GENE'S CORNER

AND OTHER NOOKS & CRANNIES

BY EUGENE MAIER

PERSPECTIVES ON MATH EDUCATION

Maier Math Foundation
Commemorative Edition

Gene's Corner and Other Nooks & Crannies

Perspectives on Math Education

Eugene A. Maier, Ph.D.

The MATH LEARNING CENTER
www.mathlearningcenter.org

Gene's Corner and Other Nooks & Crannies
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The articles in this collection first appeared as part of Gene's Corner at www.mathlearningcenter.org, 1998–2000. They were subsequently published as *Gene's Corner and Other Nooks & Crannies* in 2003.

Requests for information should be addressed to:
The Math Learning Center
PO Box 12929, Salem, Oregon 97309

QP357, QP358 P0703 P1219

Printed in the United States of America.

ISBN 978-1-60262-625-6

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Foreword

In 1979, when I moved to Oregon as a young teacher, I believed myself to be a good math instructor. I had been a successful math student up through college and felt confident I could competently pass on the needed math knowledge to my students. I taught the way I was taught, directly and authoritatively. When students struggled to understand a math procedure that I had skillfully and repeatedly explained to them, I assumed they lacked the capability to learn math. They didn't have a "math brain." Even then, I worked hard to help all my students produce enough correct answers to move to the next grade. And most of them did. As I said, I thought I was a good math teacher. I wasn't.

Around that same time, I had the good fortune of making a connection that turned out to be pivotal in my educational journey. I met Jonathan Maier, and through that friendship came to know his parents, Gene and Vaunie Maier. Gene was president of The Math Learning Center and a professor of mathematics, most recently at the University of Oregon. Over time I learned more about his work at MLC, and became more intrigued with his perspectives on education and on life.

In 1990 I joined the MLC staff and had the opportunity to observe Gene in the classroom. Initially, I found myself challenged by his approach to teaching and learning. He didn't tell people how to do math. Instead, he encouraged them to struggle with problems, to embrace mistakes, and to be resilient in finding solutions that made sense to them. To move their thinking forward, he pressed them with authentic questions. Most importantly, he believed everyone possessed the potential and capability to learn mathematics. His students seemed confident and fluent in their mathematical abilities. To them, learning math was a worthy endeavor—and an enjoyable one. I was one of those students. Being in Gene's classrooms transformed me as an educator and a person.

Gene was a leader in promoting a student-centered teaching approach. Perhaps his greater contribution to math education, however, was his use of visual models. With simple tiles and rectangles, he was able to help teachers understand the how and why of place value, number operations, and algebraic expressions. He demonstrated geometric transformations with a geoboard. He used an egg carton to help reveal the mystery of equal fractions and ratios. He arranged red and black tiles to show why subtracting a negative integer was the same as adding a positive. With manipulatives, sketches and pictures, his students made sense, for the first time, of procedures they'd previously memorized, communicated their thinking, and showed that one person's solution path could be different from another's.

MLC is privileged to have published an abundance of materials that contain Gene's work. The initial set of activities he and his colleagues wrote, the *Math and the Mind's Eye* units, are the foundation for practically all of MLC's curriculum and instructional materials, which have influenced countless teachers and students to discover their inner mathematician. During his last few years at MLC, Gene presented his provocative observations and commentary on his own page of The Math Learning Center website, Gene's Corner. His articles challenge readers to question the status quo of education and to think more deeply about teaching and learning.

With the re-publication of *Gene's Corner and Other Nooks and Crannies,* MLC is delighted to share this timeless collection. A sampling of titles in the table of contents—"Another Case of Swindling," "The PTA Does Fractions," and "How to Make a Mathaphobe"—conveys Gene's gift for reimagining instruction as a pursuit that is alive, intriguing, and connected to life's concerns. I'd be hard-pressed to identify my favorites in this trove—almost every article resonates in some profound manner. And within the "nooks and crannies" at the end of the book can be found the big ideas that Gene valued and which continue to drive MLC's mission.

I wish I could fully express the significance Gene's work has had in the world. The teaching and learning practices he shared 40 years ago are today being affirmed in classrooms, research, standards, and pre- and in-service math education courses. His contributions live on in the people he taught, in books such as this, and through the work of the newly formed Maier Math Foundation, created to expand upon Gene's work and improve mathematics education.

Thank you, Gene, for inspiring us to think differently about math, about teaching and learning, about how we relate to our world, and about seizing opportunities in a serendipitous life.

Dan Raguse
Program Director, Maier Math Foundation
Former Executive Director, The Math Learning Center
December 2019

Preface

By the beginning of 1998, I had relinquished my duties at The Math Learning Center and curtailed a half-century of teaching activity. However, my interest in mathematics education had not waned and I continued to reflect on educational trends and practices from a perspective that had unfolded over many years.

As a means of organizing my thoughts and expressing my views, I began writing short commentaries triggered by current events or past experiences. These I posted on the Internet, accessed through a link in a corner of The Math Learning Center website (www.mathlearningcenter.org) identified as Gene's Corner, which by extension became the title of the web page on which the articles appeared. The title seems appropriate given some of the other connotations of "corner": the point where an assortment of pathways come together, or a place on the edge of things from which one views the world, or the spot on the street where you find the zealot on his soapbox.

I started the Corner with a dozen or so articles in mind, and no intention of writing them on a fixed schedule. A new article appears from time to time and the old one is moved to an archive, which has grown to contain far more than a dozen articles and is becoming increasingly cumbersome to browse. Also, from time to time, I hear of an article being used as a discussion starter in a teacher education course or being distributed by a district's curriculum specialist to its teachers or being referred to by a parent in discussions with local school officials about their child's math program. Thus, it seemed appropriate to make the articles available in print form which, for certain situations, makes the articles more accessible and also allows for easier browsing.

