4}{7} \div \frac{6}{5}$ (b) $\frac{(x^2)}{2} \div \frac{x}{4}$ (c) $\frac{5}{6x} \div \frac{(x+1)}{3}$

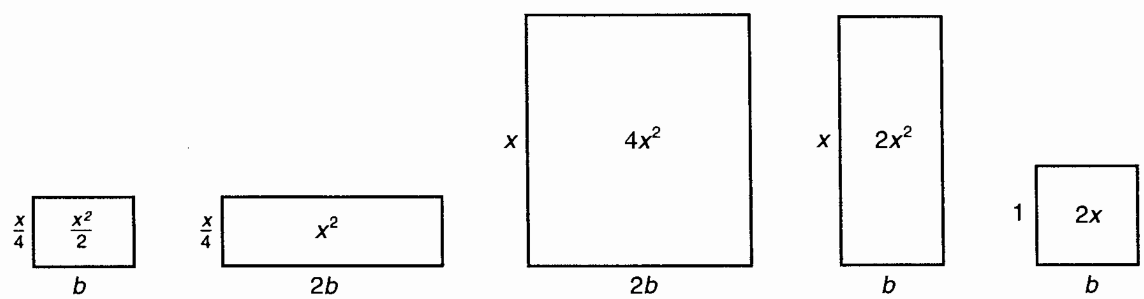
Comments

6. (a) From the following sequence of sketches, not drawn to scale, one has $\frac{4}{7} \div \frac{6}{5} = b = \frac{10}{21}$.



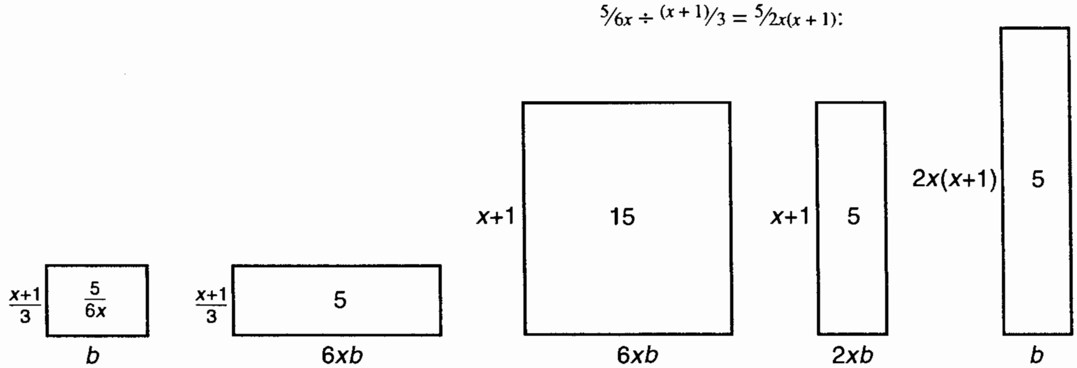
$$b = \frac{4}{7} \div \frac{6}{5} = \frac{10}{21}$$

(b) This sequence of sketches shows $\frac{(x^2)}{2} \div \frac{x}{4} = 2x$:



$$b = \frac{x^2}{2} \div \frac{x}{4} = 2x$$

(c) This sequence of sketches shows $\frac{5}{6x} \div \frac{(x+1)}{3} = \frac{5}{2x(x+1)}$:



$$b = \frac{5}{6x} \div \frac{x+1}{3} = \frac{5}{2x(x+1)}$$

Actions

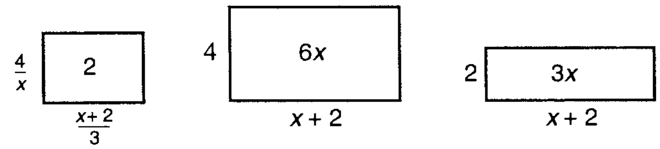
7. Ask the students to draw sketches to help them solve the following equations:

(a) $\frac{4}{x} \times \frac{x+2}{3} = 2$ (b) $\frac{x+3}{x-1} \times \frac{2}{3} = \frac{3}{4}$

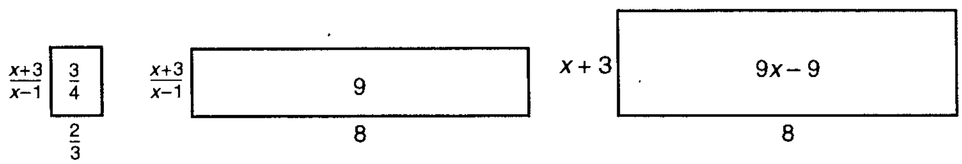
(c) $\frac{2}{x} \div \frac{3}{x+1} = 4$ (c) $\frac{2(x+1)}{3} \div \frac{8}{x+1} = 3$.

Comments

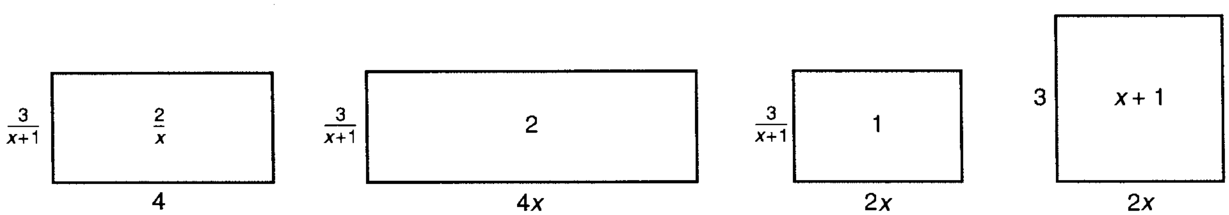
7. (a) In the sequence of sketches shown below, the second rectangle is obtained from the first by multiplying the base by 3, the height by x and, hence, the area by $3x$. The third rectangle is obtained from the second by dividing the height and area by 2. The product of the dimensions of the third rectangle is $2x + 4$ and its area is $3x$. Hence, $2x + 4 = 3x$ and, thus, $x = 4$.



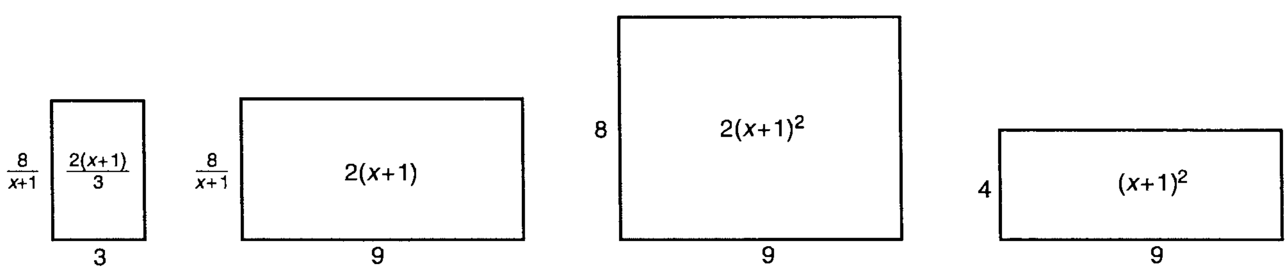
(b) The second rectangle, below, is obtained from the first by multiplying its base and area by 12. The third is obtained from the second by multiplying its height and area by $x - 1$. From the third rectangle, $8x + 24 = 9x - 9$. So, $x = 33$.



(c) From the following sequence of sketches, $6x = x + 1$. So, $x = \frac{1}{5}$.

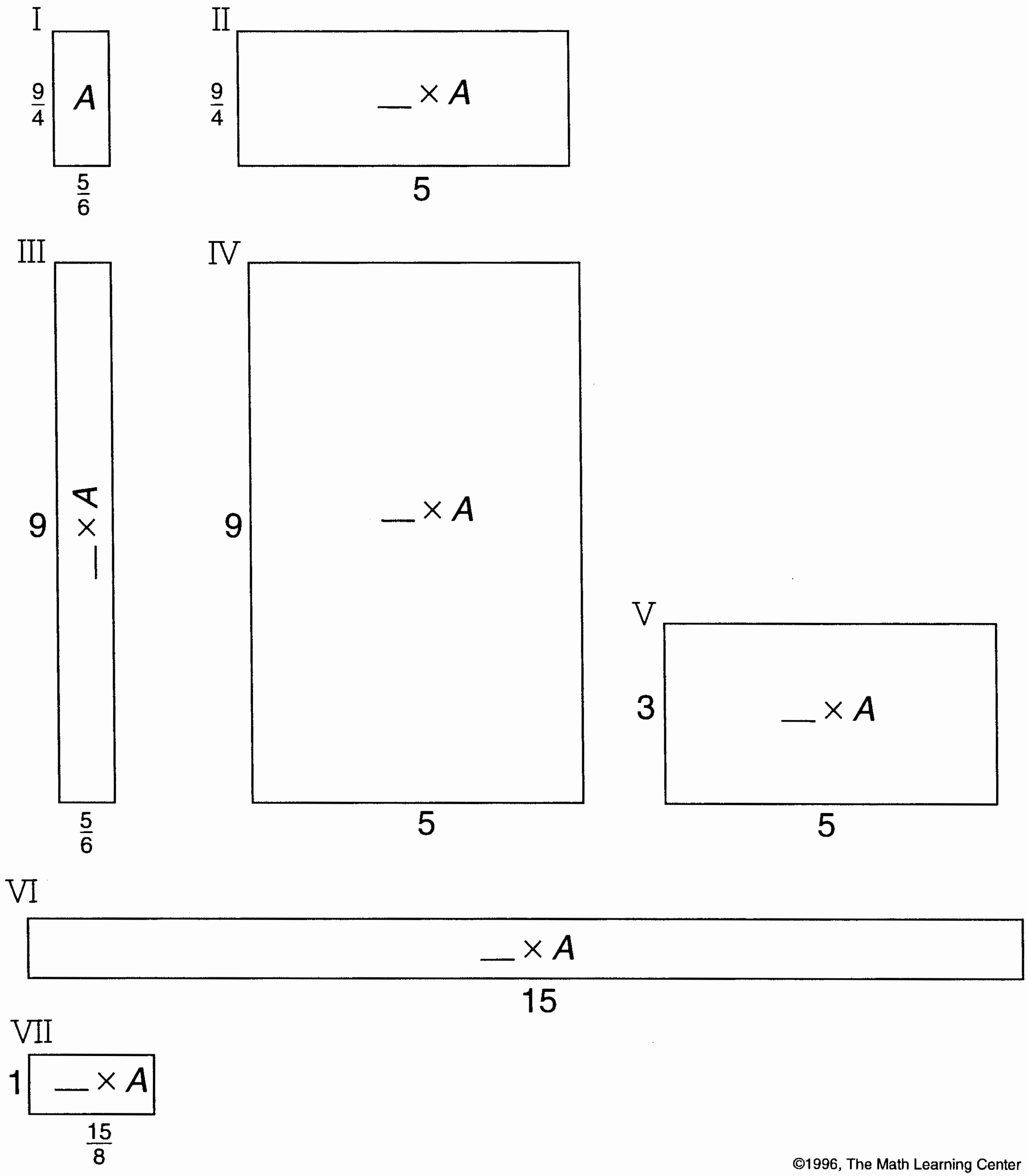


(d) The following sketches show that $(x + 1)^2 = 36$, so $x + 1$ is 6 or $x + 1$ is -6. If, $x + 1$ is 6, x is 5; if $x + 1$ is -6, x is -7. Hence, $x = 5$ or $x = -7$.



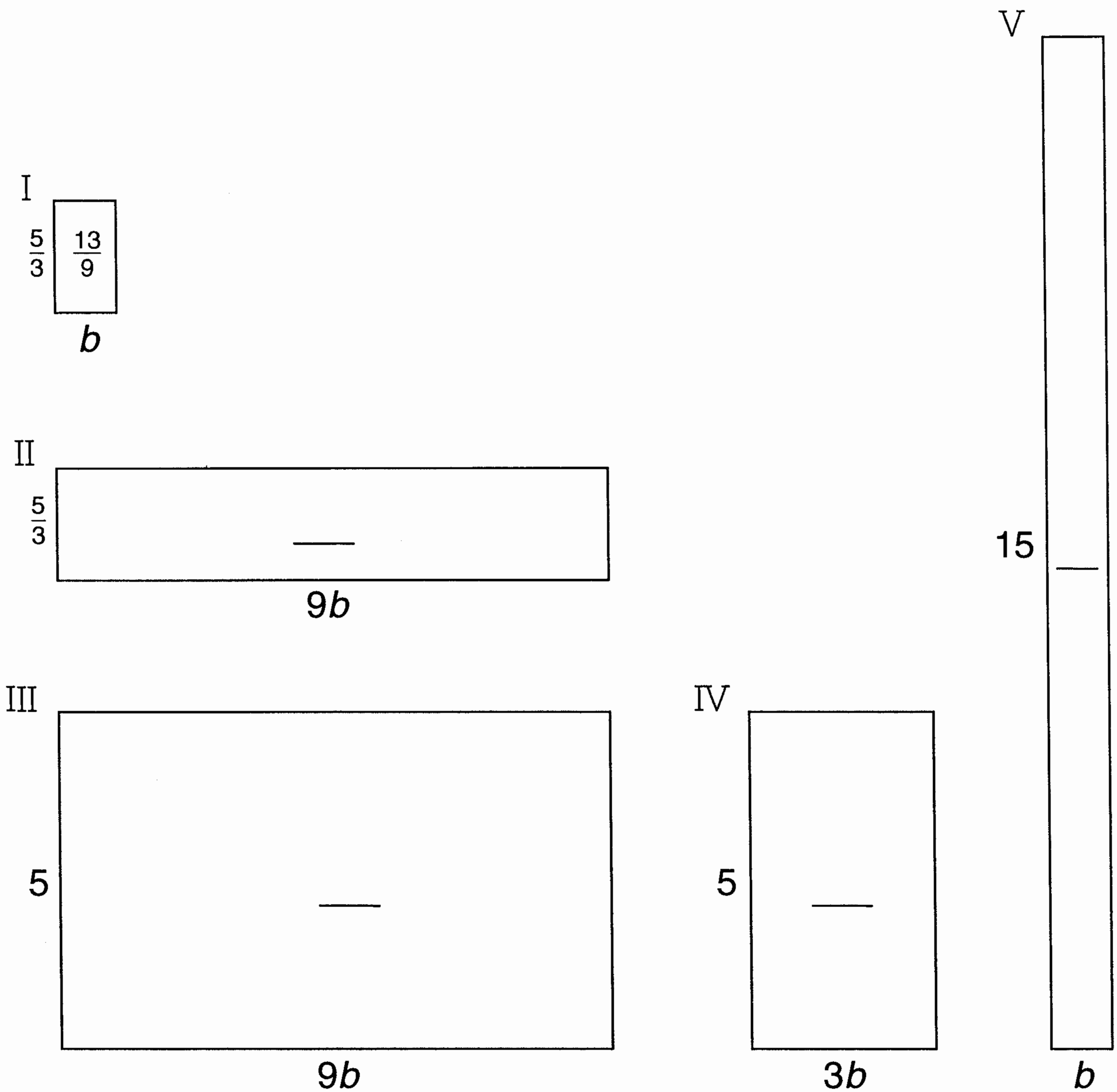
Name ______________________

Rectangle I has area *A*. All the other rectangles have areas which are multiples of *A*. Fill the blanks with the correct multiple.



Name ______________________

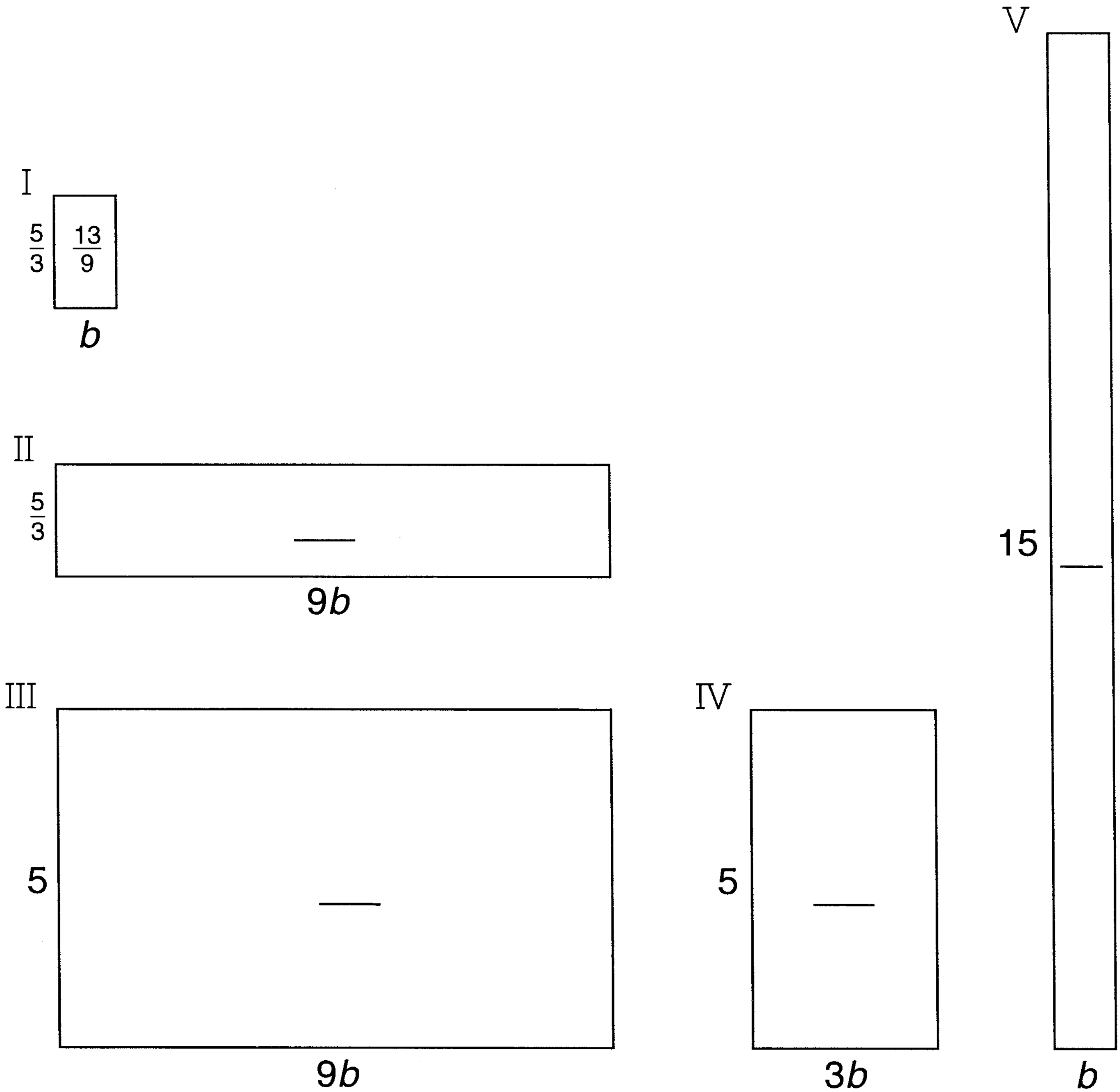
Rectangle I has area $13/9$. Find the numerical value of the areas of the other rectangles.



Name ____________________

Activity Sheet XII-4-B

Rectangle I has area $\frac{13}{9}$. Find the numerical value of the areas of the other rectangles.



Squares & Square Roots

O V E R V I E W

Methods of constructing squares of integral area are introduced. Properties of squares and square roots, including the Pythagorean Theorem, are developed.

Prerequisite Activity

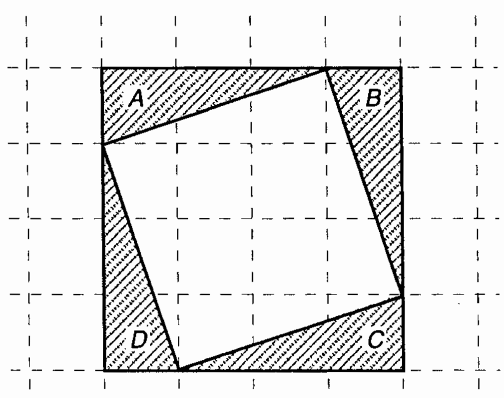
None. Unit V, Activity 2, *Geoboard Squares,* may be helpful.

Materials

Centimeter grid paper, scissors, copies of Masters and Activity Sheets as noted.

Actions

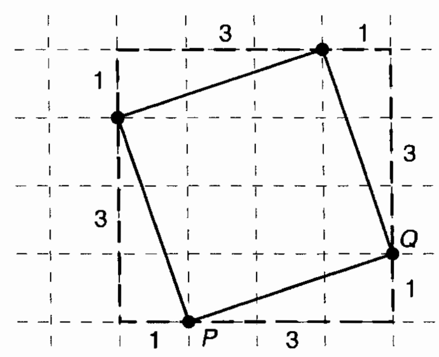
1. Distribute centimeter grid paper to the students. Tell them that 1 square represents 1 unit of area. For each of the integers 1 through 25 ask them to construct, if possible, a square whose vertices are grid intersection points and whose area is the given integer. For each square they draw, ask the students to indicate its area and the length of its side. Discuss.



Comments

1. Of the integers 1 through 25, there are 13 for which a square exists that satisfies the conditions of Action 1. Master 1 attached at the end of this activity shows all possibilities.

One way to obtain a square that fits the conditions is to pick two intersection points as successive vertices. In the instance shown below, one can get from point *P* to point *Q* by going 3 units in one direction and 1 in the other. Repeating this 3,1 pattern, as shown, results in a square.



Square generated by a 3,1 pattern.

The area of this square can be found by subtracting the area of the shaded regions from the area of the circumscribed square (see the figure). Note that regions *A* and *C* combine to form a rectangle of area 3 as do rectangles *B* and *D*. Thus, the area of the inscribed square is 16 – 6, or 10. Since the area of the square is 10, the length of its side is $\sqrt{10}$. By definition, if n is non-negative, the *square root of n,* written $\sqrt{n}$, is the length of the side of a square of area n.