My first thought was to simply print out copies of the articles, duplicate them, and bind them together with a simple spiral binding. But, as often happens, first thoughts breed second thoughts, which in turn breed third thoughts. This, coupled with the insistence of those in charge of The Math Learning Center's productions that any Center publication meet their aesthetic standards, led to a more elaborate project than I first envisioned.

The second thought was to supplement the Gene's Corner articles with one or two older articles that have retained some readership. The third thought was to go further and add a whole series of articles that would supplement the Corner articles. For the most part, these articles would predate the Corner articles and show the evolution of my views about mathematics education as I matured from the young fellow you see on the back cover—chalk in hand, ready to fill the blackboard with mathematics—to that older, and hopefully, wiser person, you see on the edge of the front cover.

My teaching style has changed as much as my appearance. I began teaching without giving much thought to the process. I taught the way I was taught. In beginning courses, I described procedures, worked examples, and answered questions on yesterday's homework. In classes of math majors, I proved theorems. Not entirely without success, if that's to be measured by the number of students who went on to earn graduate degrees in mathematics. (One of my fondest memories is of the first upper-division math class I ever taught. I was a brand new assistant professor at Pacific Lutheran College, now University. There were six students in the class. Half of them went on to earn Ph.D.s in mathematics.)

My reflecting on teaching was triggered by a number of circumstances. After I joined the University of Oregon faculty, I taught in a number of institutes for secondary mathematics teachers during the heyday of National Science Foundation support for such enterprises. I became aware of the difficulty secondary teachers encountered in teaching mathematics to all comers, regardless of their interest, aptitude, and background, and also of the tendency of some teachers, especially those with minimal background who were pressed into service as math teachers, to equate learning mathematics with the mastery of mechanical procedures.

In the '70s, my experience working with teachers broadened to include both inservice and preservice elementary teachers. I directed a five-year statewide project in which the continuing education of elementary teachers, both in mathematics and mathematics pedagogy, was a major component. Also, in my university teaching, I began teaching mathematics courses for prospective elementary teachers. I soon discovered that teaching these courses was a different enterprise from teaching students who intended to teach secondary mathematics. This latter group, for the most part, had an inclination for, and an interest in, mathematics, while the former group was similar to the population at large: A few of them liked math, while most were indifferent, if not averse, to the subject, taking it only because it was required to obtain a teaching certificate. About the same time, I became involved in Project SEED, a program in which college professors taught mathematics to elementary students, and for a period a day during one academic year, I taught mathematics to fifth graders in an economically depressed neighborhood.

This contact with elementary students and teachers led me to reflect on ways of teaching topics college mathematics professors rarely give any thought to, such as the arithmetic of fractions and the rudiments of area and other geometric notions. As I strove to provide meaningful instruction, I tried methods and settings I had not attempted before. I used manipulatives. I sat students in groups around tables where they could interact with one another. I lectured less. I sought and discussed students' insights and ideas.

Meanwhile, the statewide project ended and some of us, intent on continuing activities initiated during the project, especially our efforts to develop enthusiastic elementary mathematics teachers, formed The Math Learning Center. We viewed math as a fascinating and enjoyable—and even useful—human endeavor and believed that mathematics classrooms ought to convey this, rather than the all too prevalent notion that math was a dull and difficult subject. I resigned my professorship at the University of Oregon to devote full time to the Center's efforts to promote effective ways of doing this.

I, like others at the Center, was convinced that any normal human being could successfully deal with school mathematics given proper curriculum materials and teaching approaches. My search for paradigms to accomplish this led me hither and yon. What emerged, over time, from the various nooks and crannies I explored constitutes the second section of this book. The articles in this section, arranged in chronological order, trace the evolution of my thinking once I began reflecting on educational practices. I hope they stimulate reflection in others.

I am particularly grateful to Sue Rawls, The Math Learning Center's production manager, for overseeing the design and production of this book, to Travis Waage for designing the book, and to Tyson Smith for his whimsical, yet trenchant, illustrations. Their contributions have added greatly to the readability and friendliness of this volume.

Finally, I owe a great deal of thanks to my editor, Vaunie, who is also my wife. As my editor, she monitors my writing for clarity and conciseness with competence and forthrightness. As my wife of over fifty years—from the time she was completing a degree in journalism and I was a graduate student in mathematics—she has been, and continues to be, a loving, trusting, and patient companion as I pursue dreams and ponder philosophies.

Gene Maier
December 2002

Introduction

In the more than 25 years I have known Gene Maier, he has been to me at various time an employer, a teacher, an advisor, an editor, and a friend. As I look back over the moments we've shared, two stand out in particular. Taken together, these two moments serve as a way to introduce many of the themes you'll encounter in reading this collection of articles. I want to recount both moments in some detail, because they also show Gene Maier to be far more complex than the gifted mathematician and math educator with whom you may already be acquainted. For as you read and reflect on his words, I think you'll discover, as I have, that Gene is also a writer, a philosopher, a humanitarian, a man with a compassionate eye and a humorous wit who strives for honesty and looks for it in others and, finally, a man with deeply religious sensibilities.

The first moment occurred in the summer of 1992, well after our first meeting in 1979. Newly enrolled as a graduate student in a middle school math program created by Gene, Ted Nelson, Marj Enneking, and others at Portland State University, I sat struggling with my homework at the large table in The Math Learning Center conference room. The problems posed were more than challenging, and none of the tools in my mental kit seemed adequate to deal with them. Deep in concentration, I hardly registered Gene's presence as he strolled past the table and glanced at my desperate scribbles. When I finally looked up to meet the mild amusement in his eyes, he commented simply, "You know, I think it's possible to solve almost any of these problems by sketching a rectangle."

I knew that he was giving me a hint, but darned if I could figure out what he meant. What did rectangles have to do with the tangle of algebra I was staring at so hopelessly? To his credit, Gene stuck around for a few more minutes, giving me enough of a lead to help me into the problem, but leaving well before it was solved. In the months that followed, I learned much more about sketching my way into the insights necessary to solve all kinds of problems, including situations that seemed far too abstract to be pictured.

While the sketches and diagrams didn't always take the form of rectangles, it was amazing to learn that, with a bit of creativity, numbers—even algebraic variables!— could be portrayed in picture form, and that for me, as a visual learner, the pictures often carried far more meaning than the mathematical symbols.

Part of Gene's genius over the past 30 years has been to develop simple visual models for such abstractions as x, negative x, x squared, x cubed, $x + 5$, and the like. These are mathematical terms that, as you will discover, have stymied minds as fine as those of Winston Churchill, Yale professor and author William Lyon Phelps, and Carl Jung, who, by his own report was terrified of mathematics. While Gene's models and teaching methods are described extensively in his *Math and the Mind's Eye* materials, the articles in this collection elucidate the philosophical and pedagogical underpinnings that have fueled Gene's tireless creativity. One of his most passionately held beliefs is that to educate is to educe, to lead out, or to bring forth understandings that already exist within the learner. At its best, according to Gene, "Mathematics education evokes the inner mathematician that exists in each of us, providing nurture and support as it emerges." True to his own beliefs, Gene didn't tell me how to solve the problem, but was able to hold my obvious discomfort as he offered me a tool that would lead to the solution and serve, more importantly, to provide further insights down the road. For sketches and diagrams did, in fact, nourish my growth as a mathematician.

The second moment in memory came about eight years later as Gene and I were standing again in the offices of The Math Learning Center discussing a presentation I was planning for a group of elementary school teachers. The topic was to be algebraic thinking, and I had asked Gene how people actually use algebra in their daily lives. He paused—it wasn't the first time we'd discussed mathematics and the real world. In fact, it was well-known to me and the other folks who worked with him that the term "real-world math" was nearly guaranteed to elicit a comment, if not an entire speech from Gene. For, as you will quickly discover in reading his articles, Gene maintains that mathematics is "a legitimate, fascinating and accessible part of the world in its own right—as real as any other part of the world."

This time, Gene held his peace for a minute and then told me that if I really wanted a good answer to my question, I should watch *Dead Poets Society.* Puzzled and intrigued, I rented the film immediately. What, I wondered, would Robin Williams have to say about algebra? The answer came midway through the movie, in the form of an impassioned speech delivered by an extraordinary English teacher to a class of high school students. I quote Mr. Keating in full here because his statement about poetry reflects Gene's deepest beliefs about education in general and mathematics in specific.

> *We read and write poetry because we are members of the human race. And the human race is filled with passion. And medicine, law, business, engineering, these are noble pursuits and necessary to sustain life. But poetry, beauty, romance, love, these are what we stay alive for. To quote from Whitman, "O me! O life!... of the questions of these recurring; of the endless trains of the faithless—of cities filled with the foolish; what good amid these, O me, O life? Answer. That you are here—that life exists, and identity; that the powerful play goes on and you may contribute a verse." That the powerful play goes on and you may contribute a verse. What will your verse be?*

What Gene would tell you, and does at length in the articles that follow, is that he loves the art and science of mathematics not for its utility and function but for its sheer beauty. To him, mathematics is poetry. All the stories he's developed, the visual models he's created, the pages and pages he's written for learners of all ages, have been designed to nourish and encourage the mathematical insights that each of us holds: insights that are part of our legacy. For, as Gene explains in a speech that captures the true depth of his love and trust in every learner, "We are made in the image of God—and, as such, we reflect all aspects of God's nature. Some of us may have more inclination toward one aspect than another—but, there is no aspect we do not have."

My best wishes to you as you read this collection of writings, some of which are funny; some deeply moving, honest, heartfelt; some downright provocative. They reveal the man, clarify the foundations upon which his work rests, and expand the boundaries of mathematics and education well beyond the territory we commonly hold as "normal." May you, as students and teachers, find inspiration, nurture, and challenge in Gene's words.

Allyn Fisher
Senior Curriculum Developer, The Math Learning Center
December 2002

Gene's Corner

NUM•BER SENSE / NUMB•ER SENSE

March 17, 1998

> ***num•ber*** *\\'nem-ber\\ n (1) a unit belonging to a mathematical system and subject to specified laws of succession, addition, and multiplication; esp : NATURAL NUMBER (2) an element of any of many mathematical systems obtained by extension or analogy with the natural number system*
>
> ***numb*** *\\'nem\\ adj numb•er; numb•est (1) devoid of sensation : DEADENED (2) devoid of emotion : INDIFFERENT*
>
> Adaped from *Webster's*

A former colleague and longtime friend called a few weeks ago. "There's a new book out that I think you'll find interesting." He was right, I did. As a matter of fact, I found it fascinating.

The Number Sense: How the Mind Creates Mathematics (Oxford University Press, 1997; 274 pp.) is written by Stanislas Dehaene, research affiliate at the Institute of Health and Medical Research in Paris, France. The book traces the rapid developments that are occurring in the field of mathematics cognition—in the author's words, "the scientific inquiry into how the human brain gives rise to mathematics"—fed by innovative experiments with infants, rapid advances in brain-imaging techniques, and the ongoing study of the effects of brain lesions, which have led to a wealth of new discoveries. In a highly readable style, the author, himself a cognition researcher with a background in mathematics and neuropsychology, traces these developments, separating that which is known from that which is highly likely, while setting forth the conclusions he has reached and what they imply about the educational process.