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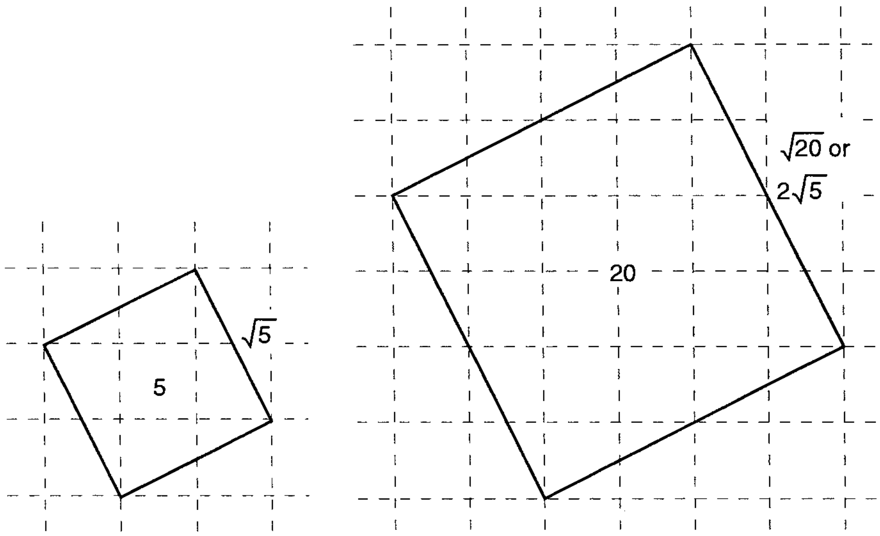
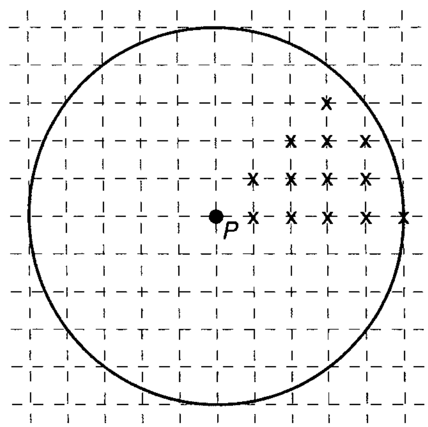
Actions



2. Have the students discuss with one another why they can be certain that the 13 integers mentioned in Comment 1 are the only ones in the range 1 through 25 for which squares exist that satisfy the conditions of Action 1.



3. Discuss with the students other approaches to constructing squares of integral area.

Comments

1. *Continued.* Shown to the left are 2 other squares, one of area 5 generated by a 2,1 pattern and the other of area 20 generated by a 4,2 pattern. Note that the side of the square of area 20 is twice as long as that of the square of area 5. Hence, $\sqrt{20} = 2\sqrt{5}$.

2. One way to see there are only 13 different areas is to note that if a square is to have area no greater than 25, then the distance between successive vertices must be less than or equal to 5. Thus, if *P* and *Q* are successive vertices, *Q* must lie on or within a circle of radius 5 whose center is at *P*. In the sketch, the 13 intersections marked with an **x** are possibilities for *Q* that lead to 13 differently sized squares. Any other choice for *Q* leads to a square the same size as one of these 13.

3. As pointed out in Action 2, not all squares of integral area can be formed using the method of Action 1. For example, a square of area 21 cannot be constructed by this method. Two methods for constructing squares of this and other areas are developed in the following Actions.

The first method involves constructing a rectangle of the desired area and then dissecting and reassembling it to obtain a square. The development of this method begins in the next Action in which students are asked, as a prelude to forming a square, to dissect a rectangle and reassemble it to form another rectangle of a given dimension.

Continued next page.

Actions

Comments

3. *Continued.* The second method involves constructing a square from 2 other squares. To use this method to construct a square of area 21, say, one starts with 2 squares whose area totals 21, squares of area 16 and 5, both of which can be constructed by the method of Action 1. This method, discussed in Actions 10 through 13, leads to the Pythagorean Theorem. Students already familiar with this theorem may suggest methods for constructing squares which depend on it.

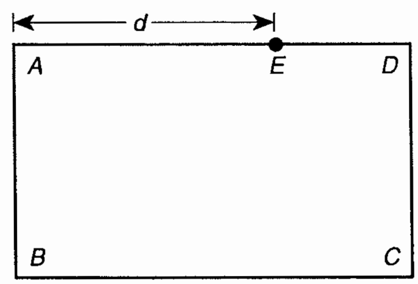
4. Distribute scissors and a copy of Master 2 to each student. Ask them to cut off the bottom portion of the page and set it aside for use in a later Action (Action 6). Then have them cut out one of the rectangles from the top portion of the page and carry out the following instructions:

a) Label one vertex of the rectangle *A*, label the adjacent vertex along the shorter side *B* and the adjacent vertex along the longer side *D*. Label the remaining vertex *C*.

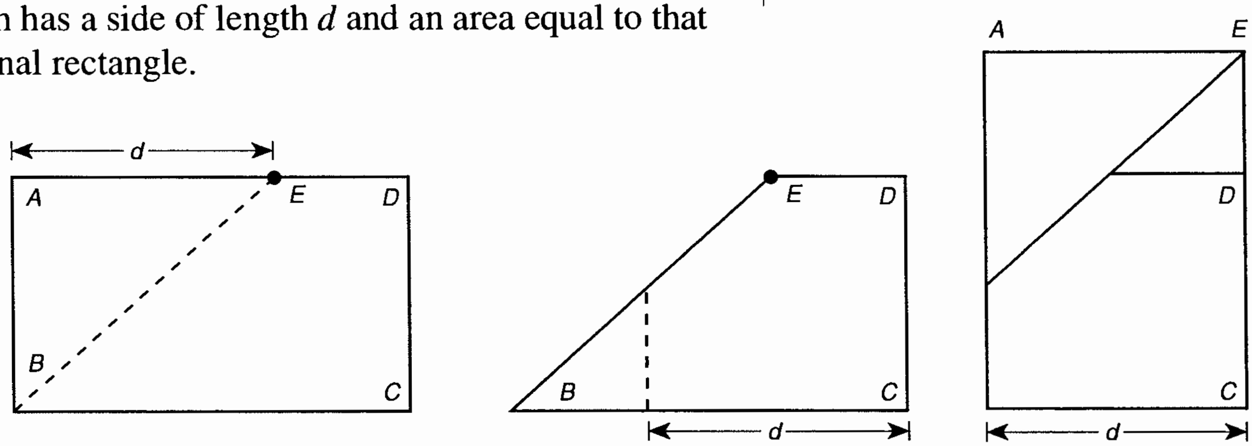
b) Pick a point along edge *AD* that is closer to *D* than *A* and label this point *E*. Let *d* be the distance from A to E.

c) Cut the rectangle along line *BE*. Then make a second cut so that the resulting pieces can be rearranged to form a rectangle which has a side of length *d* and an area equal to that of the original rectangle.

4. A labelled rectangle is shown below.



The first cut results in two pieces, triangle *ABE* and quadrilateral *BCDE*. If the latter is cut perpendicular to *BC* at a distance *d* from *C*, the resulting 3 pieces form a rectangle which has an edge of length *d*:



first cut second cut the rearranged pieces

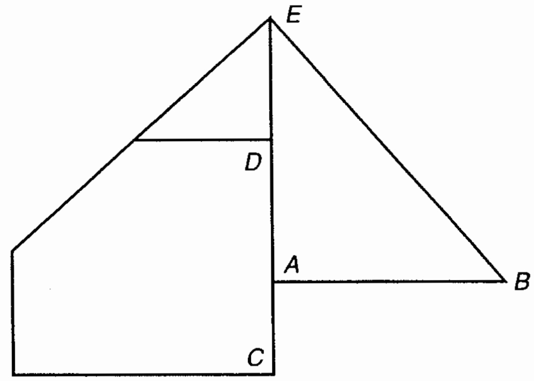
Actions

5. Ask the students to determine if the rectangle they constructed in Action 4 is a square. If not, ask them to repeat Action 4, choosing the point *E* so the resulting rectangle is more "squarelike" than the one they originally constructed.

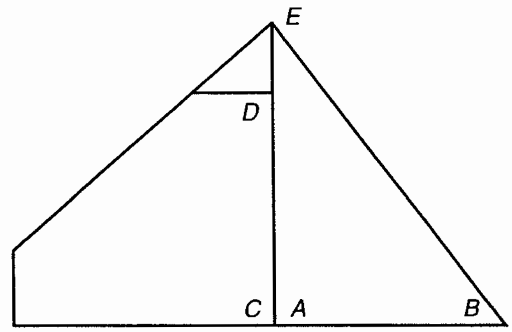
6. Discuss with the students how to choose the point *E* of Action 5 so that the dissection described in Comment 4 results in a square. Then, on the rectangle on the bottom half of Master 2, ask the students to locate point *E* and dissection lines so that when the rectangle is cut on these lines, it can be reassembled into a square with edge *AE*.

Comments

5. The sides of the rectangle constructed in Action 4 can be compared by moving the top piece of the rectangle to the position shown below. In this case a more "square-like" rectangle can be obtained by lengthening distance *AE* a bit.



6. Note that the dissection would have resulted in a square if, in the figure in Comment 5, vertices *A* and *C* coincided; that is, the above figure looked like this:



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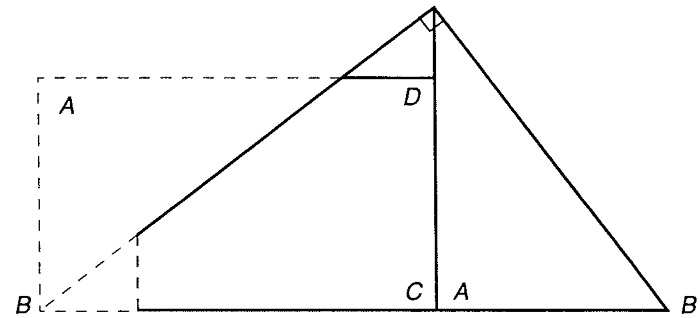
Actions

Comments

6. *Continued.* If dotted lines are added to the previous figure to show the original location of the pieces, the figure becomes:



Notice that the sum of the two angles at the top of the figure is a right angle and segments *AB* and *CD* have the same length, since they were originally opposite sides of a rectangle. These observations point to the following procedure, illustrated below, for choosing *E* so the dissection results in a square:

1. Extend side *BC* of rectangle *ABCD* and locate point *P* on the extended side so *CP* is the same length as *AB*. Also extend side CD. (See Figure 1).
2. Lay a piece of paper with a square corner on Figure 1 so the square corner lies on extended side *CD* and the sides of the corner go through vertices *B* and *P* (see Figure 2). Choose *E* to be the point where the side of the square corner intersects *AD*.

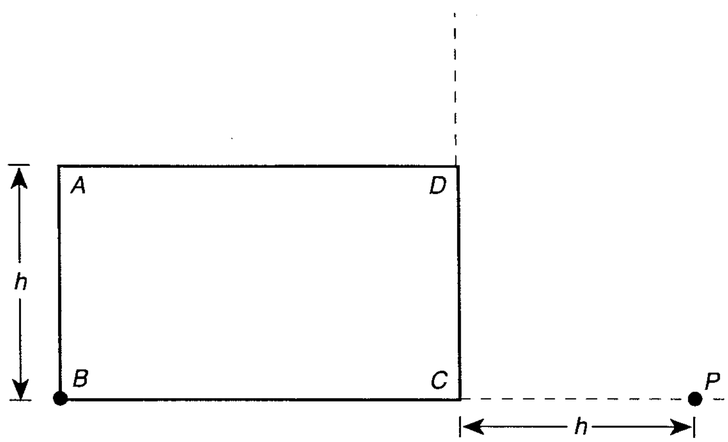


Figure 1.

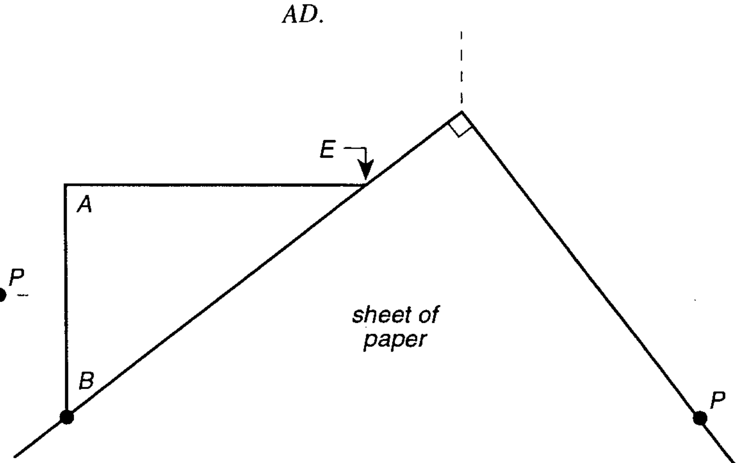


Figure 2.

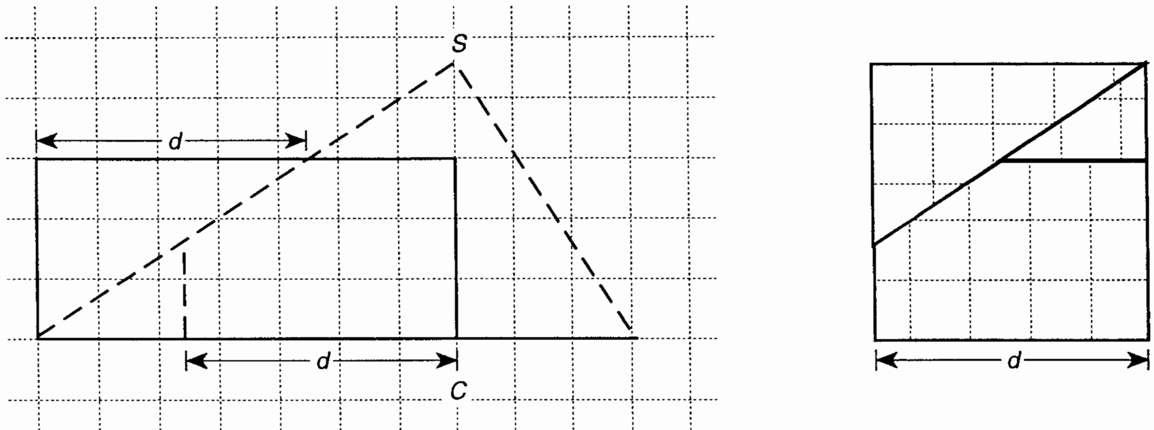
With this choice of E, the procedure described in Comment 4 results in a square.

Actions

Comments

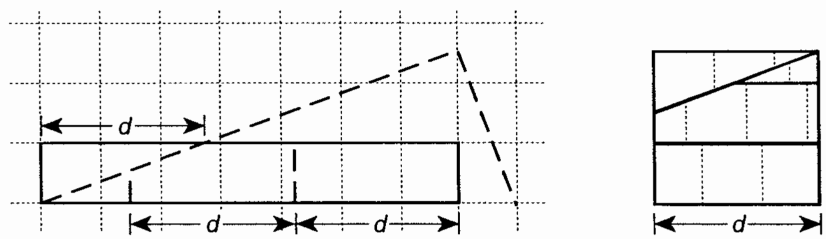
7. Distribute graph paper to the students. Ask them to draw a rectangle of area 21. Then ask them to dissect their rectangle and reassemble it to obtain a square of area 21.

7. Using the procedure described in Comment 6, a 3×7 rectangle can be dissected and reassembled as shown below to obtain a square of area 21. Note that the dimension of the square is the distance between vertex *C* of the rectangle and square corner *S*. Hence, the distance *CS* is $\sqrt{21}$.



8. Repeat Action 6 replacing 21 by 7.

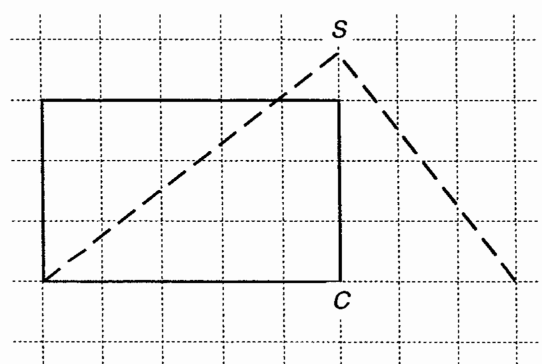
8. Forming a square from a 1 by 7 rectangle requires an additional cut.



If the rectangle is cut on the dotted lines, it can be reassembled to form a square.

9. Ask the students to construct a line segment of length $\sqrt{15}$.

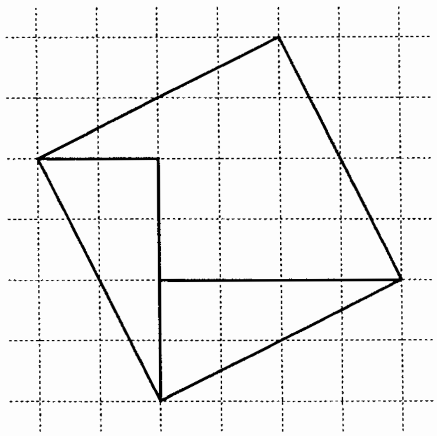
9. If the procedure of Comment 6 is carried out on a 3×5 rectangle, the result is a square of area 15. The dimension of this square is distance *CS* in the following figure. Hence, segment *CS* has length $\sqrt{15}$.



Segment *CS* has length $\sqrt{15}$.

Actions

10. Ask the students to divide one of the "tilted" squares constructed in Action 1 into 3 parts as shown below. Then ask them to cut out these 3 parts and reassemble them to form 2 squares.

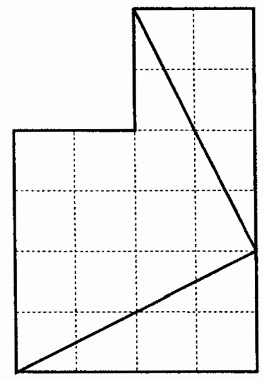


11. Ask the students to cut out a square. Then ask them to dissect their square so it can be reassembled into 2 squares.

Comments

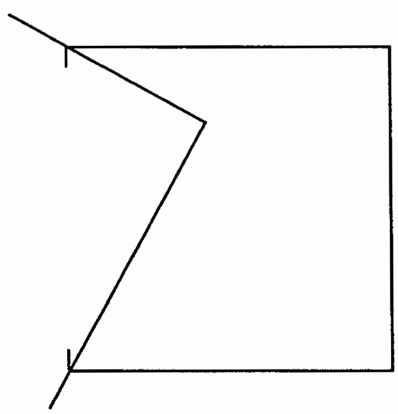
10. In Actions 10 through 13, the second method for constructing a square, mentioned in Comment 3, is developed.

Placing the 2 congruent right triangles in the positions shown below transforms the "tilted" square into 2 squares. Note that the dimensions of the 2 squares are the lengths of the legs of the congruent right triangles.

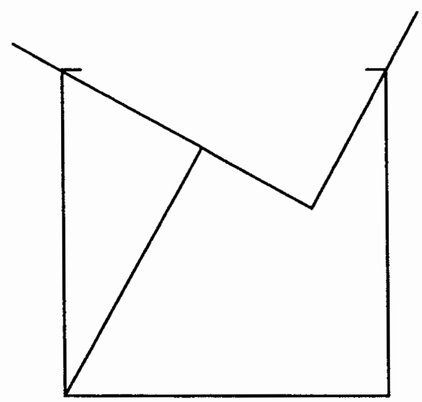


11. If the students cut a square out of grid paper, they may find it easier to turn their squares over so that the grid lines don't show.

The dissection in Action 10 can be replicated as shown:



1. Place a square corner on a side of the square so the sides of the corner pass through adjacent vertices of the square. Mark where these vertices strike the sides of the corner and trace around that part of the corner that lies in the square.



2. Move the corner to an adjacent side of the square so the marks on the sides of the corner coincide with the vertices of the square. Trace around that part of the corner that lies in the interior of the square.

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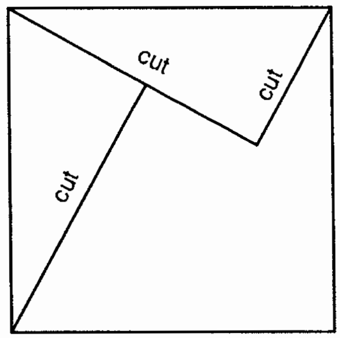
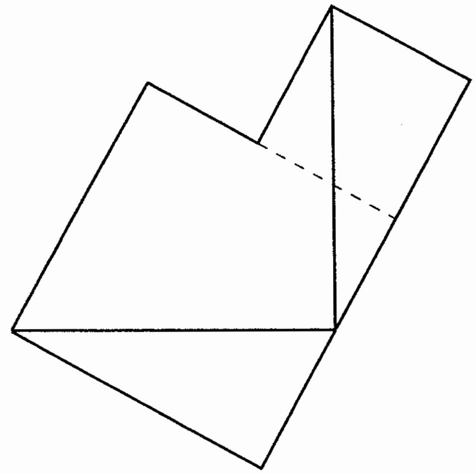
Actions	Comments

11. *Continued.*



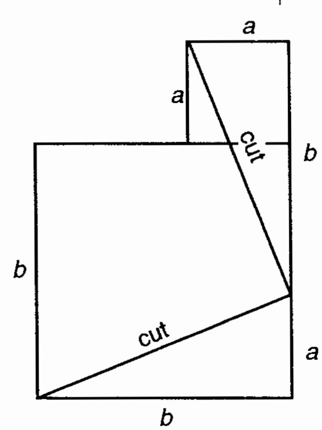
3. Cut the square on the lines traced in steps 1 and 2.

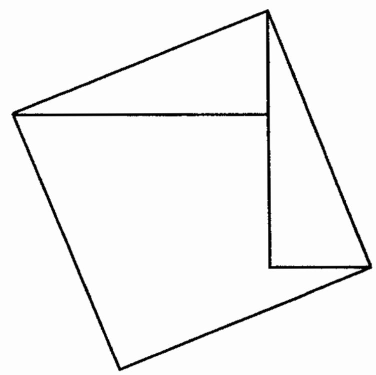


4. Reassemble pieces as shown.

12. Distribute copies of Master 3 to the students. Ask them to dissect the 2 squares so they can be reassembled to form a single square.

12. This is the converse of the dissection done in Action 11. It can be accomplished by cutting 2 congruent right triangles off the squares and relocating them as shown. The length of the legs of the triangles are the dimensions of the two squares.





Actions

13. Distribute grid paper to the students. Ask them to use the method developed in Action 12 to form a square of area 21.

Comments

13. In the first figure, squares of areas 16 and 5 are cut out, placed adjacent to one another, traced about and then dissected and reassembled as described in Comment 12 to obtain a square of area 21.

In the second figure, squares of area 13 and 8 are used.

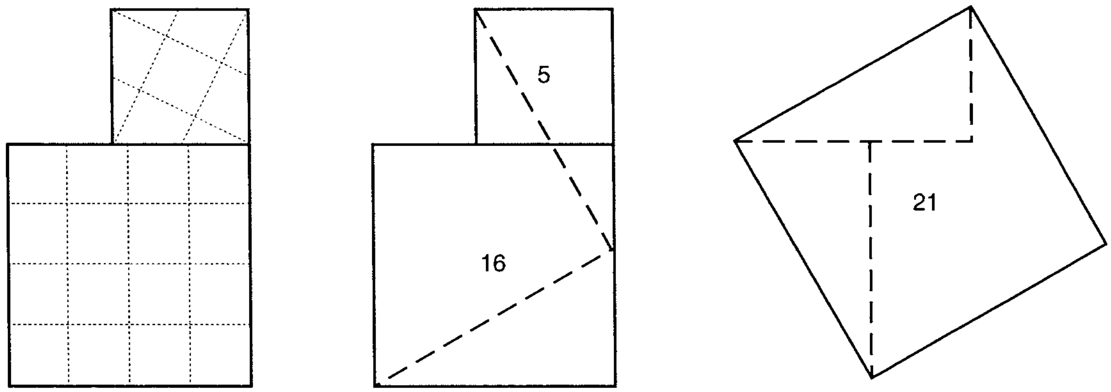


Figure 1.