The author maintains that the evidence is unmistakable that we are born with a deep, intuitive understanding of the counting numbers and that we find ways of adding and subtracting them without explicit instruction. This innate understanding apparently does not extend to other types of numbers, and to gain an intuitive understanding of these requires concrete experiences that lead to mental models. "To function in an intuitive mode," the author asserts, "our brain needs images."

Further, experimental evidence shows that the human brain has not evolved "for the purpose of formal calculations." Remembering multiplication facts and carrying out algorithmic procedures are not our brains' forte. To do this successfully we turn to verbatim memory—that is, memorization without meaning—at the expense of intuition and understanding. The danger is that we become "little calculating machines that compute but do not think."

Thus, the author suggests that we de-emphasize memorizing arithmetic tables and mastering paper-and-pencil algorithms. Instead, we should take advantage of our strength, which is our associative memory. This is what enables us to connect disparate data, use analogies to advantage, and apply knowledge in novel settings—all things that calculators don't do well. And above all, whatever we do in school, we should honor and nurture the vast amount of intuitive knowledge about numbers children bring to the educational process.

A few days before my friend's call, I had read the new "back-to-the-basics" standards adopted by the California State Board of Education, which emphasizes memorization and paper-and-pencil computations while limiting the use of electronic calculators. I was struck by the dissonance between this document and Dehaene's book. While the California standards promote memorization and drill as the basis of mathematical understanding, Dehaene is telling us that, while students can become proficient at such tasks by memorizing sequences of operations, they do so at the expense of understanding and creativity.

Dehaene suggests another course. He believes that the experimental evidence clearly shows that arithmetic algorithms are difficult for humans "to faithfully acquire and execute." Knowing that "we cannot hope to alter the architecture of the brain," he suggests we "adapt our teaching methods to the constraints of our biology." "Since," he continues, "arithmetic tables and calculation algorithms are, in a way, counternatural, I believe that we should seriously ponder the necessity of inculcating them in our children. Luckily, we now have an alternative—the electronic calculator, which is cheap, omnipresent, and infallible.…I am convinced that by releasing children from the tedious and mechanical constraints of calculation, the calculator can help them to concentrate on meaning."

So, it appears, we mathematics educators have a choice. We can foster mathematics programs that build on the extensive num•ber sense children bring to the educational process and are compatible with the development of mathematical intuition and understanding, or we can foster programs that, each year, leave our students with a numb•er sense of mathematics than they had the year before.

Another Case of Swindling

April 8, 1998

A few weeks ago I attended a meeting of the advisory board of the Interactive Mathematics Program (IMP), Rocky Mountain Region. This is a professional development and support program for secondary mathematics teachers who are implementing IMP curriculum, an integrated, problem-centered secondary curriculum that stresses the development of problem-solving skills and conceptual understanding. It requires changes in the roles of students and teachers from that of the traditional mathematics classroom. The student becomes an active rather than passive learner, and the teacher becomes a mentor rather than a lecturer.

One session of the meeting was a panel of upper-division college students who had been IMP students in high school and were describing how their involvement in IPM had benefited their college educations. They also contrasted the nature of their college mathematics courses with what they had experienced in IMP. As an IMP student, understanding mathematics concepts and attaching meaning to procedures was important. In most of their college classes, the panel members reported, that wasn't important, at least not to the students—they just wanted to know how to do the problems. And often the professors' lectures were beyond comprehension. One bright young woman who was about to graduate in international business recounted in particular a statistics class she had taken in which she had no idea what was going on. I asked her what grade she got in the course. She replied, dismissing the question as if the answer should be obvious for, after all, she was a good student, "Oh, I don't remember, it was a B or a B+." Just as I expected, I thought to myself, another case of math swindling.

I borrow the phrase from Carl Jung. In his autobiography, *Memories, Dreams, Reflections*, Jung ponders why school mathematics was so trying to him when he had no doubts about his ability to calculate. He describes an algebra class in which he was completely confused: "From time to time the teacher would say, 'Here we put the expression so-and-so,' and then he would scribble a few letters on the blackboard. I had no idea where he got them and why he did it—the only reason I could see was that it enabled him to bring the procedure to what he felt was a satisfactory conclusion. I was so intimidated by my incomprehension that I did not dare ask any questions." However, Jung tells us, "I was able to get along, more or less, by copying out algebraic formulas whose meaning I did not understand, and by memorizing where a particular combination of letters had stood on the blackboard. And," he continues, "thanks to my good visual memory, I contrived for a long while to swindle my way through mathematics, I usually had good marks."

I was near the top of the class when, in reality, I had no idea what was going on. I could state all the definitions and prove all the theorems, but, in so doing, I was relying on my memory and not my understanding.

Swindling one's way through mathematics. I know exactly what Jung meant. I've done it, too—even as a graduate student in mathematics. There are courses I have taken, where if you would look at the scores recorded on test papers, I was near the top of the class when, in reality, I had no idea what was going on. I could state all the definitions and prove all the theorems, but, in so doing, I was relying on my memory and not my understanding.

Others, I have discovered, have also been in on the scam. On a number of occasions I have asked an audience of adults if anyone ever swindled their way through a math class, that is, taken a math class, done the required work and passed—perhaps with an above-average grade—and afterwards wondered what the class was all about. Hands go up all over the room. And if I ask about their experiences, tales of memorized procedures, rote learning, and repetitive drill abound. A lot of us have been math swindlers.

That's not to say we swindlers have done anything dishonorable. We played by the rules of the game. We figured out how to give our teachers the answers they wanted. Within us, we knew that our understanding was superficial. But that was of small concern compared to passing courses, earning diplomas, and getting scholarships.

While swindling may be prevalent, it's not universal. I have been in a number of school settings where swindling is not an issue. Concepts are introduced with concrete examples, students discuss their understandings with one another, meaningful questions are asked, procedures are developed and tested, problems are posed and readily tackled, and students willingly talk about what they know and don't know. And students are evaluated on something other than their ability to successfully carry out routinized procedures.

The next time you look at a set of test scores, ask yourself what's been measured: the test taker's mathematical expertise or their skill at swindling.

Long Division—Dead as a Dodo Bird?

April 27, 1998

Over sixteen years ago, the banner headline on page 4 of the January 19, 1982, edition of *Education Week* proclaimed: "New Technology to Render Long Division 'Dead as a Dodo Bird.'" The accompanying article reported on remarks made by Richard Anderson, then president of the Mathematical Association of America, during a symposium on "The Changing Role of the Mathematical and Computer Sciences Precollege Education" at a meeting of the American Association for the Advancement of Science.

According to the article, Anderson predicted: "when computers and calculators truly come of age in the schools, paper-and-pencil long division will probably be 'as dead as a dodo bird.'" He maintained that calculators eliminate the need for laborious paper-and-pencil computations and are "changing the nature of what is important in arithmetic." He also predicted that opposition to these changes is likely and educators will stick to the traditional methods they themselves learned.

Anderson was pretty much right on all counts. Many educators are sticking to traditional methods and calculators do eliminate the need for laborious pencil-and-paper procedures. And as far as the extinction of long division, Anderson knew enough about the educational system to add that "coming of age in the classroom" caveat.

Coming of age seems to take a lot longer in the classroom than in other parts of the world. When I was in high school in the early '40s, I was forbidden to turn in a typewritten essay even though typewriters had been around for over 50 years, and there had been one in my family's household for as long as I could remember. Using a typewriter, we were told, would atrophy our handwriting skills. (My handwriting skills atrophied anyway and I never did learn to type properly.)

The tenacity with which long division holds sway in the classroom and the prejudice against the acceptance of the calculator are epitomized in recent documents issued by the California State Board of Education. Under the heading NUMBER SENSE in the grade 5 section of the *California Mathematics Academic Content Standards* as adopted by the California State Board of Education and posted on the Web February 2, 1998, is the statement: "By the end of the fifth grade...students...are proficient with division, including division with positive decimals and long division with multiple digit divisors." In a chapter on the use of technology in the September 5, 1997, draft of the *Mathematics Framework for California Public Schools*, the statement is made that "everyone should be able to do arithmetic with facility and without reliance on calculators."

These statements, I think, reflect some common misconceptions. They suggest that skill at long division promotes the development of number sense while the use of a calculator—like, I was told, the use of a typewriter—is debilitating. To promote arithmetical facility, they ban the calculator which, of all universally available computational tools, enables one to do arithmetic with the greatest ease. In the interest of developing arithmetical facility, it makes more sense to me to ban paper and pencil.

Consider the process of division. You might reflect on the last time you had occasion to find the quotient of numbers outside of a school setting. The first instance that occurred to me was when, some days ago, I wanted to know the gas mileage I was getting on my new car. The trip indicator, which I had set at 0 the last time I got gas, said 347 and the gas pump switched off at 13.7 gallons. So, to determine the mileage, I wanted to divide 347 by 13.7.

There are a lot of computational tools I could have used to help me make the computation—an abacus, base ten pieces, a calculator, or paper and pencil, to mention a few. However, driving away from the station, using any of these tools was inconvenient, so I did it in my head. (There are a lot of ways to do this mentally. I don't recall how I did it at the time—as I sit here writing this, I did the following: ten 13.7's are 137, so 20 are 274. Thus I got 20 miles to the gallon with 73—the difference between 347 and 274—miles to spare. Half of 137 is 68.5, so there are five more 13.7's in 347. Thus I got 25 miles per gallon with 4.5 miles unaccounted for which is about one-third of 13.7, so I knew I got about $25\frac{1}{3}$ miles to the gallon—my calculator says $347 \div 13.7 = 25.32846$.)

If I were at home sitting at my desk and I wanted to perform that computation, I would have used a calculator. The computational mode I use depends on the circumstances and what's available. If I am teaching and want to develop understanding and number sense, I use base ten pieces. If I want to perform a multidigit computation, I use a calculator. If I have a series of computations to perform, I can create a spreadsheet. And, yes, I might use paper and pencil, to record my thinking, or to draw a sketch or diagram that aids my thinking, but rarely for the purpose of carrying out some algorithmic method I learned in school. I can't remember any instance when I have used the paper-and-pencil long division algorithm I learned in school—I suspect I have only used it a handful of times in my adult life. I have no use for it, I have other quicker and more efficient ways to calculate and besides, if I ever have need for a paper-and-pencil algorithm for long divison, I have enough knowledge—as does any moderately mathematically literate person—to create my own.

In most cases, however, when I want to do a computation and the numbers aren't too large, I do it mentally. I find that the most convenient—I don't need paper and pencil, a calculator, or any other tool—all I need is my mind, which hopefully is with me most of the time. And least of all, do I need the paper-and-pencil algorithms I learned in school. Rather, I have to thrust them from my mind—if I want to multiply 37 × 25 in my head, starting out by thinking "5 × 7 is 5 and carry 3" gets me nowhere. (On the other hand, a bit of number sense tells me that it takes four 25's to make a 100 and there are 9 groups of four and 1 more in 37, so 37 × 25 is 925.) That's why I say it's a misconception that learning a long-division algorithm, or some other paper-and-pencil procedure, has something to do with developing number sense. It is very likely to have the opposite effect.