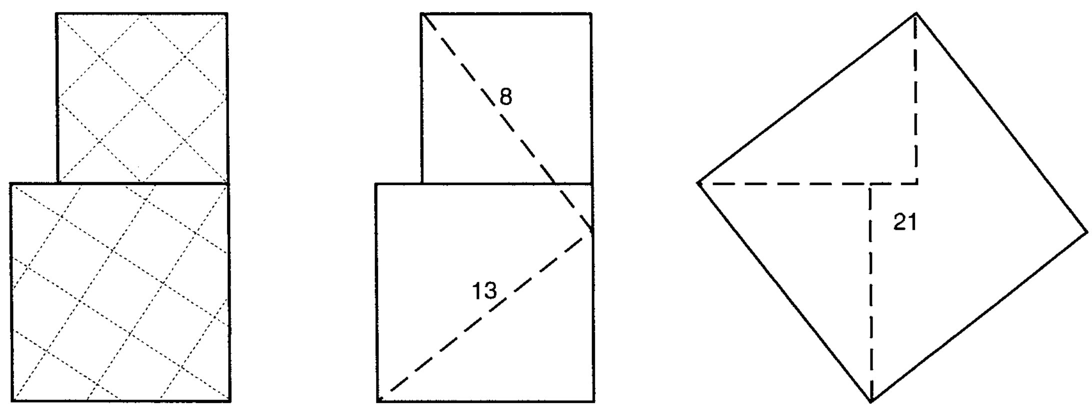
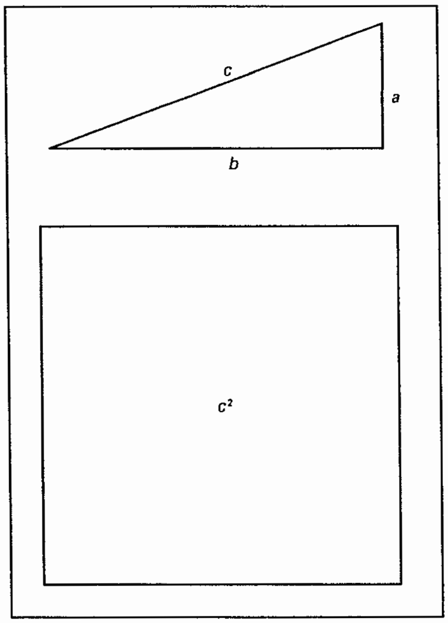


Figure 2.

Actions

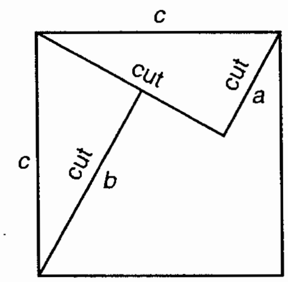
14. Distribute copies of Master 4 (shown below) to the students. Ask them to dissect the square so that it can be reassembled into 2 squares, one of area a^2 and one of area b^2 Point out to the students that this dissection establishes the Pythagorean Theorem.



Comments

14. Note that the right triangle has a hypotenuse of length c and the square has area c^2. Thus, the hypotenuse and the side of the square have the same length.

The dissection can be accomplished as shown in steps 3 and 4 of Comment 11. The cut lines in step 3 can be determined by cutting out the triangle, laying it on the square so the hypotenuse of the triangle coincides with a side of the square and then tracing along the legs of the triangle.



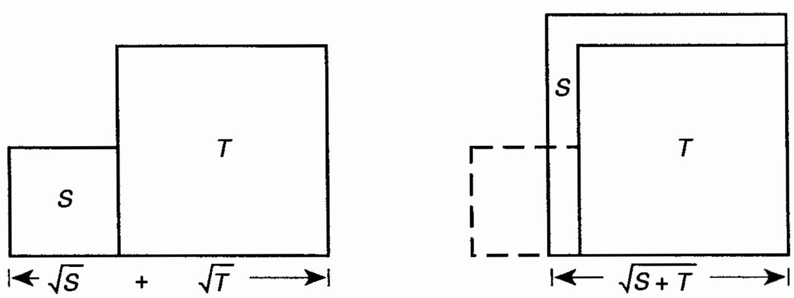
This dissection shows that a square whose edge is the hypotenuse, c, of a right triangle, can be dissected and rearranged into 2 squares whose edges are the legs, a and b, of the right triangle. Hence, $c^2 = a^2 + b^2$.

15. (Optional). Ask the students to compare $\sqrt{S} + \sqrt{T}$ and $\sqrt{S+T}$ for positive S and T. Discuss their conclusions.

15. In this and the next action, the square root of sums, products and quotients is investigated.

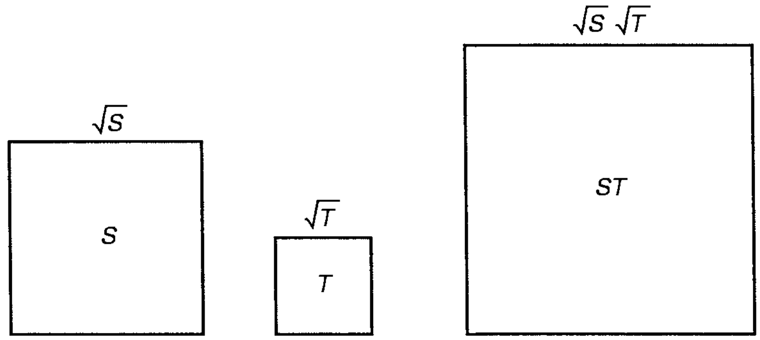
For positive S and T, $\sqrt{S} + \sqrt{T} > \sqrt{S+T}$, that is, the sum of the square roots of positive numbers is greater than the square root of their sum. The students may arrive at this conclusion by comparing the values of $\sqrt{S} + \sqrt{T}$ and $\sqrt{S+T}$ for specific values of S and T, for example, when $S = 4$ an $T = 9$, $\sqrt{S} + \sqrt{T} = 5$ and $\sqrt{S+T} = \sqrt{13}$, which is less than 4.

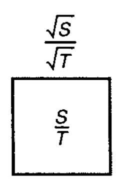
You can urge the students to provide a general argument for their conclusion. One argument, based on the sketches shown below, begins by noting that $\sqrt{S} + \sqrt{T}$ is the combined length of the edges of squares of areas S and T respectively. If the square of area T is enlarged to a square of area $S + T$, the edge of the enlarged square would be less than the combined length of the edges of the original squares, as shown in the sketch.



Actions

16. (Optional) Repeat Action 15 for (a) $\sqrt{ST}$ and $\sqrt{S}\sqrt{T}$; (b) $\sqrt{S}/\sqrt{T}$ and $\sqrt{S/T}$.





Comments

16. The students may reach the conclusion that the square root of the product of two numbers equals the product of their square roots by comparing the values of $\sqrt{ST}$ and $\sqrt{S}\sqrt{T}$ for specific values of *S* and *T*.

A general argument can be given by noting that if the length, $\sqrt{S}$, of the side of a square of area *S* is changed by a factor of $\sqrt{T}$, its area, *S*, is changed by a factor of $\sqrt{T} \times \sqrt{T}$, or *T*. (See Comment 1, in Activity 3, *Fraction Sums and Differences.*) The result is a square of area *ST* whose side has length $\sqrt{S}\sqrt{T}$. On the other hand, the length of the side of a square is the square root of its area, in this case, $\sqrt{ST}$. Thus, $\sqrt{ST} = \sqrt{S}\sqrt{T}$.

In the above argument, given squares of area *S* and *T*, a new square has been constructed, the length of whose side is the product of the lengths of the sides of the two original squares. Making a scale drawing of this new square requires constructing a line segment whose length *ST* is the product of the lengths *S* and *T* of two given line segments. For those interested, a method for carrying out this construction is given in the appendix at the end of this activity.

The situation for quotients is similar to that of products. If a new square is obtained from a square whose area is *S* by dividing the length of its side, $\sqrt{S}$, by $\sqrt{T}$, the area of the new square is the area *S* of the original square divided by *T*, i.e., S/T. Hence, $\sqrt{S/T} = \sqrt{S}/\sqrt{T}$, that is, the square root of the quotient of two numbers is the quotient of their square roots.

The appendix also includes a method for constructing a line segment whose length is the quotients of the lengths of two given line segments.

The results of this Action are sometimes used to "simplify radicals." For example, $\sqrt{45} = \sqrt{(9)(5)} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$; $\sqrt{7/5} = \sqrt{35/25} = \sqrt{35}/\sqrt{25} = \sqrt{35}/5$.

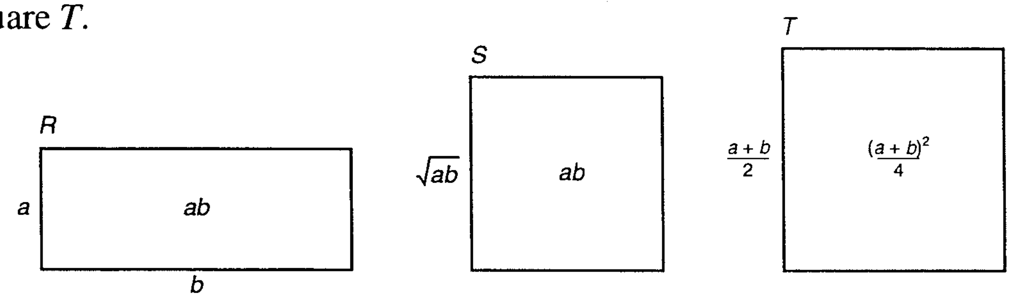
Actions

17. (Optional). Show the students a non-square rectangle *R* whose sides have length *a* and *b*. Tell them that square *S* has the same area as rectangle *R* and square *T* has the same perimeter as rectangle *R*. Ask the students to find the length of the sides of square *S* and square *T*.

18. (Optional.) Define the *arithmetic mean* and the *geometric mean* of two positive numbers. Discuss their relationship.

Comments

17. The length of the side of square S is $\sqrt{ab}$. The length of the side of square *T* is $a+b/2$.



18. If *a* and *b* are the lengths of the sides of a rectangle, the *geometric mean* of *a* and *b* is the length of the side of a square which has the same area as the rectangle; the *arithmetic mean* is the length of the side of a square which has the same perimeter as the rectangle. That is, the geometric mean of *a* and *b* is $\sqrt{ab}$; the arithmetic mean of *a* and *b* is $a+b/2$. Since square *S*, above, is smaller than square *T* (see below), the geometric mean of two positive numbers is less than their arithmetic mean.

The reason *T* is larger than *S* is that the area of *S* is the same as the area of rectangle *R* while the area of *T* is larger than the area of *R*. To see this, cut rectangle *R* into 2 pieces as indicated in Figure 1. Form shape *L* with these 2 pieces as shown in Figure 2. Note *L* and *R* have the same area and the longer side of *L* is the same length as the side of *T*. Thus *L* is *T* with a corner removed. Hence, *T*'s area is greater than *L*'s.

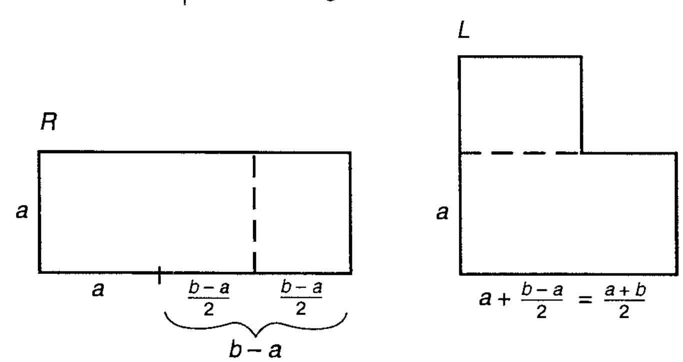
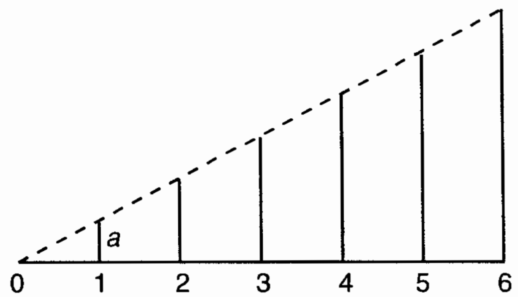


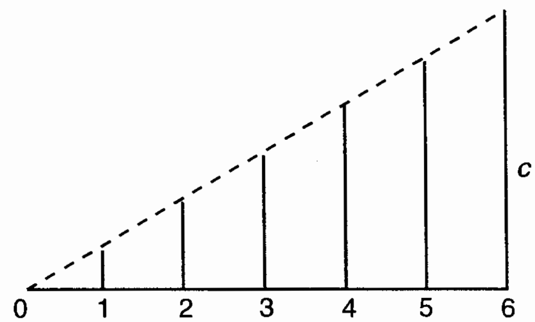
Figure 1. Figure 2.

Appendix: Constructing Products and Quotients

Actions

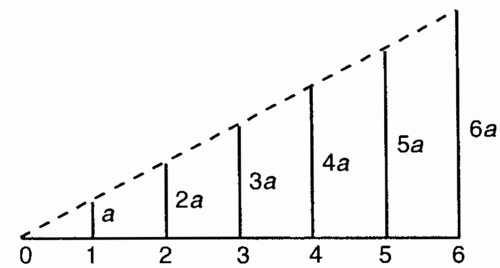
1. Distribute Activity Sheet XII-5-A. The sheet contains two sketches, each of which is a set of line segments perpendicular to a number line. In each sketch, the length of one of the perpendicular segments is given: in the first sketch, the length of the first segment is *a* and, in the second sketch, the length of the last segment is *c*. Ask the students to find the lengths of the remaining segments. Discuss the methods they use.

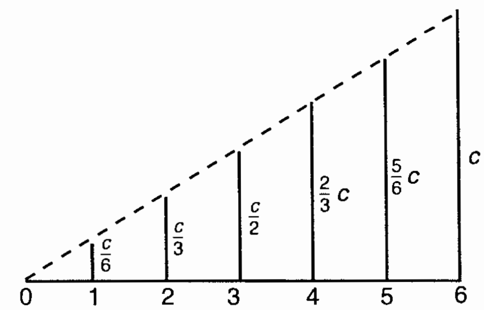




Comments

1. The height of the remaining segments are shown below. The students will use a variety of methods to determine these lengths.

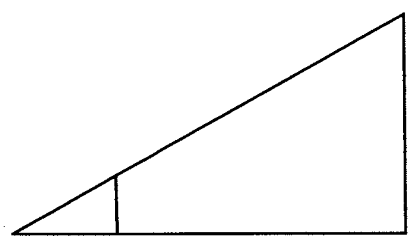




If it doesn't arise in the students' discussions, point out that if the length of the side of a triangle is changed by a certain factor while the shape of the triangle is retained, then the length of the other sides also change by that factor. For example, expanding the base of the triangle by a factor of 3, while retaining the triangle's shape, also expands the height and the hypotenuse by a factor of 3. Reducing the base of a triangle to $^1/_3$ its original length, while retaining the shape of the triangle, also reduces its height and hypotenuse to $^1/_3$ of their original lengths; or, to put it another way, if the length of the base of a triangle is divided by 3, while retaining the shape of the triangle, then the lengths of the height and hypotenuse are also divided by 3.

Actions

2. For each of the two sketches on Activity Sheet XII-5-A, ask the students to locate 3.5 on the number line and draw the perpendicular segment from that point to the dotted line. Then ask them to find the lengths of these segments. Discuss.



Comments

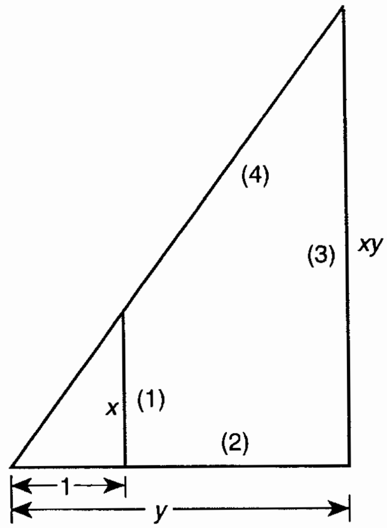
2. The students can draw the perpendiculars by eye or, if they desire greater precision, they can use the corner of a sheet of paper or a notecard to draw the perpendiculars.

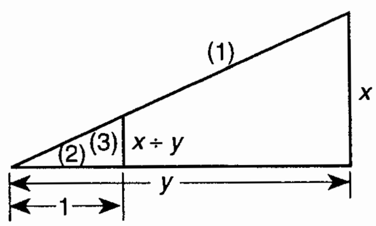
There are several ways to determine the lengths of the segments. One might observe that the lengths of the perpendicular segments grow uniformly as one moves to the right on the number line and, thus, since 3.5 is halfway between 3 and 4, the length of the perpendicular at 3.5 will be halfway between the length of the segment at 3 and that at 4. In the top sketch, the segment at 3 has length $3a$ and that at 4 has length $4a$, so that the segment at 3.5 has length $3.5a$. In the bottom sketch, the length of the desired segment is halfway between $^1/_2c$ and $^2/_3c$ or, converting to twelfths, halfway between $^6/_{12}c$ and $^8/_{12}c$. Thus, the length of the segment is $^7/_{12}c$.

Alternatively, one might observe that the large triangle shown in each of the sketches on the left, is obtained by extending the base of the small triangle by a factor of 3.5 while retaining its shape. Hence, the height of the small triangle is extended by a factor of 3.5 to obtain the height of the large triangle. This means, in the first instance, the height is extended from a to $3.5a$ and, in the second instance, from $^1/_6c$ to $3.5(^1/_6c)$, or $^7/_{12}c$.

Actions

3. Distribute copies of Activity Sheet XII-5-B to the students. Ask them to make the indicated constructions.



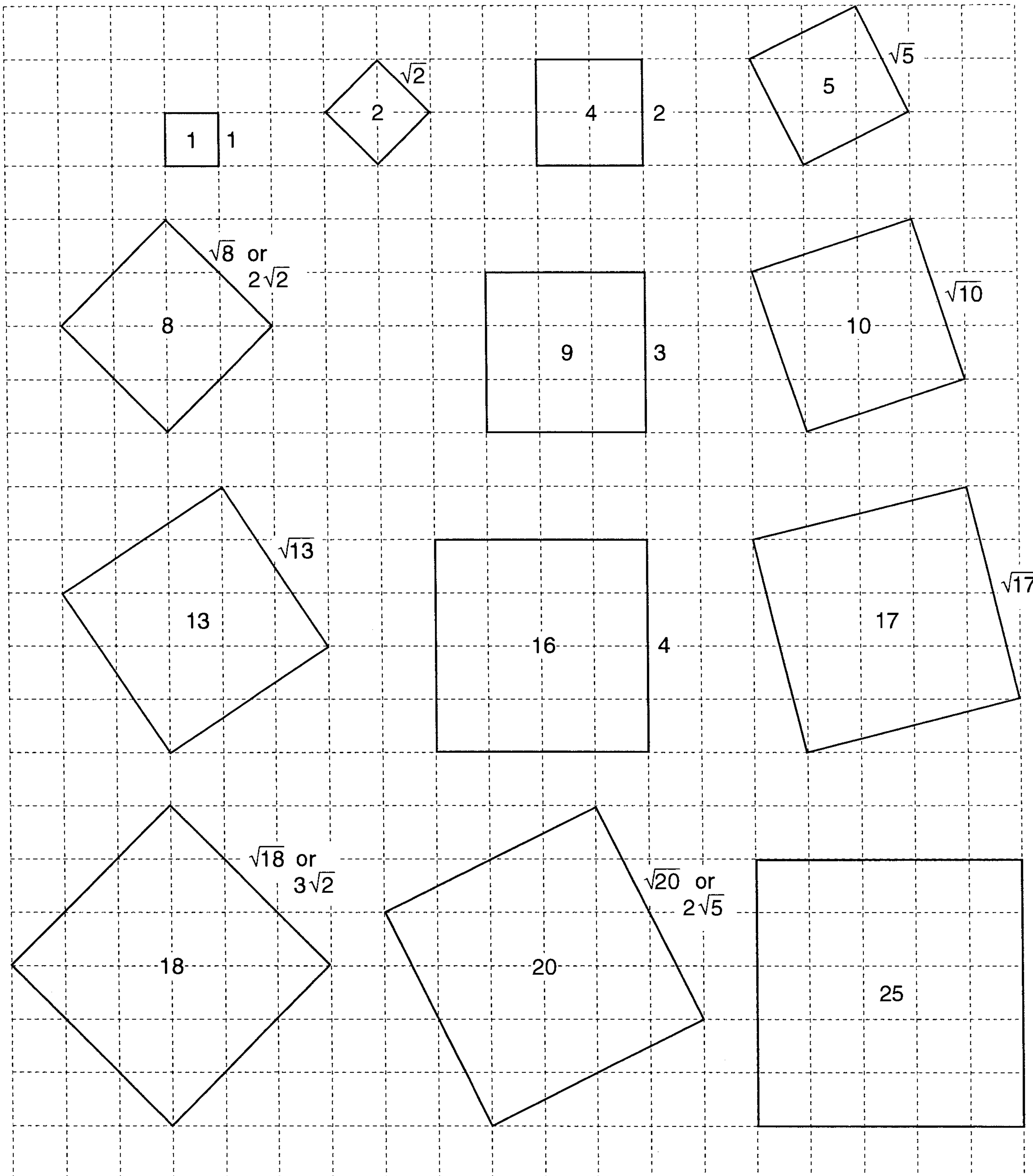


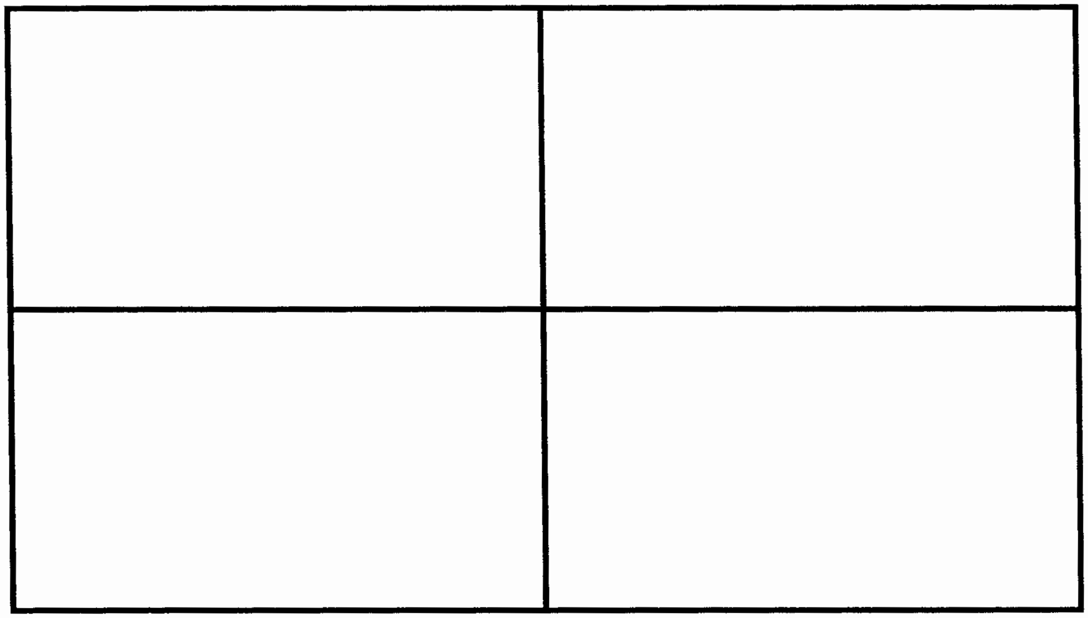
Comments

3. The edge of a paper or notecard can be used to mark off lengths and one of its corners can be used to draw perpendiculars.

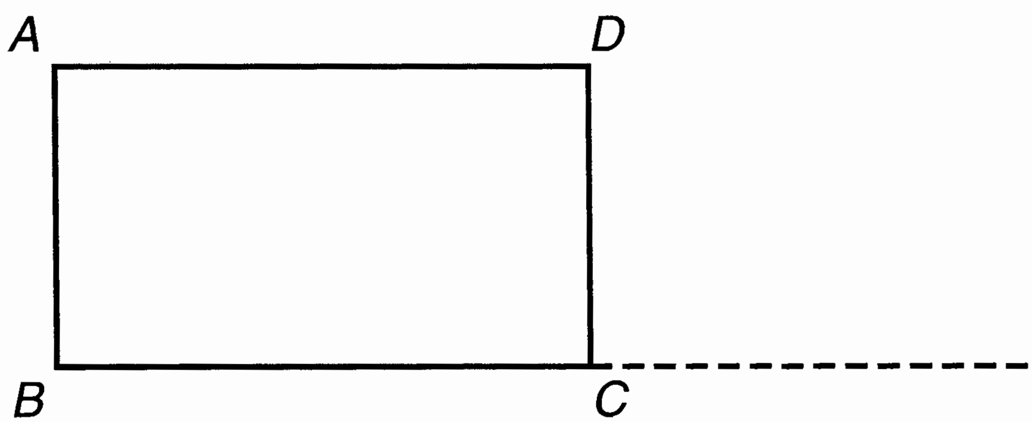
One way of doing the first construction: (1) draw a perpendicular of height x at the end of the segment of length 1, (2) extend the segment of length 1 to length y, (3) draw a perpendicular at the end of the extended segment, (4) extend the hypotenuse of the triangle with base 1 and height x to meet this perpendicular. The result is two triangles of the same shape (see the figure). The base of the larger triangle is the base of the smaller triangle multiplied by y. Thus, the height of the larger triangle is the height, x, of the smaller triangle multiplied by y.

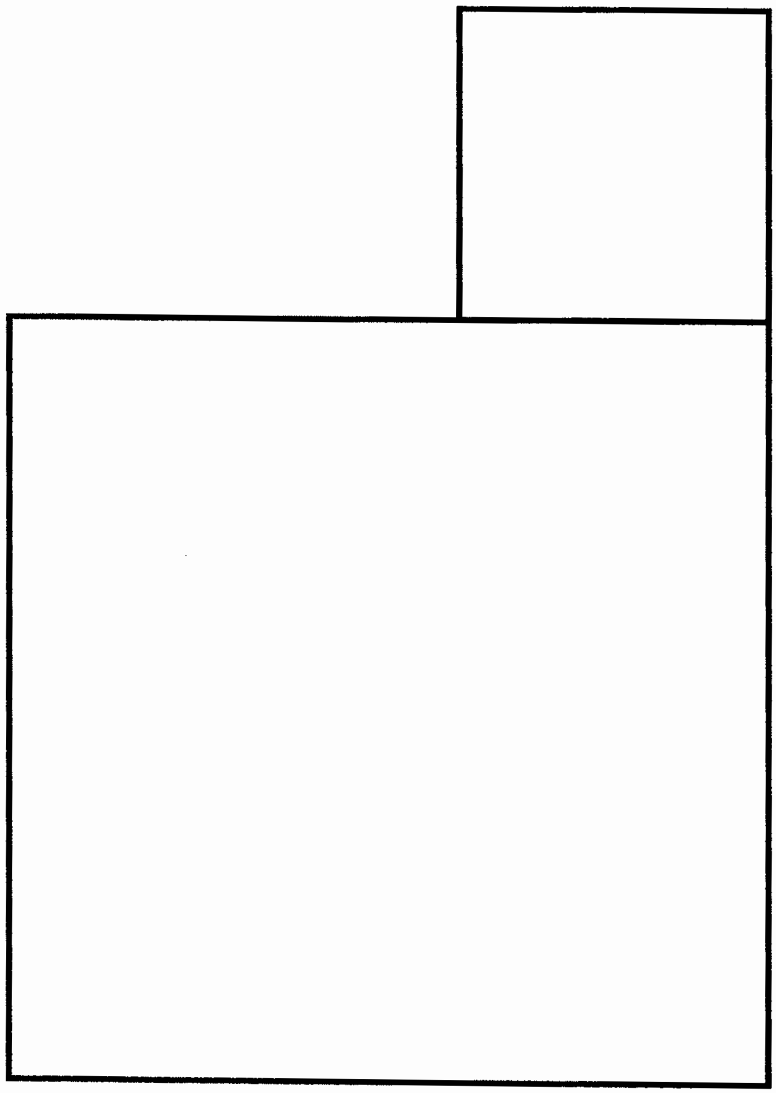
The second construction can be done as follows: (1) construct a right triangle with base y and height x, (2) locate a point on the base 1 unit from the hypotenuse, (3) draw the perpendicular segment from this point to the hypotenuse. The result is two triangles of the same shape. The base of the smaller triangle is the base of the larger triangle divided by y. Thus, the height of the smaller triangle is the height, x, of the larger triangle divided by y.

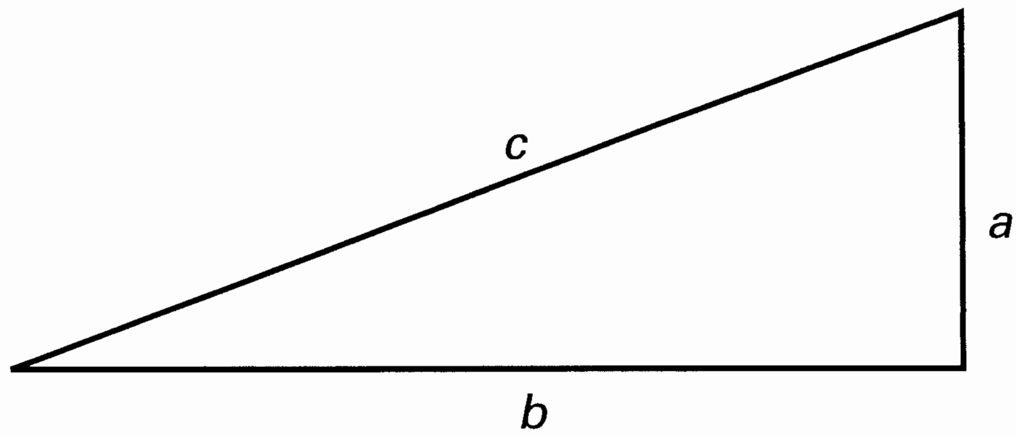

1
1
2
√2
4
2
5
√5
8
√8 or 2√2
9
3
10
√10
13
√13
16
4
17
√17
18
√18 or 3√2
20
√20 or 2√5
25
5

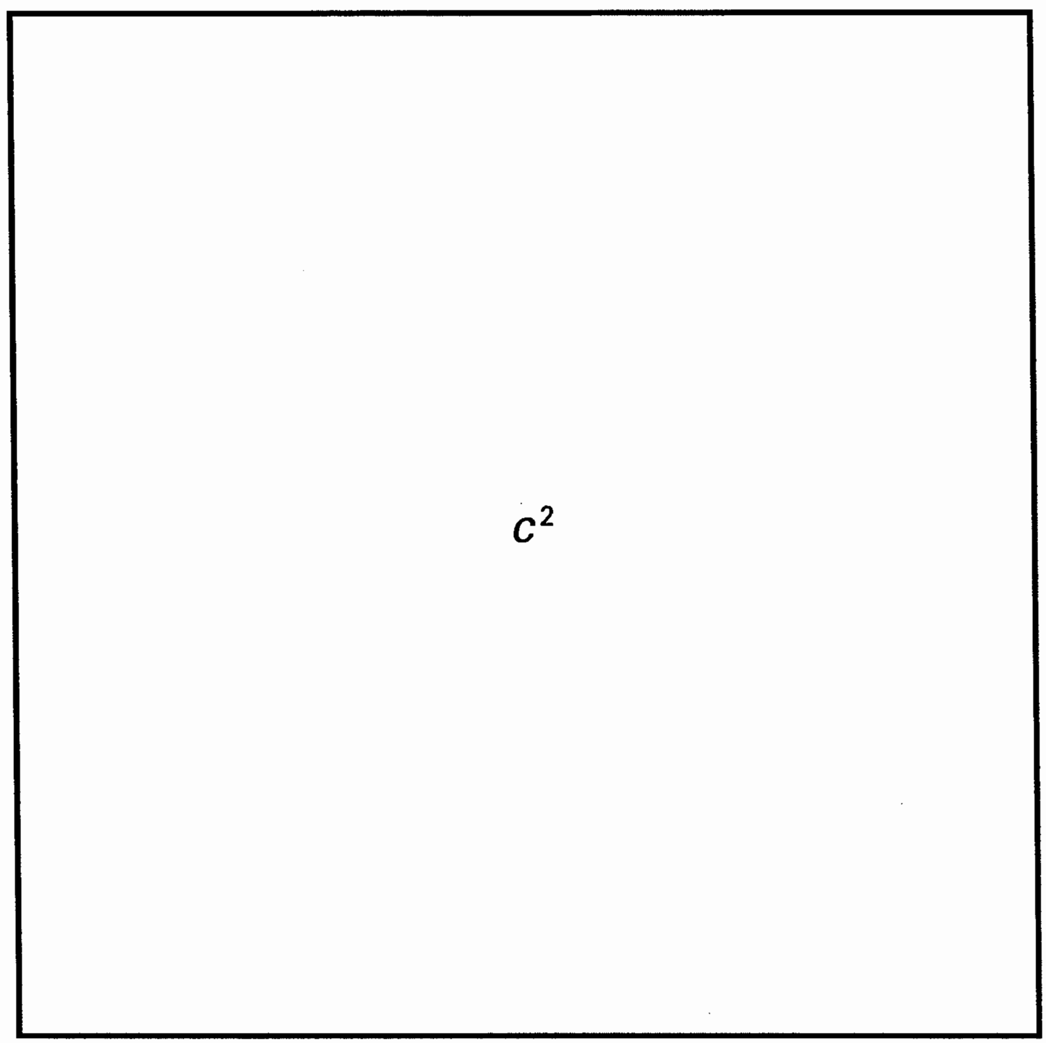


cut



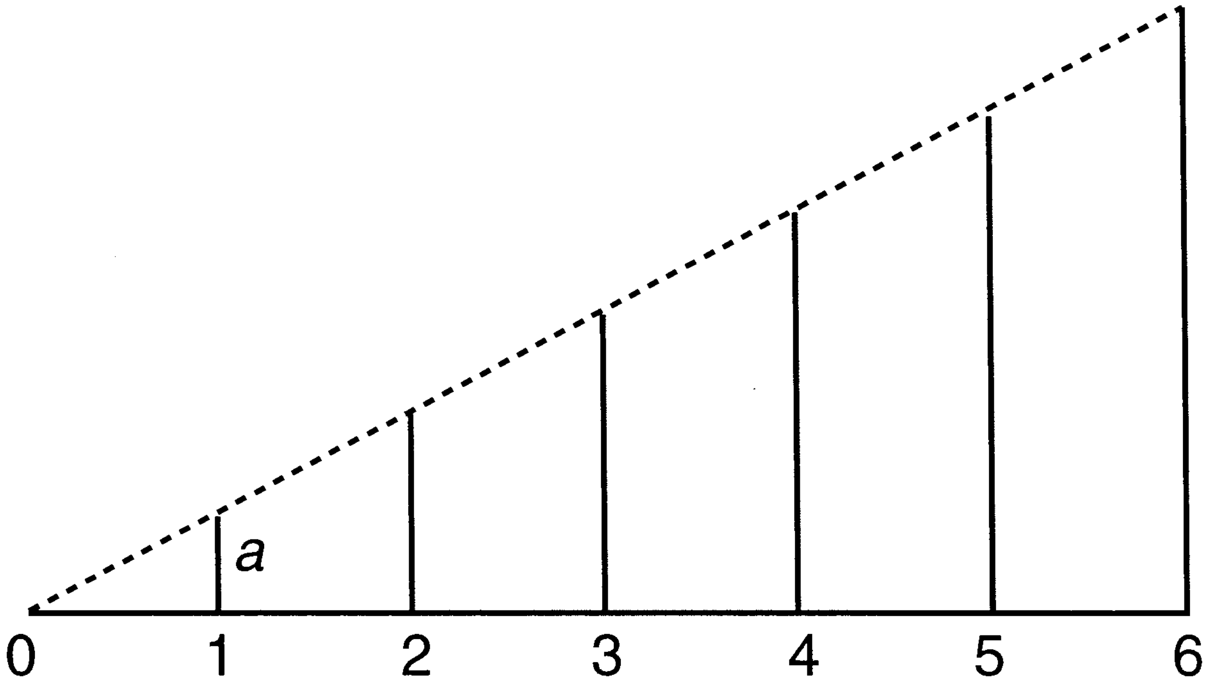


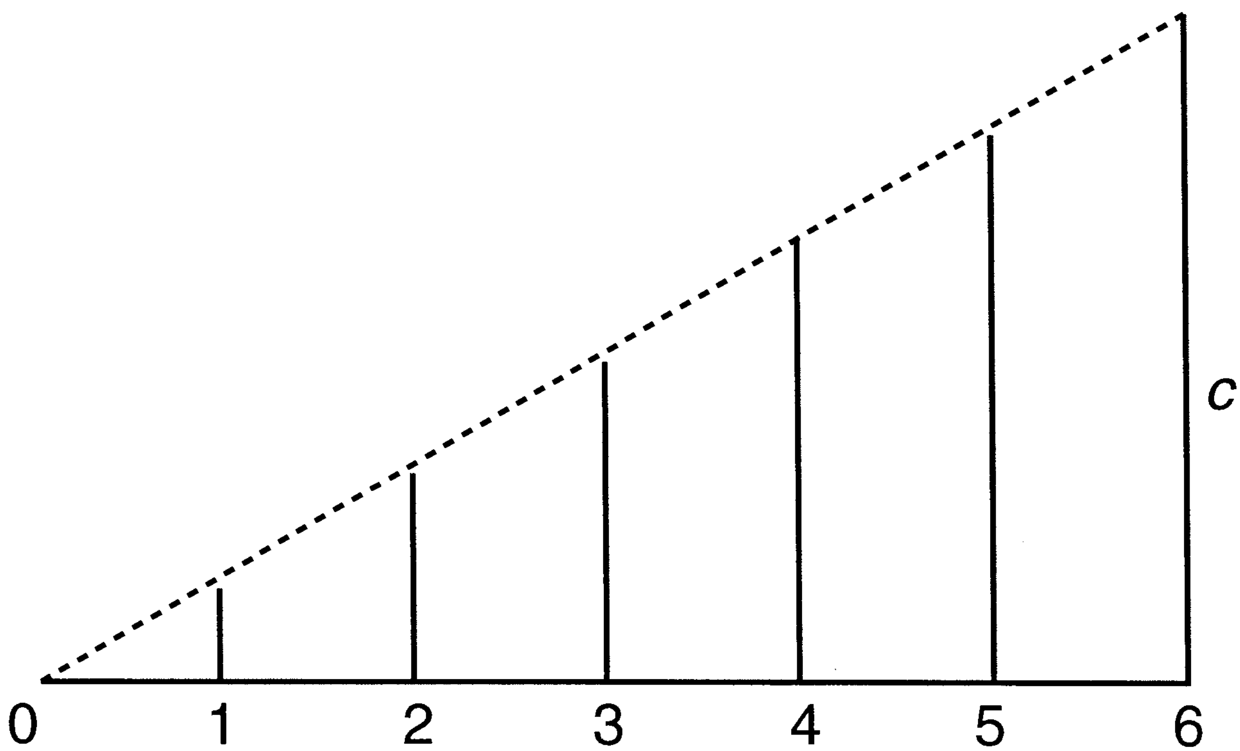

c
a
b


c^2

Name ____________________________

Activity Sheet XII-5-A





Name ______________________

Construct a line segment where length is xy.

Construct a line segment where length is $x \div y$.

Complex Numbers

OVERVIEW

Green and yellow bicolored counting pieces are used to introduce complex numbers and their arithmetical operations.

Prerequisite Activity

Unit VI, *Modeling Integers.*

Materials

Black and red bicolored counting and edge pieces, green and yellow bicolored counting and edge pieces. Black, red, green and yellow overhead pieces. Masters, as needed.

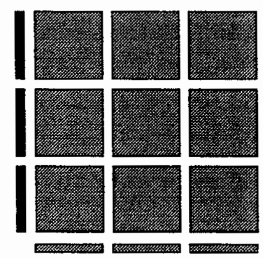
Actions

1. Distribute black and red counting and edge pieces to the students. Ask them to construct a square array, with edge pieces, whose value is –9. Discuss the existence of a square root of –9.

2. Distribute green and yellow counting pieces to the students. Tell the students the purpose of these pieces is to provide square roots for negative numbers. In particular, if adjacent edges of an array are green, the array will be red. Illustrate.

Comments

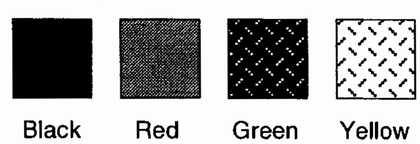
1. A red array will always have one black edge and one red edge:



Using only black and red pieces, a red square can not have edges with identical values, that is to say, –9 does not have a square root.

2. Cardstock tile with different colored sides can be printed using Master A, found at the end of this activity. Printing Master 1 in black on red cardstock provides black/red tile. Printing it in green on yellow cardstock provides green/yellow tile. If colored printing is not available, printing Master 2 on yellow cardstock provides tile which can be distinguished from solid black tile—and which can be referred to as green tile, if you like. Red, green and yellow tile for the overhead can be cut from colored transparency film.

In this write-up, colored tile will be represented as follows:



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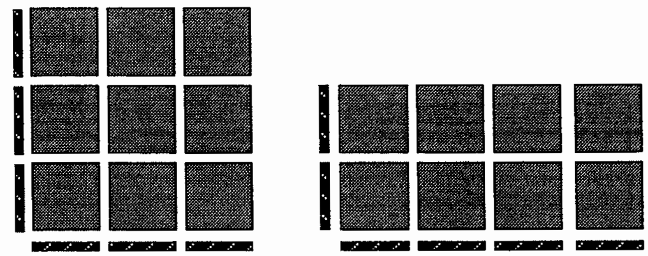
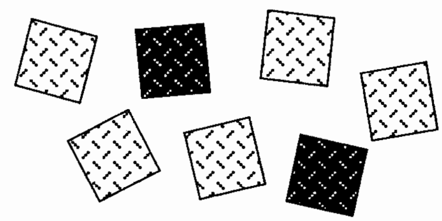
2. *Continued.* Green pieces are introduced to provide square roots for negative numbers. To this end, if both edges of an array are green, the array will be red:

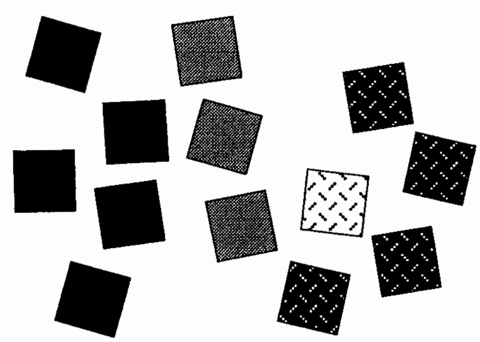


3. Discuss the net values of collections of counting pieces involving green/yellow and black/red pieces. Introduce the standard notation and terminology for these values.

3. Green and yellow are opposites. Thus, the net value of the following collection is 3 yellow.



The net value of the following collection is 2 black plus 3 green.



In order to distinguish between net values in the black/red system and net values in the green/yellow system, the letter i will be used to indicate net values in the green/yellow system. Thus, a collection of 4 black tile has net value 4 while a collection of 4 green tile has net value $4i$. A collection of 4 red tile has net value -4 while a collection of 4 yellow tile has net value $-4i$.

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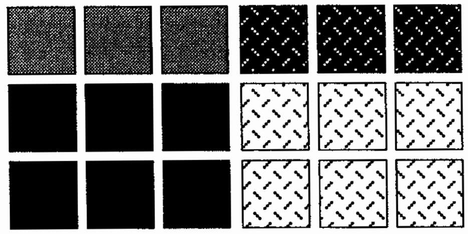
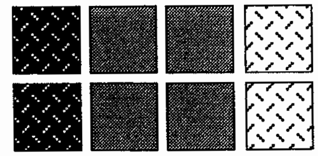
Actions

Comments

3. *Continued.* Shown below are two other collections with their net values.



$3 - 3i$

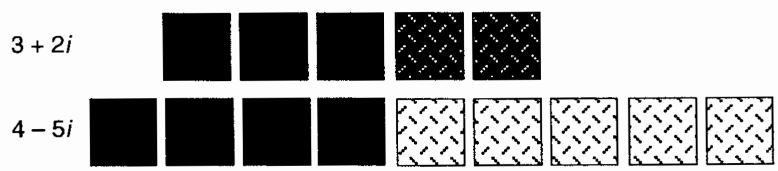


-4

Numbers of the form $a + bi$ are called *complex* numbers. If a and b are integers, $a + bi$ is called a *complex* or *Gaussian* integer. An *imaginary* number is a complex number for which $b \neq 0$. A *real* number is a complex number for which $b = 0$, a *pure* imaginary number is an imaginary number for which $a = 0$.

4. Ask students to use counting pieces to find the sum and difference of $3 + 2i$ and $4 - 5i$. Discuss.

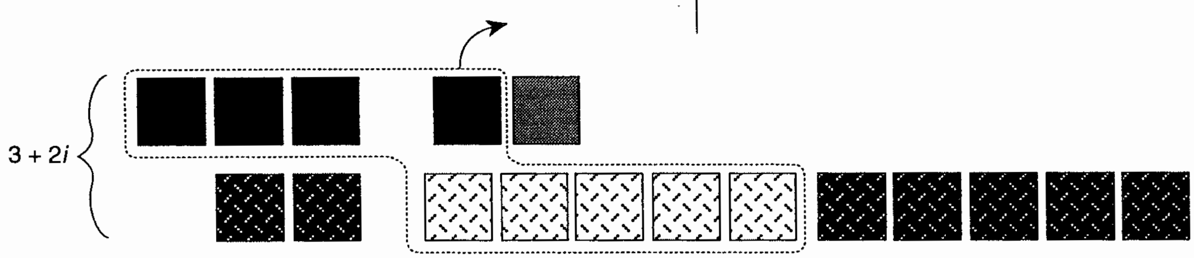
4. The sum $(3 + 2i) + (4 - 5i)$ can be found by combining a collection whose value is $3 + 2i$ with a collection whose value is $4 - 5i$ and then finding the value of the combined collection.