Ask the adults in your neighborhood to mentally multiply 37 by 25. I suspect you will find that many of them have been so heavily schooled in paper-and-pencil algorithms they believe that carrying out these algorithms is what arithmetic is all about. Their natural number sense, rather than being nurtured, has been so constricted that, when it comes to arithmetic, they have no recourse but to reproduce the paper-and-pencil processes they have been drilled in. It's just as well we had taught them to do their arithmetic on a calculator, their number sense is likely to have suffered less and we would have saved a lot of time and energy.

So I await the day when calculators come of age in the schools, as they have in other parts of the world. My children marvel at my being forbidden to use a typewriter in school. Hopefully, my grandchildren will marvel at why their parents weren't handed a calculator instead of spending the fifth grade drilling on long division.

A Modest Proposal: SOQME

May 14, 1998

The past few days' electronic mail brought three messages, all of a similar vein.

A teacher reported his experiences when, earlier in his career, he was involved in a peer teacher setting. "It was so exciting," he said, when students who had worked with base ten pieces were able to see base ten representations of two-digit multiplications in their mind's eye and find the products in their head. "Unfortunately," he continues, "the teacher always wanted the students to learn the algorithm even after seeing how the student really understood the math that was taking place. They were afraid that the teacher the next year would think they didn't do their job if the student didn't know how to do the long paper-and-pencil method of multiplication."

Another teacher was involved in a workshop with the math staff of a middle school. He reports: "During one particular session, the issue of calculators came up and the teachers shared their frustrations with the kids mindlessly reaching for calculators to do every type of computation, including single digit. Rather than examine why they were doing that and examine ways to empower kids to think for themselves, I find it interesting that they blamed the calculator itself as the culprit. Consequently, midway into the school year, after kids had unrestricted use of calculators in their math and science classes up to that point, the staff and administration agreed to ban calculators and their use in schools! I was incredulous!…What is it going to take to really, truly change things?"

The third communication, actually a series of communications over a period of four days, came from a teacher in the throes of textbook adoption. The first message reported that the middle school math department had not been able to get the approval of the school board to use the materials they had chosen. "It gets pretty comical," she says, "because the board has no idea what is in any of the books, they just know that if we suggest the best book comes in little booklets and is not in hard cover, it cannot be good. We were told yesterday that our middle school is too advanced and we need to wait for elementary and high school to catch up with us. What they don't know is that we are not near where we want to be."

The second message was a report on how well her students had performed on the American College Test and an alert that a fax was on its way. The fax was a letter to the editor printed in the local paper under the heading "The Public Speaks." The tone of the lengthy letter is captured by the following excerpt: "The key phrase for education should be 'repetition, repetition, repetition.' Anyone who has ever successfully coached a winning team will tell you that practice consisted of the repetition of basic skills. That's exactly what our children need in any math class."

The third in the series of messages was a report of the School Board's action: "The vote was 11–2...against [the department's choice]. Their minds were made up before they even walked into the boardroom. It is pretty scary that after many hours of studying the board can make a textbook decision based on ignorant parents.…It is just very hard understanding why people can't see the value of what we are doing."

In pondering the imponderable questions raised by these messages, it occurred to me that the problem might be us rather than them; that we might be addicted to the reaching of unreachable goals. So to deal with this addiction and bring us back to earth, I propose the formation of an organization with the acronym SOQME, pronounced "sock-me" and standing for the Society of Quixotic Mathematics Educators.

Membership is open to all those who are idealistic enough to believe that curriculum decisions are based on student understandings and not on some misguided rationale for teaching a particular algorithm. Or that student displays of mindlessness in the mathematics classroom have something to do with their interactions with other human beings, foremost of whom is the teacher, rather than their interactions with a machine. Or that school board members make decisions based on an enlightened view of what constitutes mathematical literacy rather than the appeasing of misinformed patrons.

Training in windmill jousting is available after school in the athletic practice field behind the high school.

The Words of Education

June 9, 1998

Several years ago I acquired a Dover republication of Ernest Weekley's *An Etymological Dictionary of Modern English*. Originally published in 1921, Weekly's dictionary, according to the blurb on the back of my copy, is "easily the finest such work ever produced." One evening, while browsing my new acquisition, I looked up the origins of as many of the words of education as came to mind.

As one might expect, many of these words no longer carry their original meanings, although one might wish that they did. Here are some of the things I discovered.

The word "student" comes from the Latin *studere*, to be zealous, which in turn stems from *studium*, eager attention. Would that it were so, especially in the presence of the "teacher" which, in Old English, is the one who shows or guides.

"School" comes from a Greek word for leisure. "A sense," Weekley notes in one of the tongue-in-cheek comments that dot his dictionary, "passing into that of otiose [i.e., idle] discussion, place for holding such." Weekley's comment sent me off to the library to see what more I could find. In his etymological dictionary, *Origins*, Eric Partridge traces the development of the meaning of "school" from the Greek word *skhole,* "originally a halt, hence a rest, leisure, hence employment for leisure, especially such employment for children." Thus school is what you did if there was nothing else to do.

The word "administer" literally means "to minister" or "to serve." The Latin minister, servant, stems from the Latin *minor*, lesser, conveying the sense that a minister is one who serves or assists other persons of higher rank. Like teachers, I presume.

"Test" and "examine" both evolved from the world of commodities which, I suppose, is appropriate if the educational establishment is viewed as a business enterprise. "Test" derives from the Latin *testum*, an earthen vessel, which evolved into the Middle English *test*, a vessel in which metals were assayed, whence being *put to the test*, that is, evaluated. "Examine" stems from the Latin *examen*, the needle, or tongue, of a balance, used in weighing.