$(3 + 2i) + (4 - 5i) = 7 - 3i$

The difference $(3 + 2i) - (4 - 5i)$ can be found by combining a collection whose value is $3 + 2i$ with a collection whose value is the opposite of $4 - 5i$ and then finding the value of the combined collection.

Alternatively, the difference can be found by forming a collection with net value $3 + 2i$ from which a collection with value $4 - 5i$ can be removed:



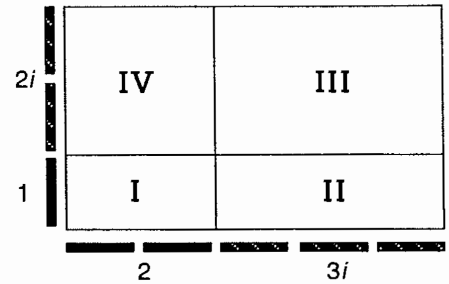
$(3 + 2i) - (4 - 5i) = -1 + 7i$

Actions

5. Ask the students to form a counting piece array for which one edge has value $1 + 2i$ and the other edge has value $2 + 3i$. Discuss. Then ask the students to use their array to find the product $(1 + 2i)(2 + 3i)$.

Comments

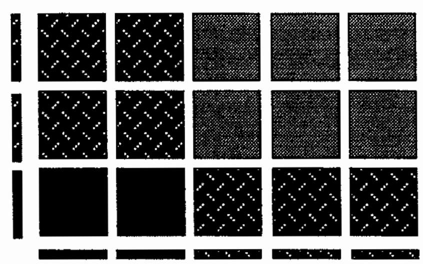
5. An array whose edges have the given values has four sections as indicated below.



The tile in section I are black since both edges of this section are black. The tile in section III are red since both edges are green.

Sections II and IV both have one black and one green edge. The students may question what color tile to place in these sections. Note that section II represents the product $1 \times 3i$. If, in the complex numbers, multiplication by 1 is to leave a number unchanged, just as it does in the real numbers, then the value of section II is $3i$, in which case it contains green tile. To maintain consistency, the tile in section IV, which also has one black and one green edge, must also be green.

The completed array appears below. Its value is $-4 + 7i$.



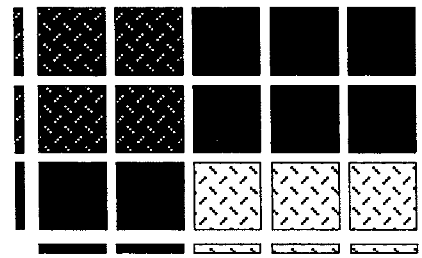
$(2 + 3i)(1 + 2i) = -4 + 7i$

Actions

6. Have the students use the array with edge pieces formed in Action 5 to find $(1 + 2i)(2 - 3i)$. Repeat for $(1 - 2i)(2 - 3i)$.

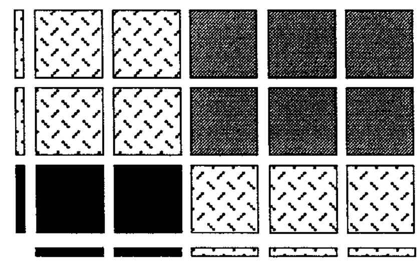
Comments

6. Turn over the last 3 columns of the array to obtain the following:



$(1 + 2i)(2 - 3i) = 8 + i$

Then, turn over the top 2 rows:



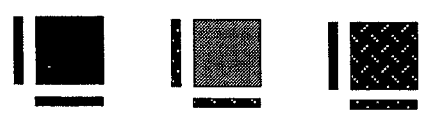
$(1 - 2i)(2 - 3i) = -4 - 7i$

Actions

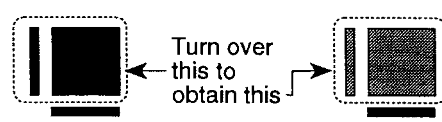
7. Ask the students to summarize what they have discovered about the relationship of the color of a tile in an array to the colors of its edge pieces. Discuss.

Comments

7. If both the row and column that a tile is in have black edges, the tile is black. If they both have green edges, the tile is red. If one has a black edge and the other a green edge, the tile is green. These 3 situations are illustrated below.



The situation for other combinations of edges can be determined by turning over pieces in one of the above situations. For example, turning over the row in the situation on the left, we see that if one edge is red and the other black, the tile is red:



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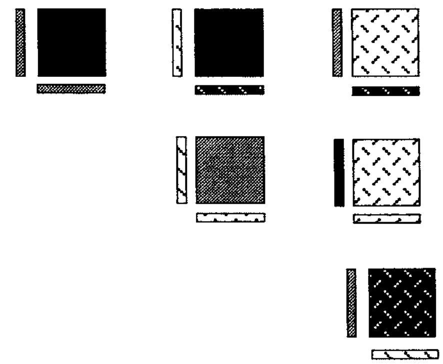
Actions

Comments

7. *Continued.* Here are oher possible situations, obtained by turning over a row and/or column in the three situations shown above:



These results can be summarized in tabular form:

edge 2 \ edge 1	B	R	G	Y
B	B	R	G	Y
R	R	B	Y	G
G	G	Y	R	B
Y	Y	G	B	R

Some observations:
- If both edges are black or red, the tile is black.
- If both edges are green or yellow, the tile is red.
- If one edge is black, the tile has the same color as the other edge.
- If one edge is red, the tile has the opposite color of the other edge.
- If one edge is green and the other yellow, the tile is black.

Rewriting the above table in terms of values rather than color produces the multiplication table to the left.

x	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Actions

8. Ask the students to use counting pieces to find the following:

(a) $(3 - 2i) + (4 + 3i)$ (b) $(4 + 6i) - (-6 + 4i)$

(c) $(4 - 5i) - (6 - i)$ (d) $(2 - 3i)(4 - 2i)$

(e) $(3 + i)(3 - 2i)$ (f) $(-1 + 3i)(1 + 3i)$

(g) $5i \div (1 + 2i)$ (g) $(16 + 2i) \div (2 - 3i)$

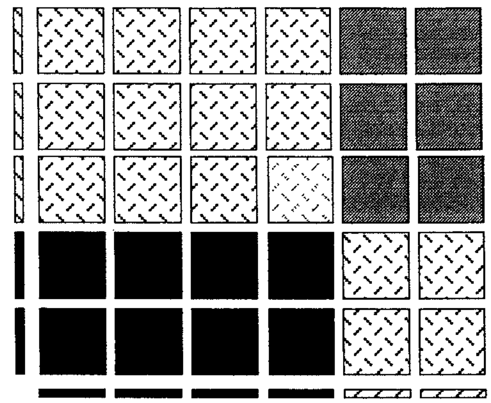
Comments

8. (a) $7 + i$

(b) $10 + 2i$

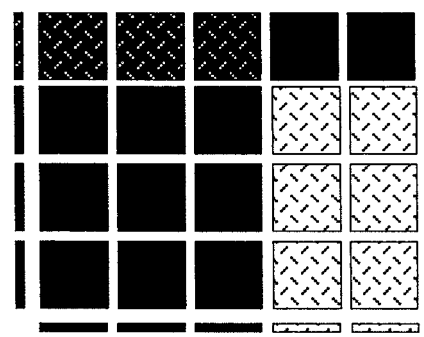
(c) $-2 - 4i$

(d)



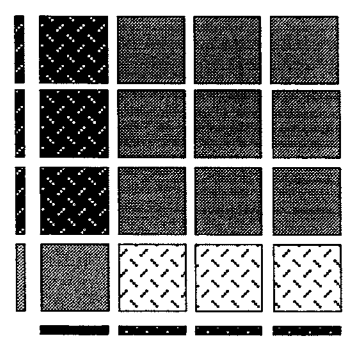
$(2 - 3i)(4 - 2i) = 2 - 16i$

(e)



$(3 + i)(3 - 2i) = 11 - 3i$

(f)



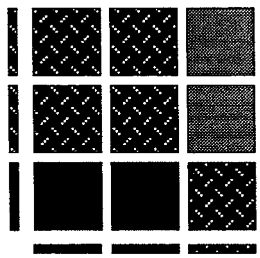
$(-1 + 3i)(1 + 3i) = -10$

Continued next page.

Actions

Comments

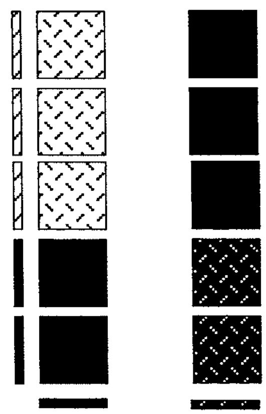
8. *Continued.*
(g) If an array whose value is $5i$ is constructed so one edge has value $1 + 2i$, the value of the other edge is the desired quotient. Since the real part of $5i$ is 0, the array must have an equal number of black and red pieces.



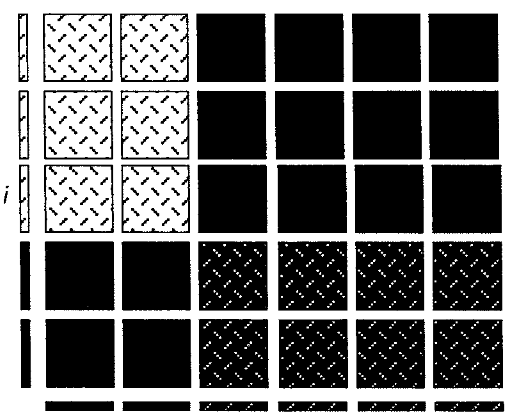
$5i \div (1 + 2i) = 2 + i$

The array shown here has value $5i$. Its left edge has value $1 + 2i$. The value of the other edge is $2 + i$. Hence, $5i \div (1 + 2i) = 2 + i$.

(h) The students will devise various strategies for constructing an array whose value is $16 + 2i$ and has an edge whose value is $2 - 3i$.



One way to proceed is to lay out an edge of 2 black and 3 yellow and then consider which of black or red and which of green or yellow must be in the other edge. Since the resulting array must contain at least 16 black tile, colors should be chosen that produce both black and green tile, with more of the former. As shown on the left, a selection of black leads to a column of 2 black and 3 yellow and a selection of green leads to a column of 2 green and 3 black. Two of the former and 4 of the latter will produce an array whose net value is $16 + 2i$, as shown below.



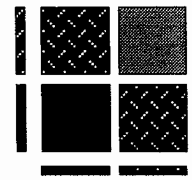
$(16 + 2i) \div (2 - 3i) = 2 + 4i$

Actions

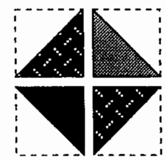
9. (Optional.) Ask the students to use counting pieces to find a square root of $2i$. Then ask them how they might use counting pieces, along with a scissors, to find a square root of i. Discuss.

Comments

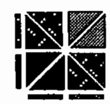
9. The following array, with two adjacent edges of the same value, shows that $1 + i$ is a square root of $2i$.

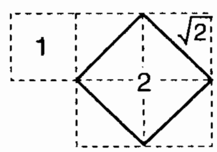


By cutting tile in half, one obtains a square whose value is i:



If the tile are cut in half again, one can rearrange the above square into a square with two adjacent edges of equal value:

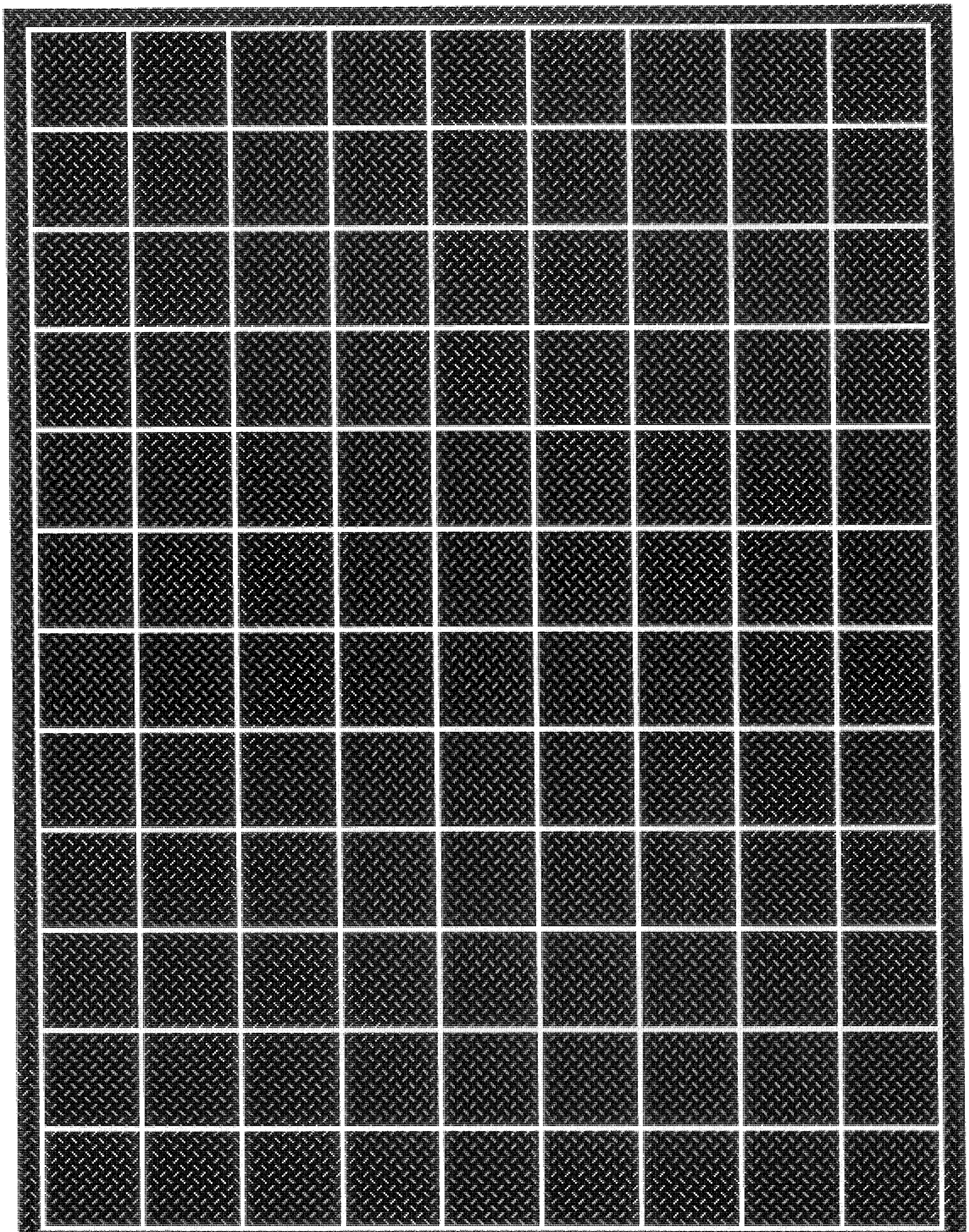




The square shown here has area 2. Its edges are diagonals of unit squares.

The length of each edge piece is half the diagonal of a unit square, or $\sqrt{2}/2$ (see the sketch). Thus, $\sqrt{2}/2 + \sqrt{2}/2\, i$ is a square root of i. Since each edge could consist of a red and a yellow edge piece, rather than a black and a green, $-\sqrt{2}/2 - \sqrt{2}/2\, i$ is also a square root of i.

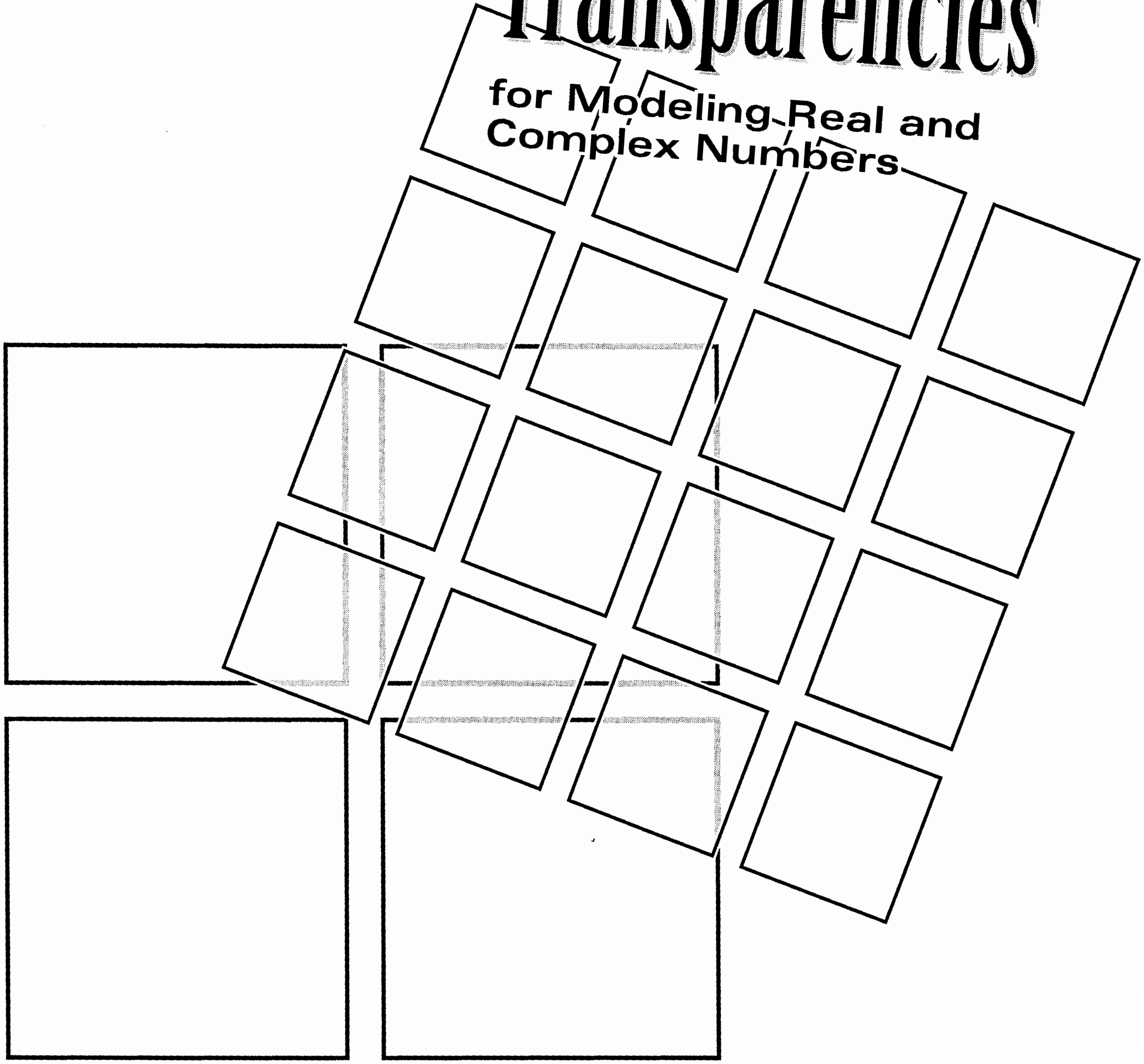
Note that no new colors—only scissors—are necessary to obtain square roots for i, that is, no new colors are needed to solve the equation $x^2 = i$. This is an illustration of the *Fundamental Theorem of Algebra*: Every polynomial equation with complex numbers as coefficients has solutions which are complex numbers.