"Discipline" has a variety of meanings today: field of study, self-control, punishment. The first of these is closest to its original meaning. The word derives from the Latin *discere*, to learn, akin to *docere*, to teach. The latter gives rise to such words as "docent" and "docile." According to Webster's, the primary meaning of docile is "easily taught" and only secondarily did docile come to mean "tractable," that is "easily managed." If one were to remain faithful to its original intent, the goal of discipline would be teachability, not tractability.

What struck me most, however, in my evening's browsing in Weekley's was the contrast in the original meanings of the words "educate" and "train." Whatever their current usage, and I suspect that some use the words interchangeably, their root meanings bring into focus the difference between two styles of teaching mathematics.

"Educate" stems from *educere*, to lead out, a Latin word that lives in Modern English as "educe." "Train" is from the French *traóner*, to drag behind one, as in "bridal train." Thus, we have two disparate metaphors for teaching: leading out or dragging behind.

Reflecting on the differences between education and training—in their original sense—these two metaphors suggest, brought several things to mind. In education, the basic raw material of learning comes from within [see the article Num•ber Sense, Numb•er Sense]—it's there, waiting to be educed, that is, as Webster's defines the word, to be brought out, as something latent. In training, the raw material comes from without the learner; it is imposed by whoever is doing the dragging—there is not much the one being dragged can do about it, except to dig in one's heels.

Education is lasting; training is transient. Once something is drawn out, it stays emerged and can be put to use whenever needed. However, if one is being dragged about—being told repeatedly to do thus-and-so—one only knows how to do thus-and-so. If one is required to do something else, all the practice one has doing thus-and-so is of no avail. As a matter of fact it may be a hindrance—one may be so entrenched in doing thus-and-so, one does it automatically, no matter what the situation.

Mathematics training drags the learner along a predetermined course disregarding the learner's readiness or preferences for the paths taken. Mathematics education evokes the inner mathematician that exists in each of us, providing nurture and support as it emerges.

And so the word journey ended. But the images remain.

The Inner Mathematician

July 10, 1998

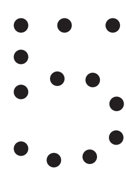
One of the tenets of The Math Learning Center is that every human being has an innate mathematical spirit that harbors a natural sense of number and space—an affinity for things numerical and geometrical. The possession of this spirit is as normal as having two eyes, walking upright, experiencing emotions. It is part of what it means to be human.

The evidence of this mathematical spirit abounds. Toddlers are fascinated by numbers and shapes. As they learn to talk, number names are sprinkled throughout their vocabulary. They are quick to imitate the counting process, perhaps a bit nonsensically from the point of view of adults who don't consider eleventy-six a number. But gradually the child sorts out the correct sequence of counting numbers, learns how they are connected to the number of objects in a set, and how they combine to form other numbers. Although the vocabulary may not be there, the child distinguishes between properties of various geometrical shapes, realizing that balls roll and blocks don't, and that you can't fit round pegs into square holes. And, all too soon perhaps, the child has the lay of their surroundings firmly in mind, devising strategies to reach all sorts of nooks and crannies in an ever-widening range of space explorations.

This everyday evidence of the toddler's natural mathematical inclinations are confirmed by the results of experiments by researchers studying mathematics cognition. Neuropsychologist Stanislas Dehaene [see the article, Num•ber Sense, Numb•er Sense] is convinced that all human beings, within their first year of life, have a well-developed intuition about numbers.

Like all infant life, the nascent inner mathematician is fragile and requires nurturing. Left to its own devices it can starve; given the wrong care, it can be strangled into unconsciousness. Many adults profess they are terrible at mathematics and have no aptitude for the subject, when in reality the mathematician within them was never allowed to flourish. Rather than having no mathematical nature, the truth more likely is someone, in the interest of providing it with their version of a proper mathematical diet, choked it to death.

Of course, that wasn't the intention. This strangling of a mathematical nature can happen in subtle, unnoticed ways. I'm sure I've contributed to a number of gasps for breath by my students, and scarcely noticed when it happened. Perhaps such instances are clearer to the eye of the classroom observer. I remember visiting a first grade class. The teacher displayed a calm and caring attitude towards her young charges. The day's arithmetic lesson dealt with writing numerals. The teacher had placed several dots on the board, like so:



and asked if anyone could take a piece of chalk and fill in between the dots to obtain the numeral 5. Several hands shot up and the teacher called on a child who eagerly went to the board and drew the numeral in one continuous motion, starting at the dot in the upper right hand corner. The teacher, in her kindly voice, said no, that wasn't right, could anyone do it correctly. Whereupon another child went to the board, started in the upper left, drew the lower part of the numeral first and, in a second stroke, drew the top line of the numeral. The teacher said yes, that was right, and cemented her verdict with a little verse about putting the cap of the 5 on last.

My, I thought, this may be how it all starts. How the seed is being sown for those oft repeated phrases, "I'm not good at math"; "I quit taking math as soon as I could"; "Math never made sense to me." Imagine being a first-grader during arithmetic period. You have just done something that made perfectly good sense and you're told you're wrong. The task must be done this other way. What does one conclude? That math is an arcane subject, governed by mysterious rules, revealed by the teacher, which are to be observed under all circumstances. To be successful, I must abandon my way of doing things and adopt the teacher's, even though my way works just fine and makes more sense to me. I better quit listening to my inner voice.

Fortunately, the mathematical spirit is resilient and can be resuscitated, even after years of dormancy. The evidence for that also abounds. If one can manage to entice adults into math workshops where their inner mathematician is honored and heeded no matter how feeble and constrained its voice, renaissances occur. A teacher becomes aware of her belief, formed as a consequence of her own schooling, that she was "not only a failure in mathematics, but incapable of learning it" discovers that she can, indeed, make sense of mathematics. A struggling sixth grade math teacher, who thought of herself as a "math dummy" and "didn't care diddly squat about math" finds an unexpected enthusiasm for math and is professionally recognized for her math teaching.