Unit XII / Math and the Mind's Eye

Transparencies

for Modeling Real and Complex Numbers



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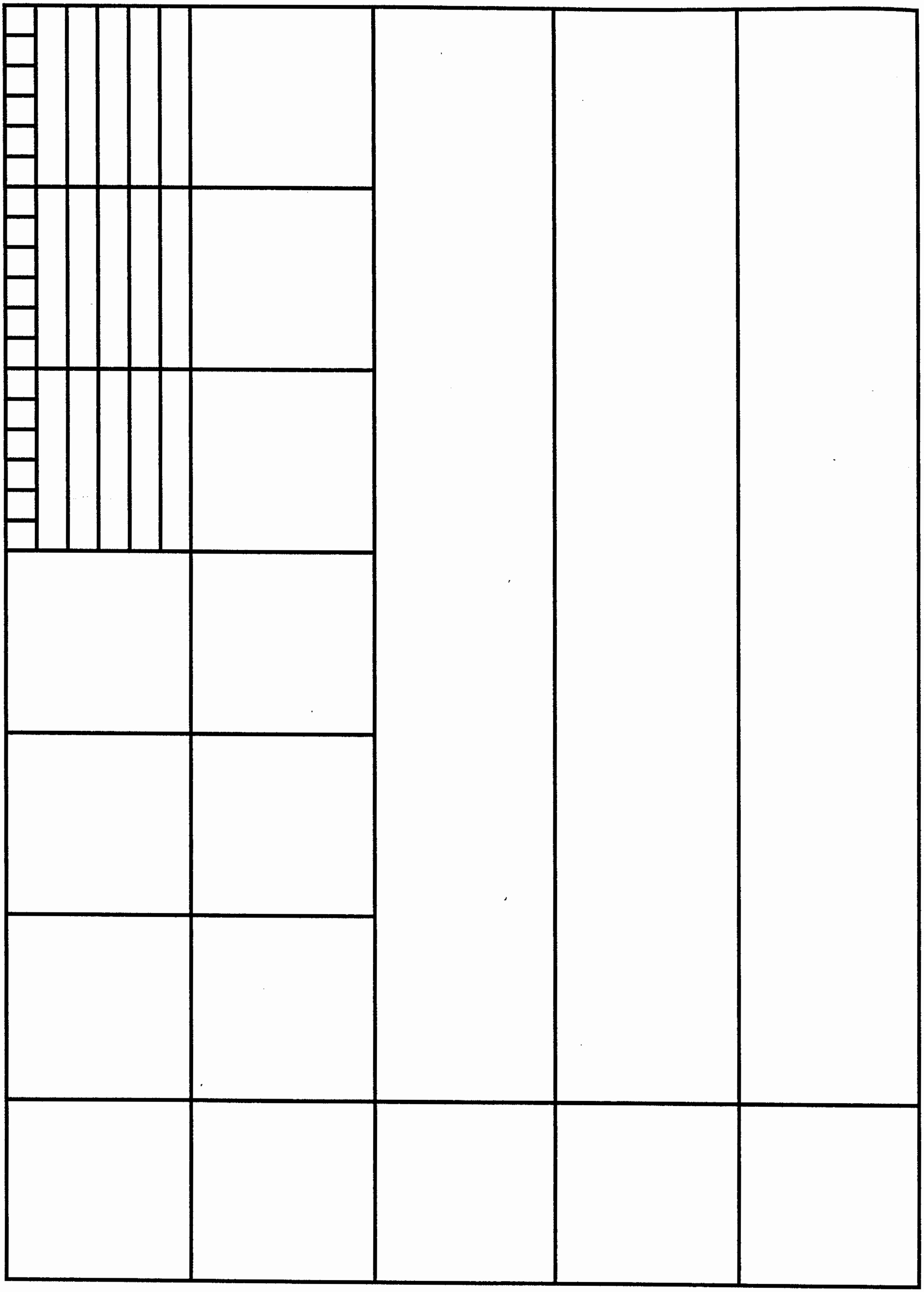
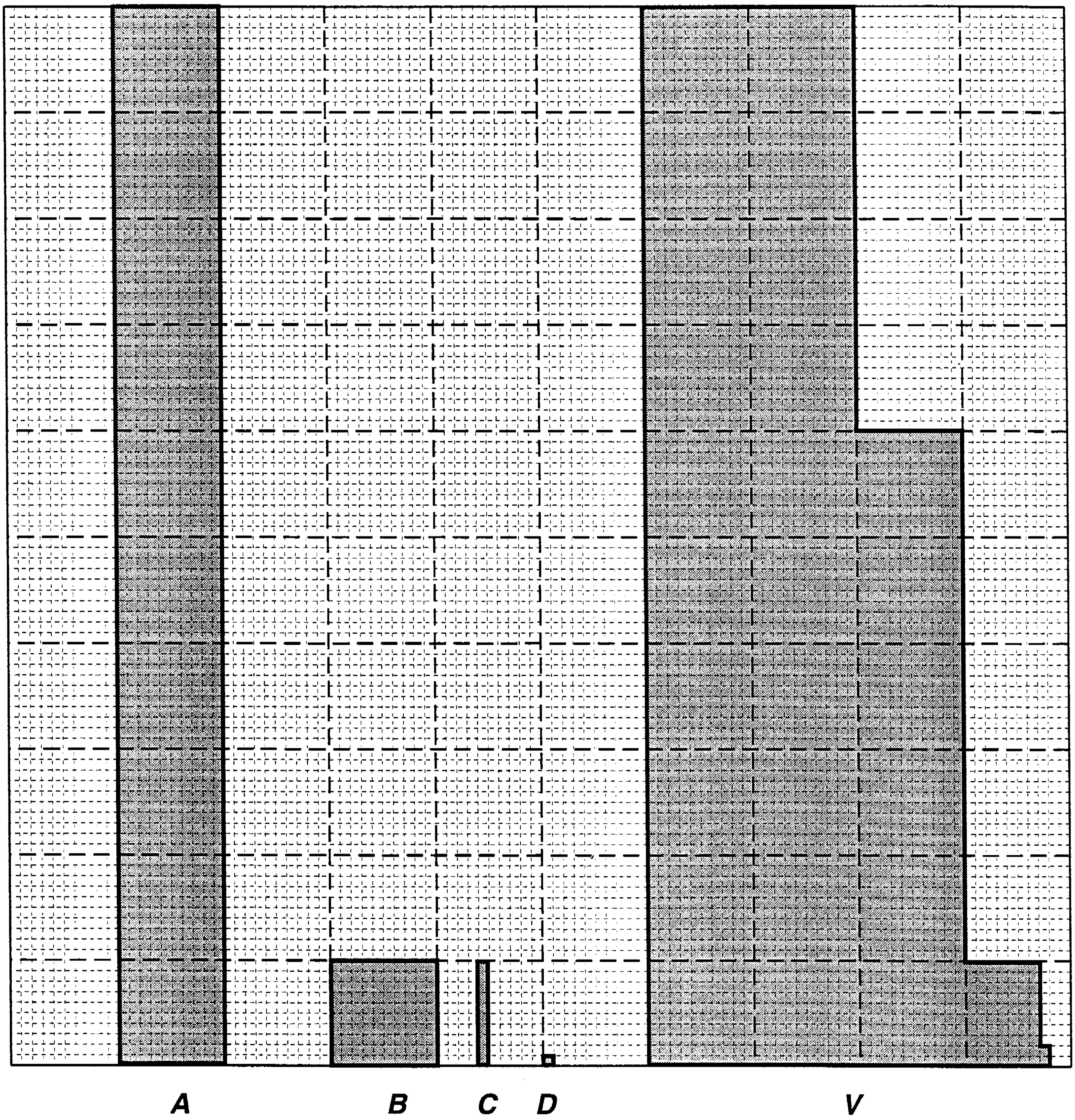
 Cut across page on above lines. Discard this piece.

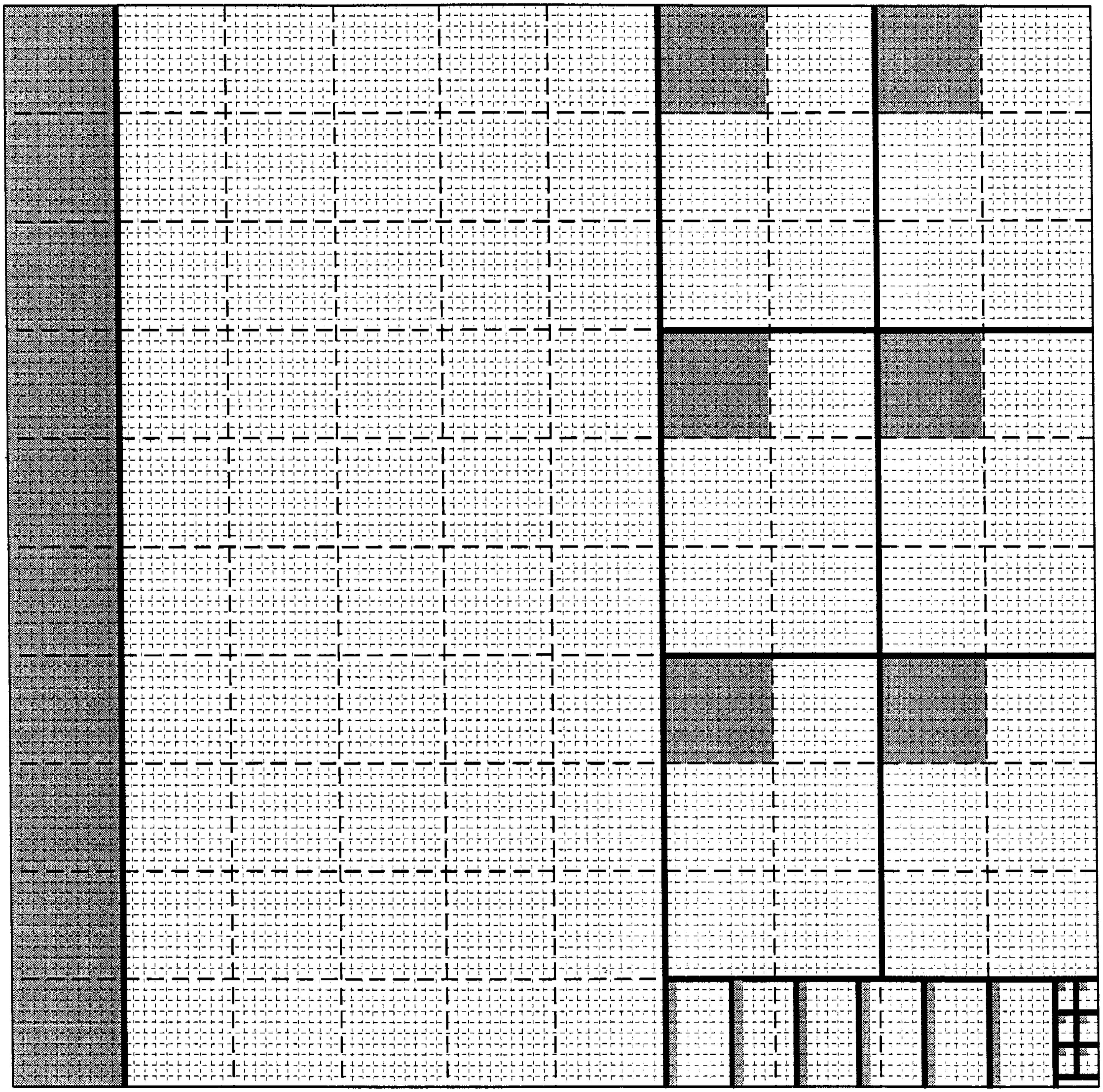
XII-1 Master 2 Cut across page on above lines. Discard this piece.



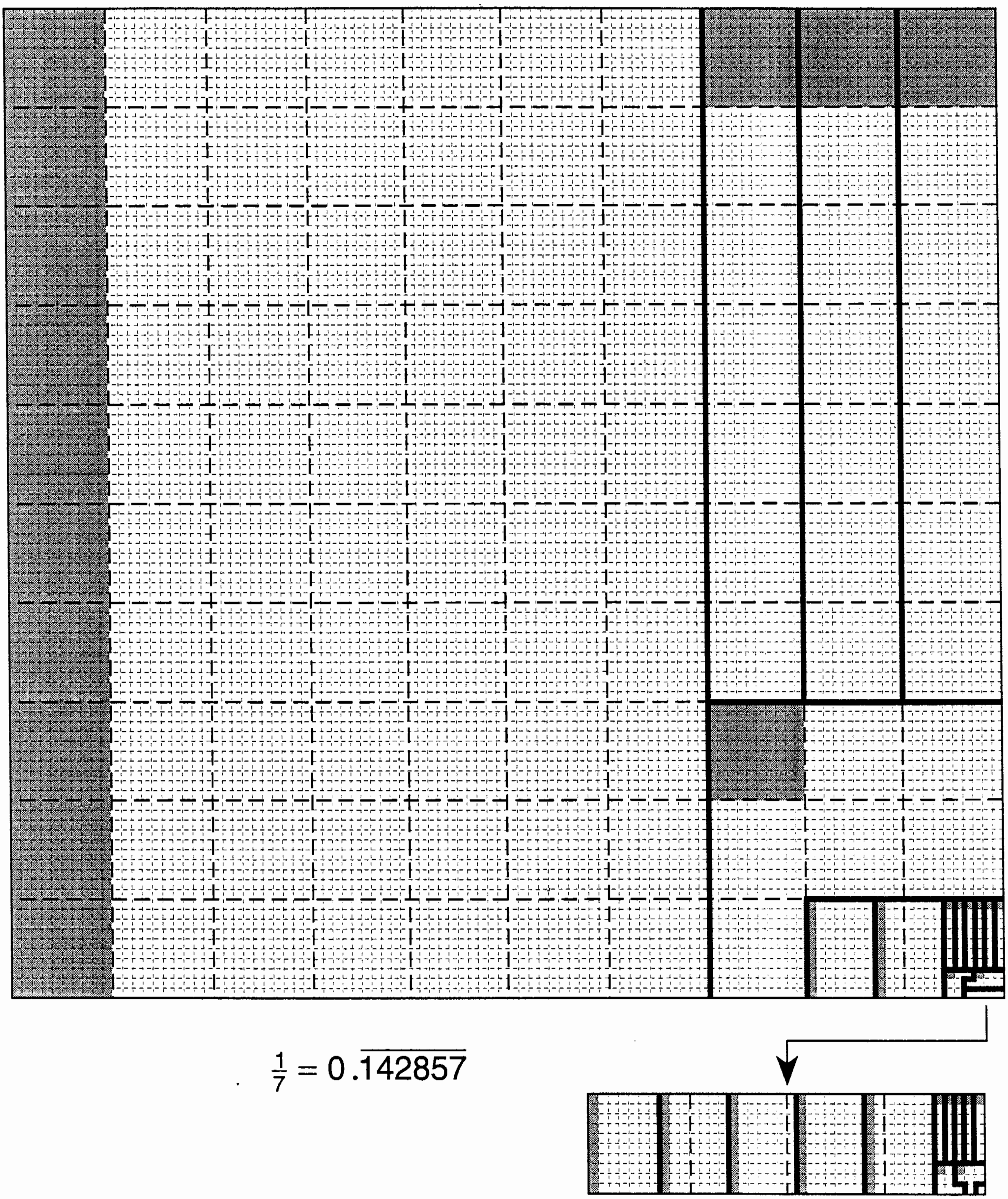
XII-1 Master 3 Cut on lines.


A
B
C
D
V

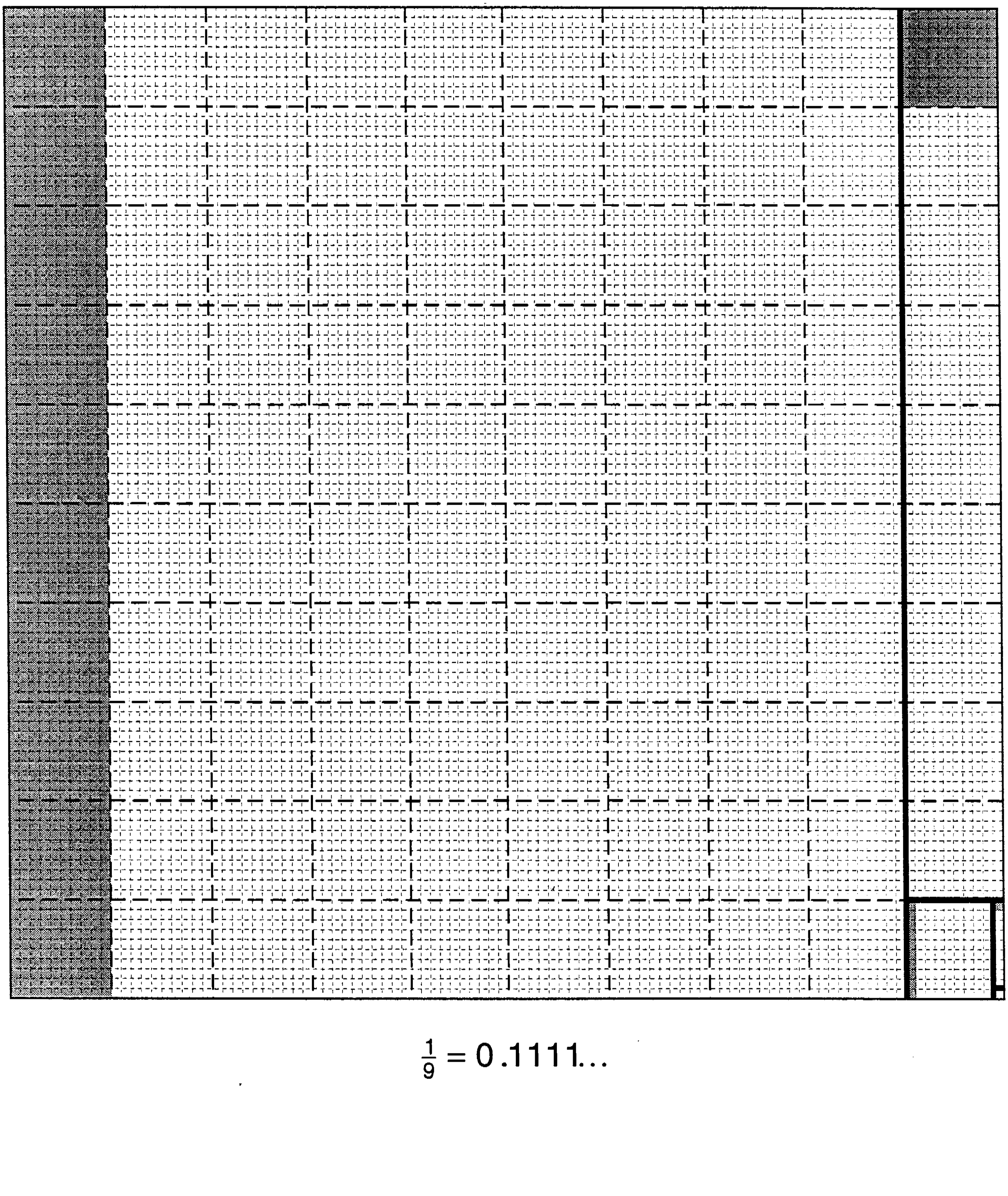
Decimal Grid Paper



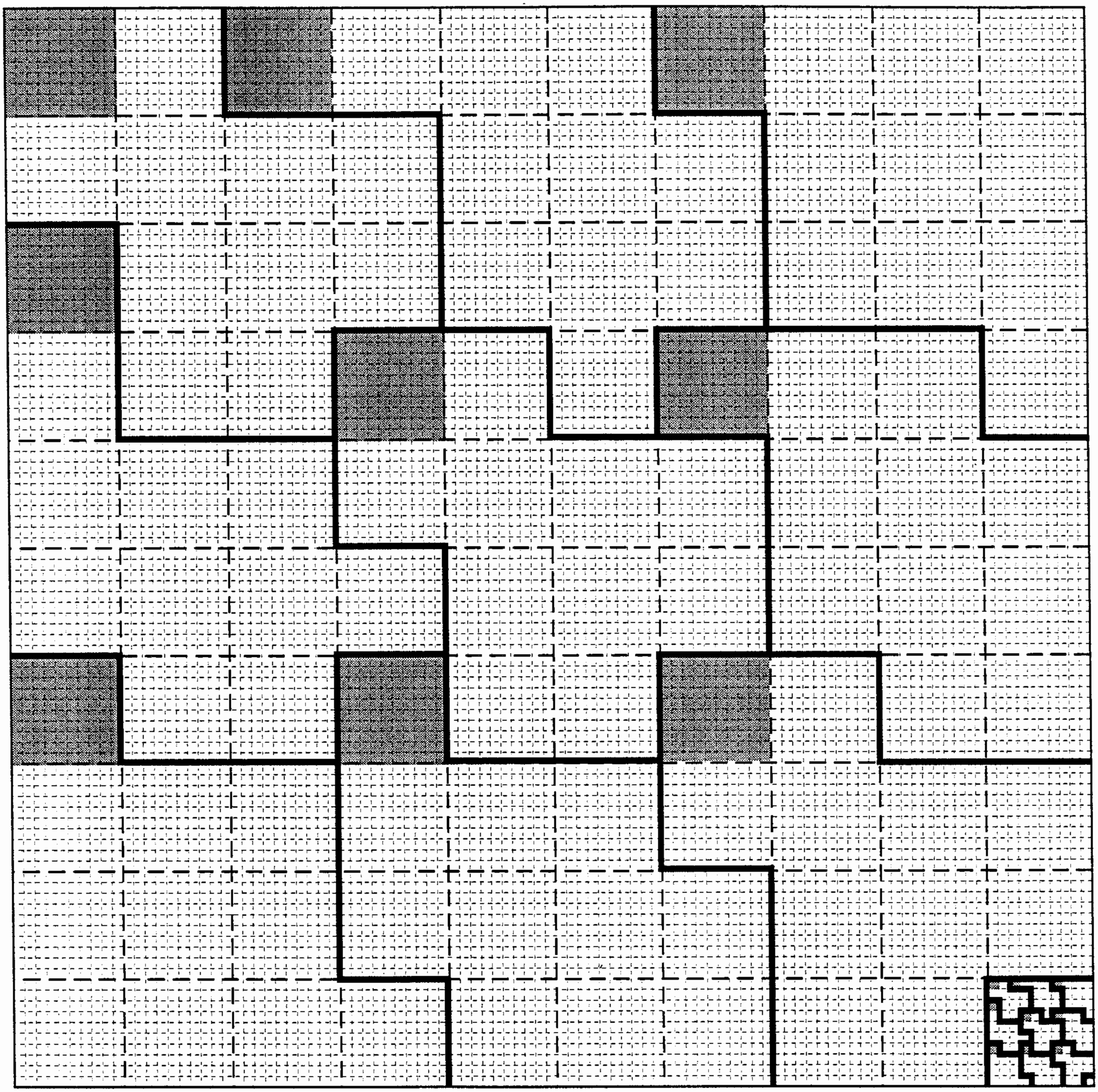
$\frac{1}{6} = 0.1666\ldots$



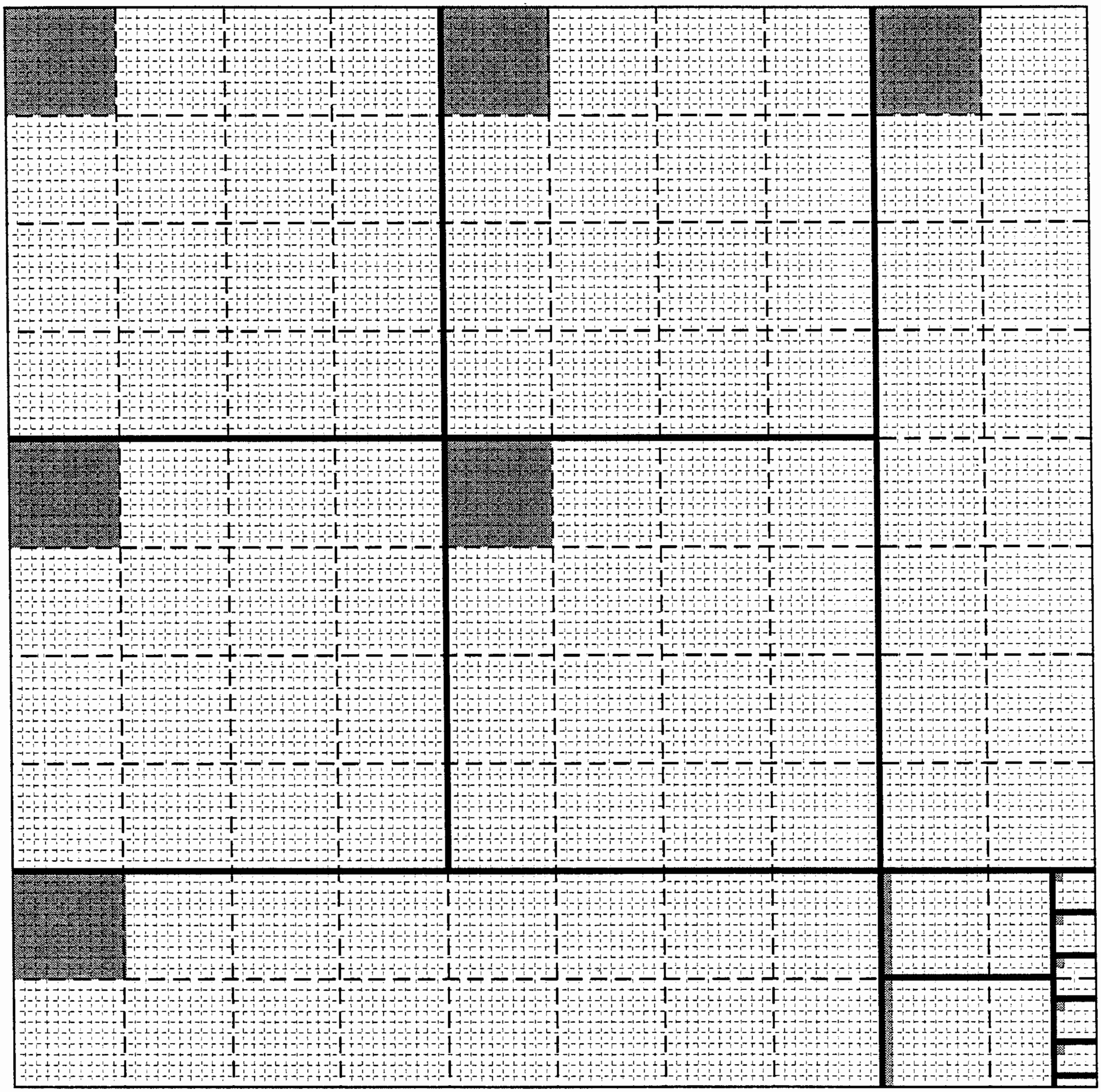
$\frac{1}{7} = 0.\overline{142857}$



$\frac{1}{9} = 0.1111...$

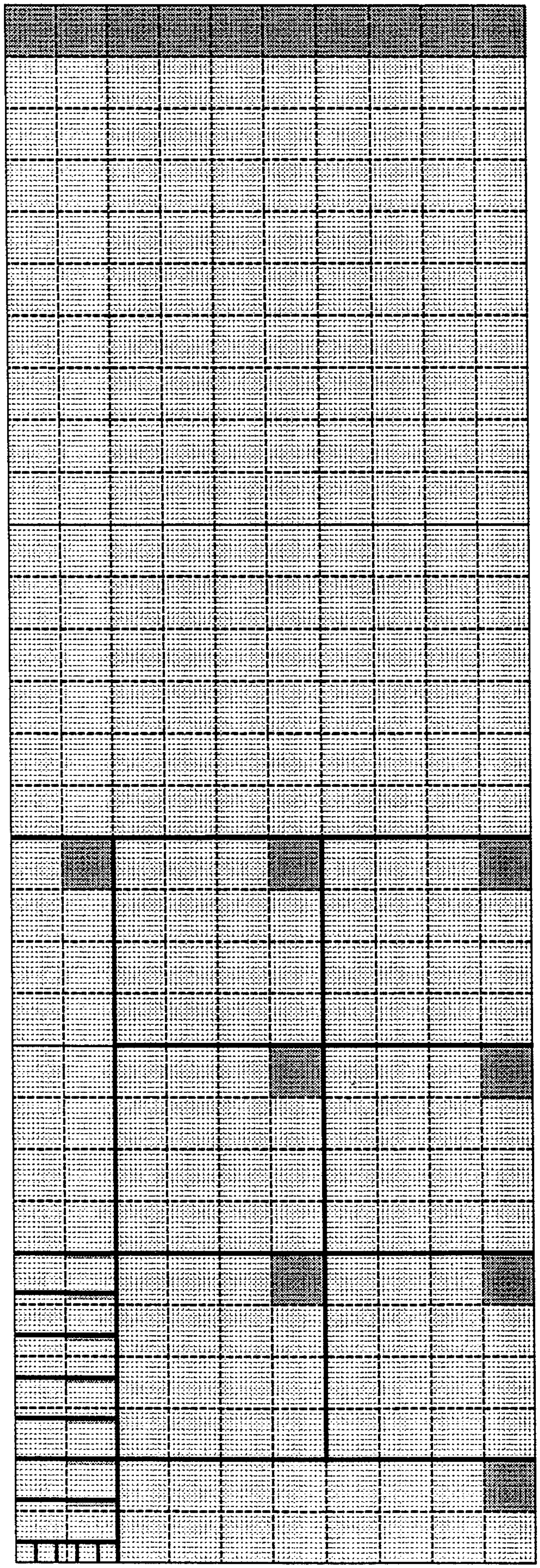


$\frac{1}{11} = 0.0909\ldots$



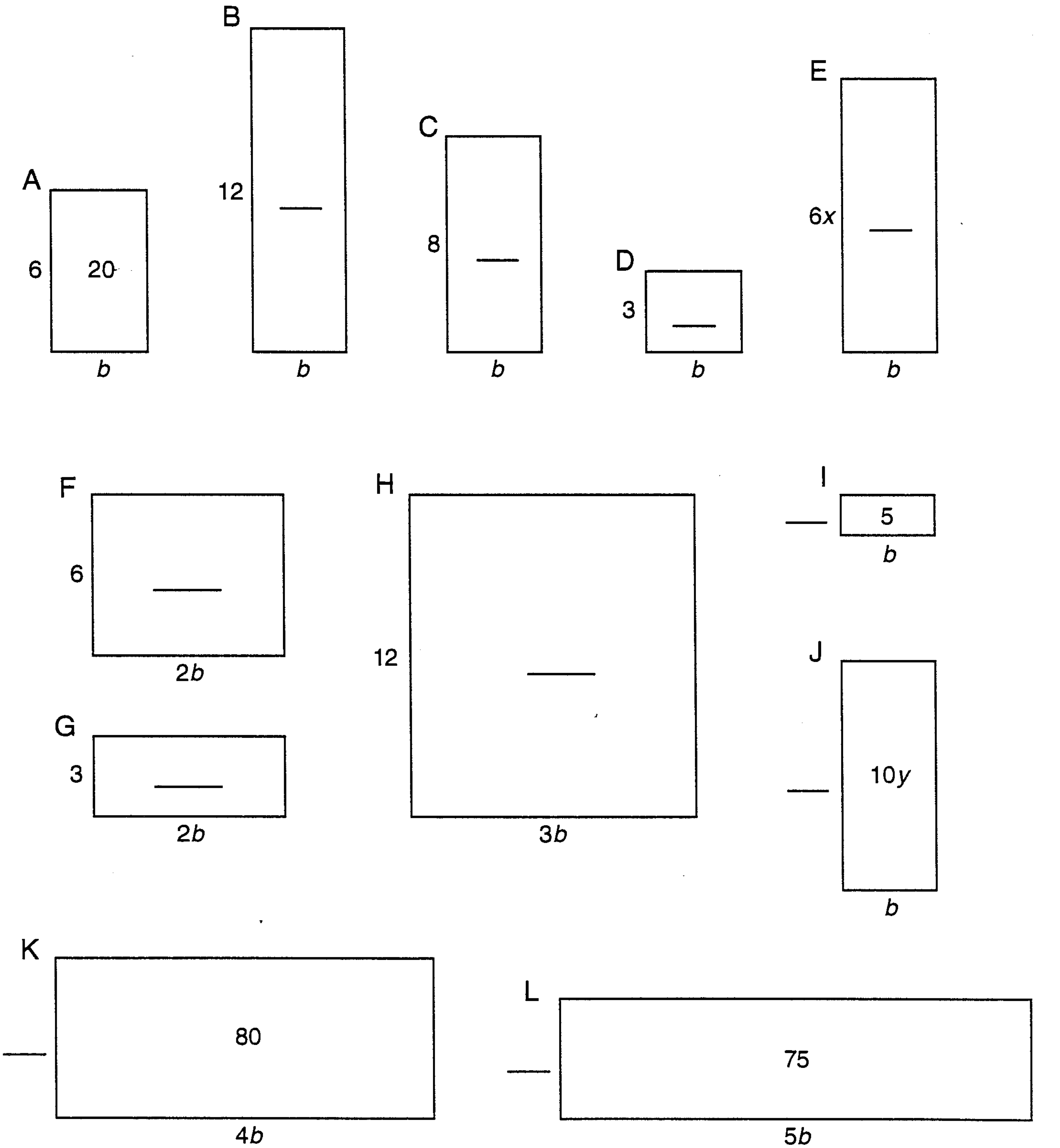
$\frac{1}{16} = 0.0625$

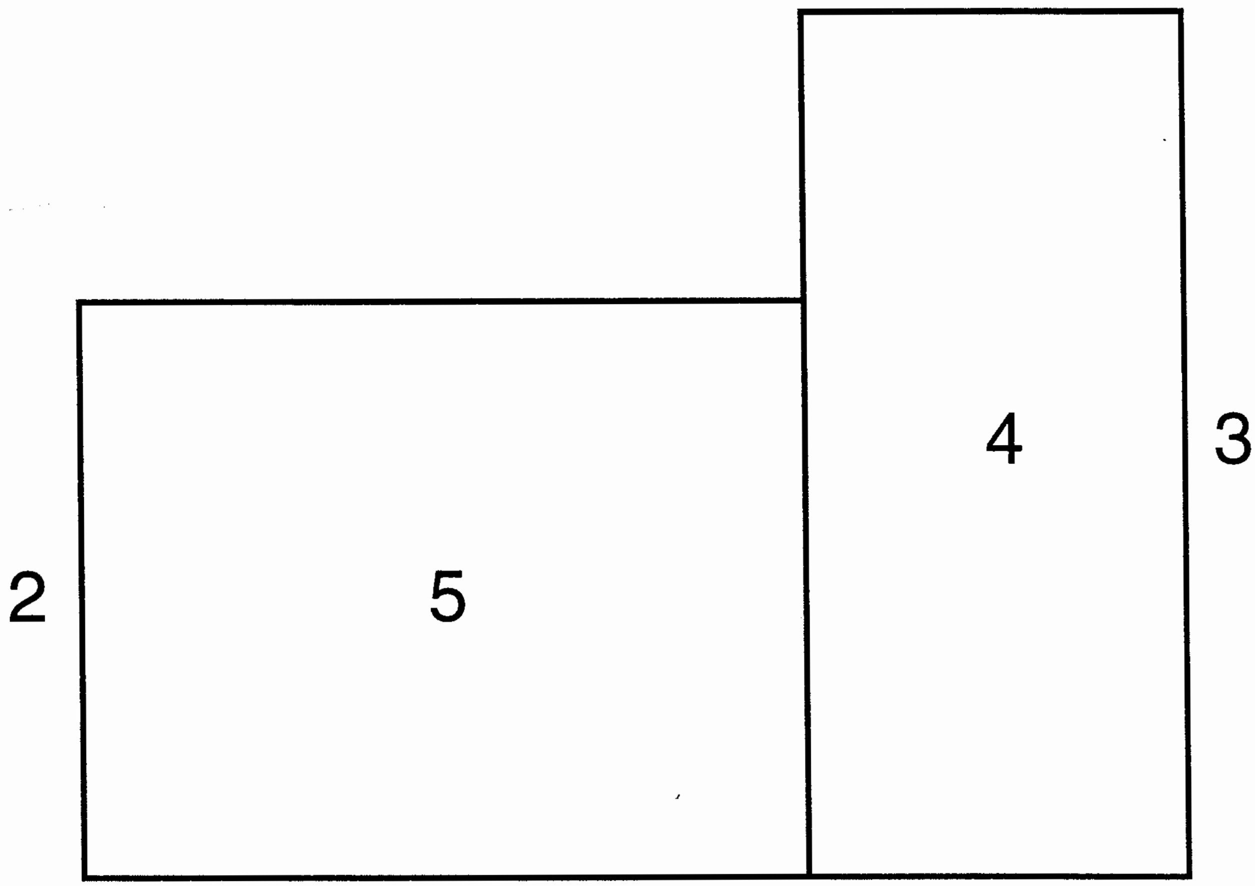
$\frac{3}{16} = 0.1875$

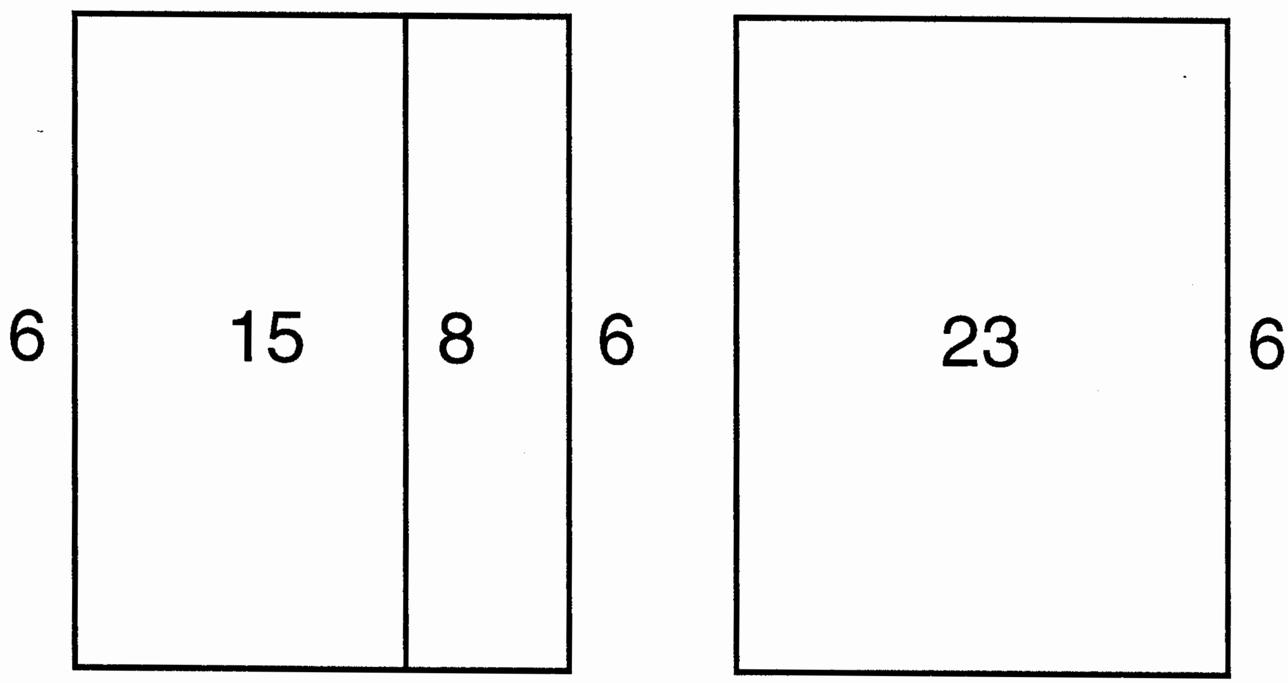


Name ____________________

Rectangle A has area 20, height 6 and base *b*. For the other rectangles, either the area or the height is missing. Fill in the missing area or height with either a numerical value or an algebraic expression that does not involve *b*.

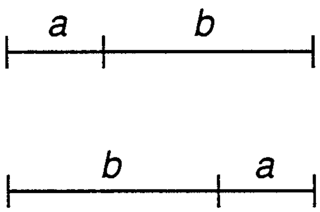



4
3
2
5

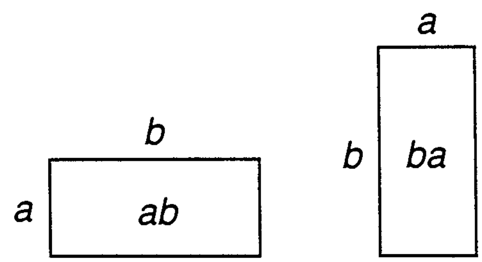

6
15
8
6
23
6

Picturing Properties

I. Commutative Properties

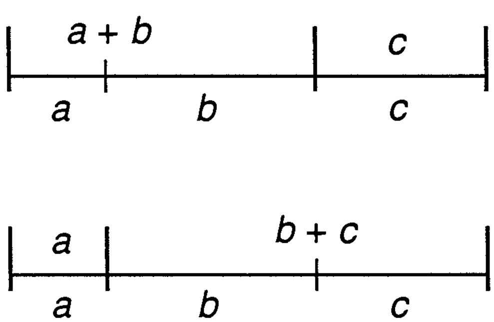


$a + b = b + a$

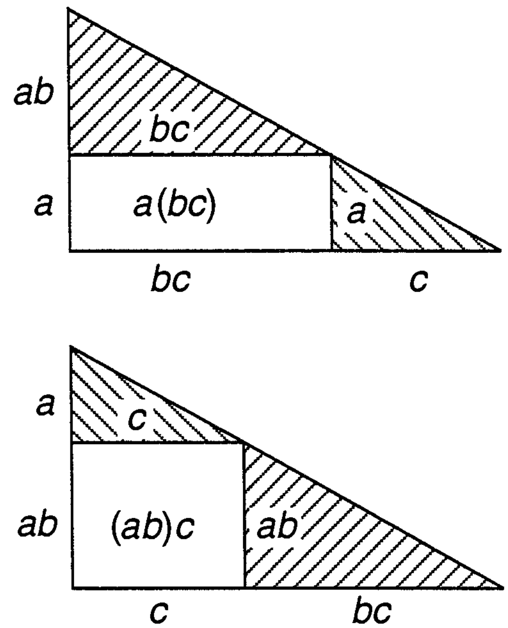


$ab = ba$

II. Associative Properties

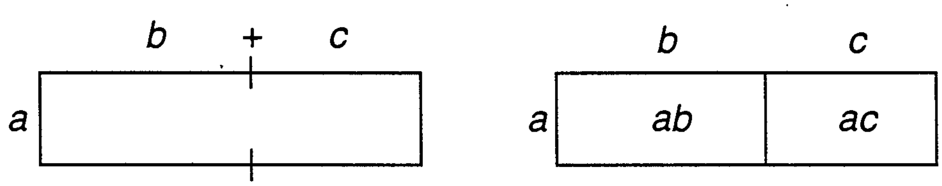


$(a + b) + c = a + (b + c)$



$a(bc) = (ab)c$

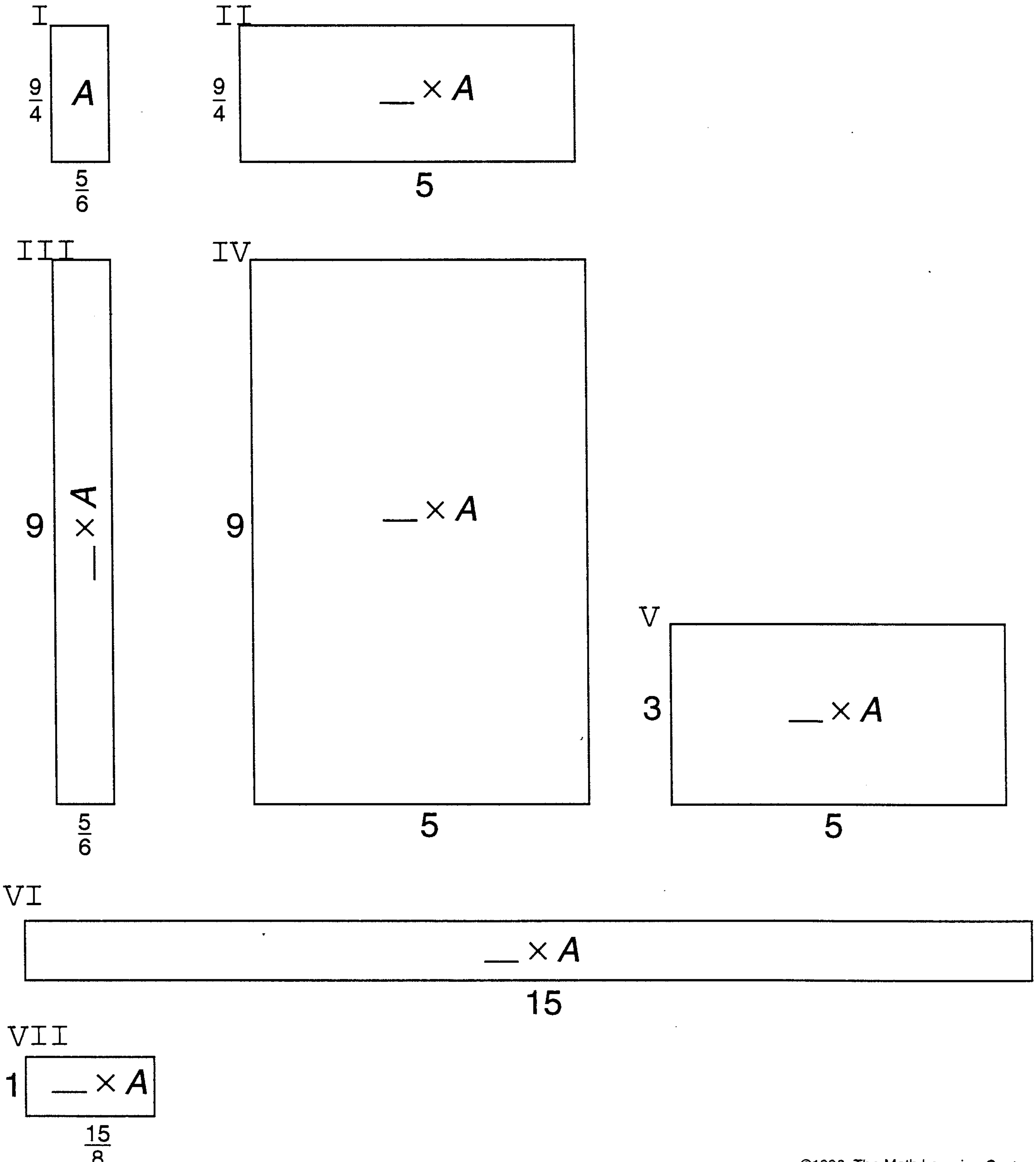
III. Distributive Properties



$a(b + c) = ab + ac$

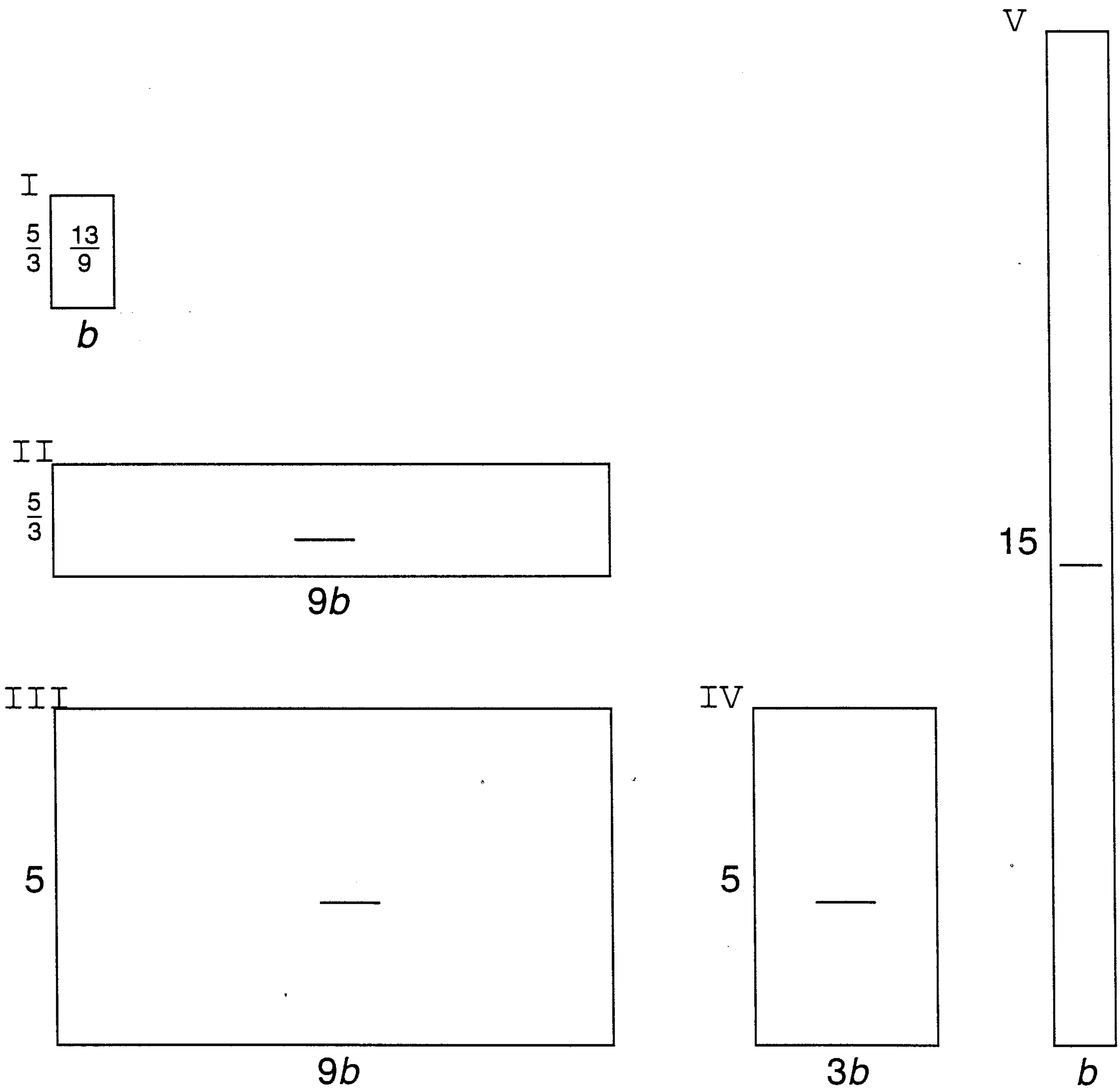
Name ______________________________

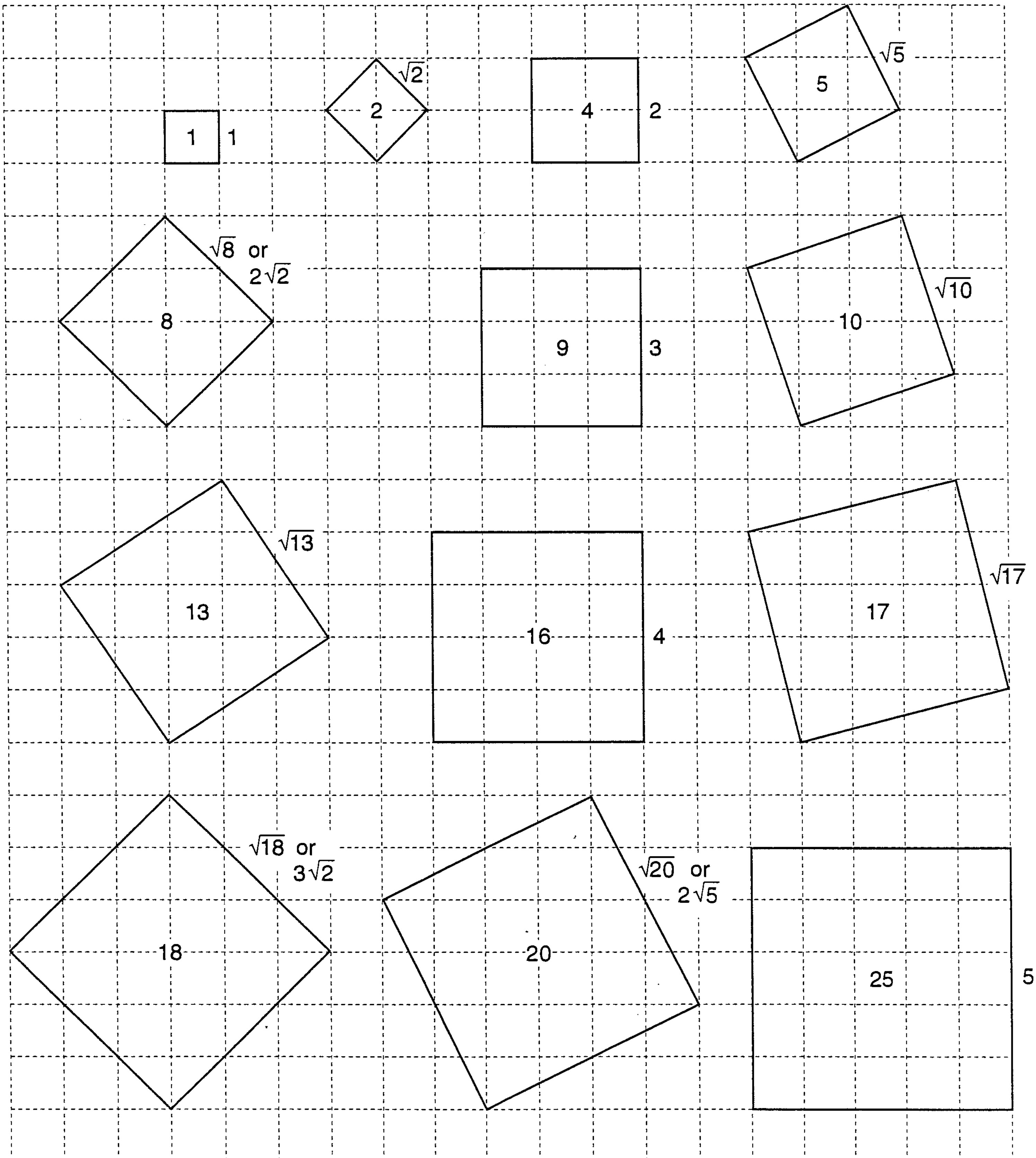
Rectangle I has area *A*. All the other rectangles have areas which are multiples of *A*. Fill the blanks with the correct multiple.

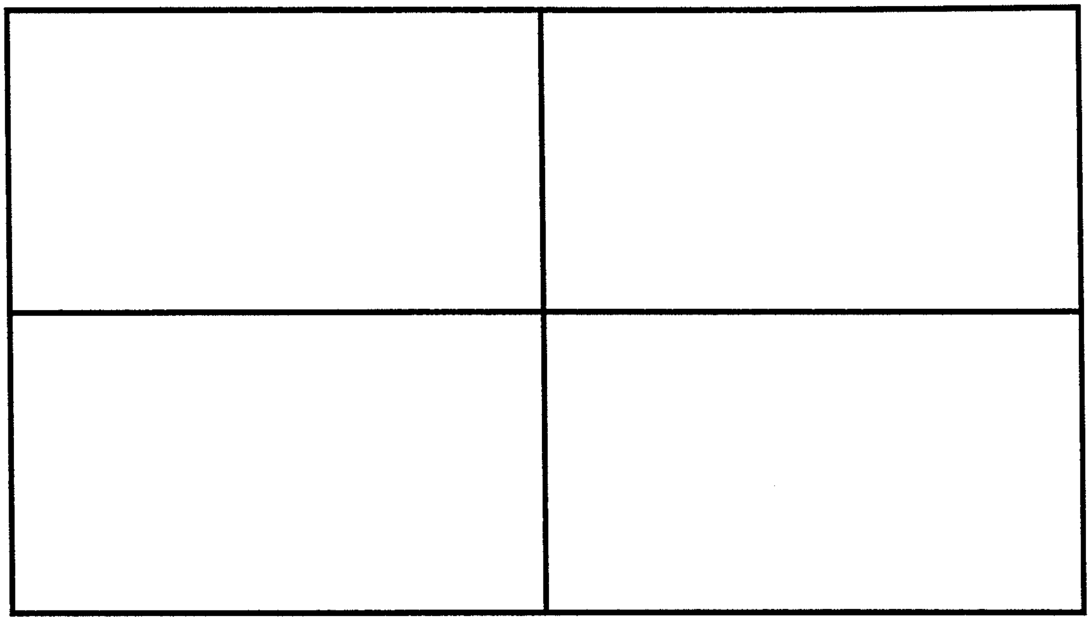


Name ____________________________

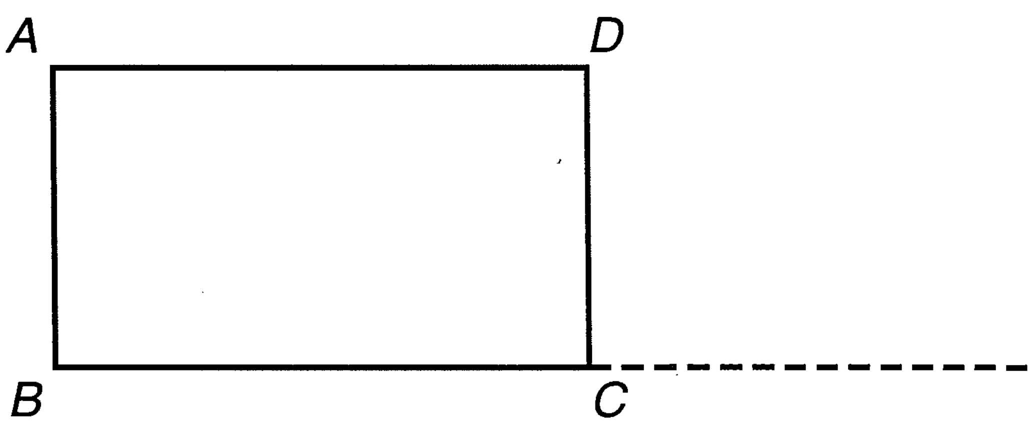
Rectangle I has area $13/9$. Find the numerical value of the areas of the other rectangles.

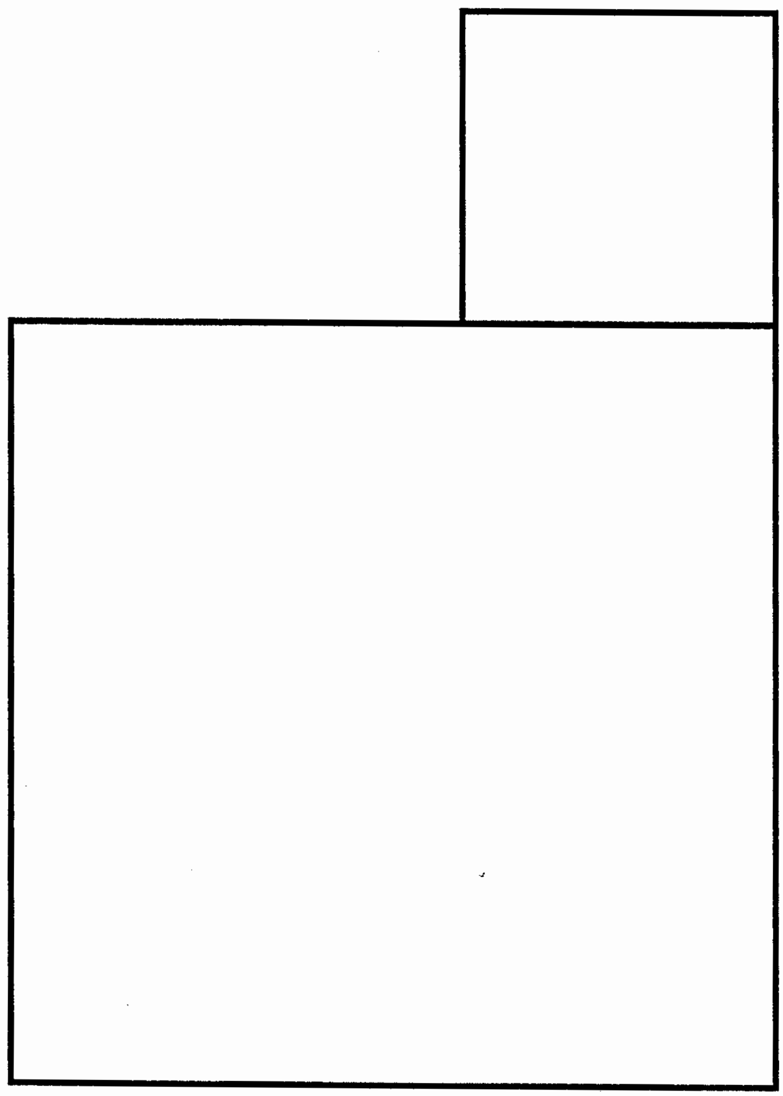


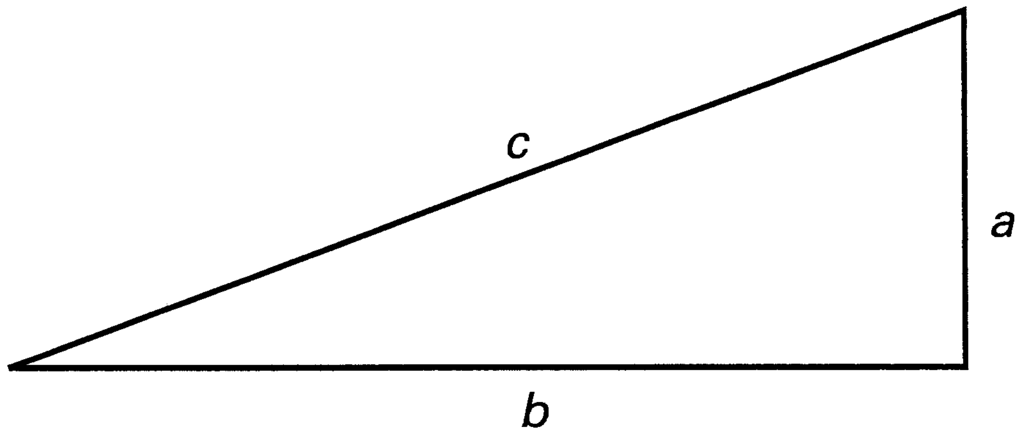

1
1
2
√2
4
2
5
√5
8
√8 or 2√2
9
3
10
√10
13
√13
16
4
17
√17
18
√18 or 3√2
20
√20 or 2√5
25
5

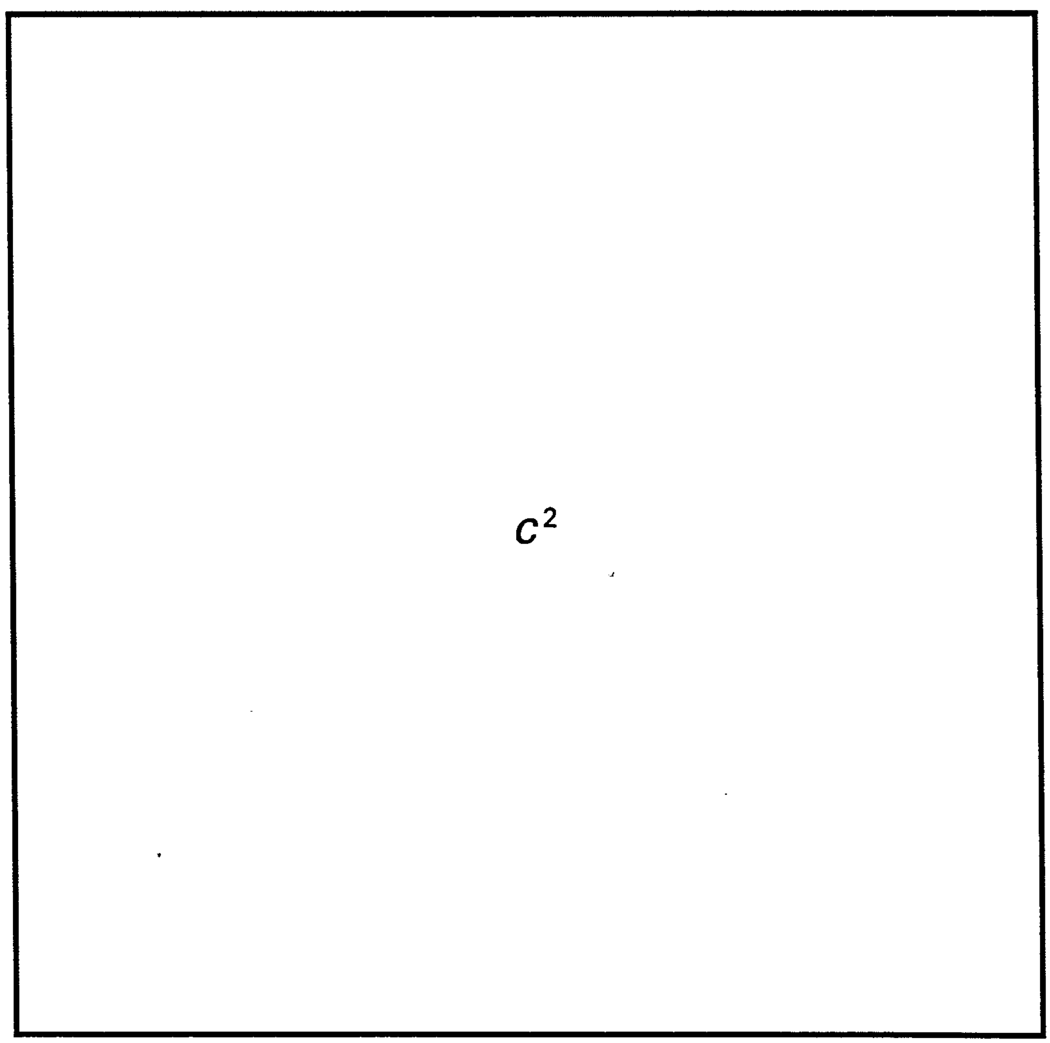


cut

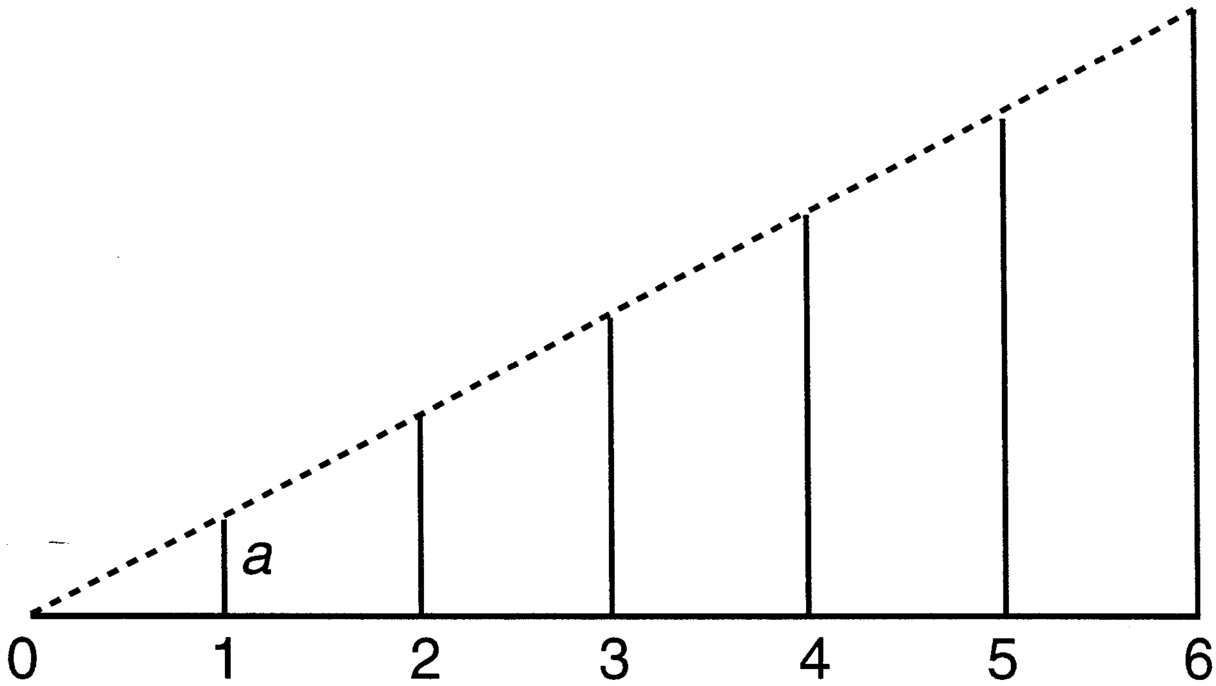


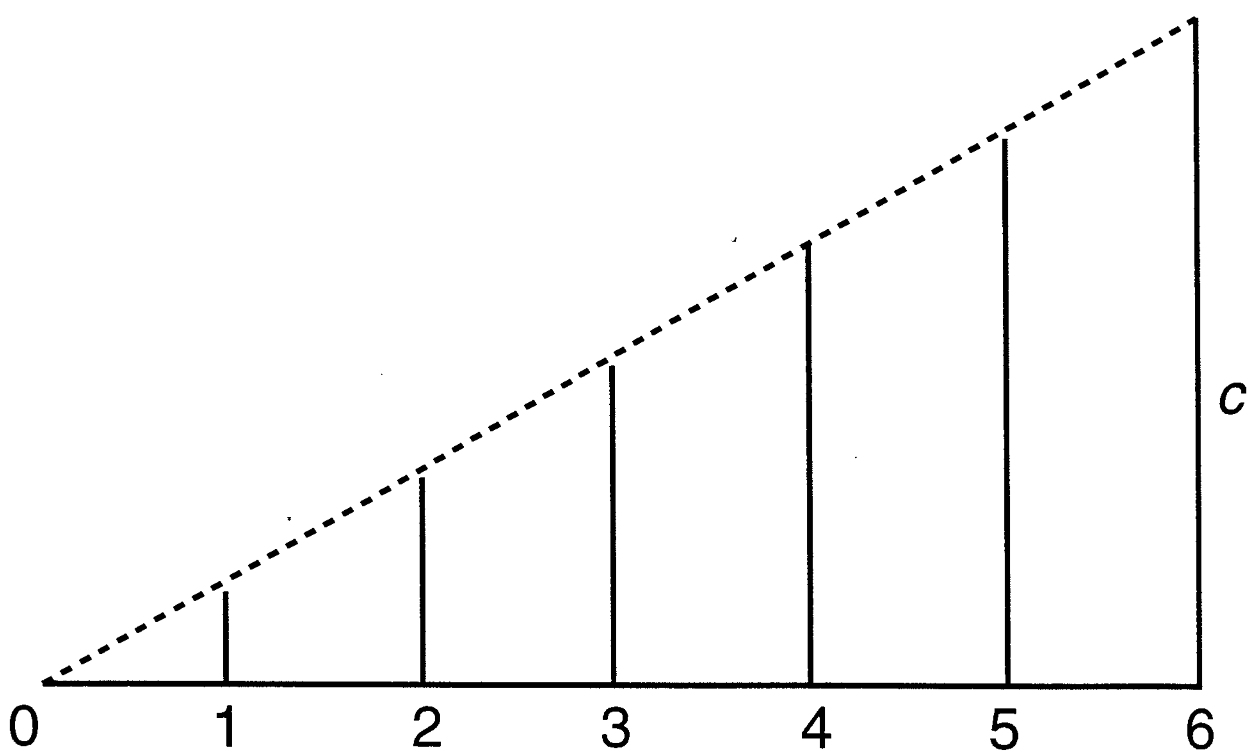



c
a
b


c^2

Name ______________________________





Name ______________________________

1

x y

Construct a line segment where length is xy.

1

x y

Construct a line segment where length is $x \div y$.