While it is exciting to see these revivals of mathematical spirits, one has to believe that it were better that the stifling of one's inner mathematician never occurred. Certainly one doesn't wish that for one's students. There is no magic elixir I know of that we can feed our students to keep their inner mathematician healthy. But rather than constraining it and attempting to conform it to our image, we can listen to its voice, allow it room to exercise and explore, and provide it with a menu of mathematical activities that promotes its growth and broadens its understanding.

The Real World

August 10, 1998

The term “the real world” bothers me, especially when it is used in discussions about teaching mathematics. It suggests to me that there is some other, more authentic place I ought to be rather than the place I am, especially if that place is the mathematics classroom.

The term often appears in discussions of problem solving. A case in point is NCTM Standard 1: Mathematics as Problem Solving. I have no quibble with the Standard; it’s the statements made in the accompanying rationale for the Standard that bother me. Statements like: “[Children] should encounter problems that arise from both real-world and mathematical contexts.” “A balance should be struck between problems that apply mathematics to the real world and problems that arise from the investigation of mathematical ideas.” “Problem situations, which for younger students necessarily arise from the real world, now often spring from within mathematics itself.”

These statements imply to me that there is the real world and then there is something else called mathematics. So where does that leave me, someone who has been involved with this thing called mathematics for the past half-century? Have I been marginalized? Shipped to an alien planet? Am I drifting around in dreamland? If mathematics is not part of the real world, then where is it? What other world is there? Where in the world am I?

I can understand how this otherworldly view of mathematics arises. Many would agree with Churchill who, in his autobiography *My Early Life*, described his experience of school mathematics as being in “an Alice-in-Wonderland world” inhabited by all sort of strange creatures like quadratic equations and sines and cosines which, when he finally completed his math requirements, "passes away like the phantasmagoria of a fevered dream" never to be encountered again. But even Churchill, as mathophobic as he was, caught a glimpse of the mathematical world that was hidden from him during most of his schooling. "A much respected Harrow master," he tells us, "convinced me that Mathematics was not a hopeless bog of nonsense, and that there were meanings and rhythms behind the comical hieroglyphics."

If mathematics seems remote to students, it's not the fault of mathematics, but rather of the way it's taught. I suspect that those who wrote the standards would agree. But I don't think the situation is improved by suggesting that mathematics is unreal and by embroidering its occurrences outside the classroom.

Most of the problems I've seen that carry a "real-world" label are contrived for the classroom and don't really reflect what goes on elsewhere. Take, for example, the "real-world problem situation" cited in the NCTM Standards: "In a two-player game, one point is awarded at each toss of a fair coin. The player to first attain n points wins a pizza. Players A and B commence play; however, the game is interrupted at a point at which A and B have unequal scores. How should the pizza be divided fairly?" Now, just where in the world did that scenario take place?

It's not that the problem is uninteresting or not instructive. Problems on the division of gambling stakes have attracted attention for a long time. But why give the impression that anybody out there is making repeated tosses of a coin to determine who wins a pizza? Our students know that doesn't happen and to pretend that it does, besides being untruthful, feeds the very notion that one is trying to counter, namely, that school mathematics is out of touch with reality.

Also, I don't want to give the impression to my students that mathematics is not a legitimate, fascinating, and accessible part of the world in its own right—as real as any other part of the world. It does have connections to many other parts of the world and it is true that exploring these connections can be a useful and informative learning experience.

I suggest we abandon our attempts to contrive "real-world" problems for the classroom and concentrate on presenting "real" problems. By that, I mean, problems that are instructive and interesting; problems that students will put energy into investigating. If they connect to other parts of the world, well and good. If they don't, that's okay too. If it helps to clarify them, cast them in nonmathematical language and relate them to students' past experiences. But, don't pretend they replicate situations one is likely to encounter outside the school world.

Very little of what goes on in the mathematics classroom is ever going to be encountered per se in the average person's life. That's not the point. The point is to develop one's mathematical competence and confidence so that whenever one does encounter a mathematical situation later in life, whatever it may be, one is willing to tackle it and has some background and knows some strategies for doing that.

The world of mathematics need not be an "Alice-in-Wonderland world" nor a morass of "comical hieroglyphics." There are "meanings and rhythms" in the world of mathematics and we can unfold those to our students. Mathematics need not be divorced from the stuff of life. One can use common experiences to lend meaning to mathematical discourse. I can, for example, talk about arithmetic progressions in terms of stair steps. But I do this not because carpenters consciously apply a knowledge of arithmetic progressions when building a flight of stairs, but because students' familiarity with them provides useful images that help them grasp the concepts being introduced.

There is only one world, and mathematics—and the mathematics classroom—is a very real part of that world: a part of the world in which many of us spend a good deal of our time and most of our energy. Let's not deny its existence.

The End of the Trail

September 19, 1998

It's the time of year when students across the country have resumed their trek along the educational trail—a journey that's likely to occupy them until they reach majority and beyond. Given the time and energy involved, it's worth asking, "What's the end of the trail?"

Here in Oregon, we've enacted the Oregon Educational Act for the 21st Century. I suspect your state has a similar Goals 2000 plan. There's no question about the end of the educational trail envisioned in the Oregon Plan. The Plan envisions "a work force equal to any in the world by the year 2010." It provides a "continuous connection of learning for each student from preschool through postsecondary entry into the workforce." It provides for redesigning Oregon schools at every step: "preschool, kindergarten through high school, higher education, and school-to-work transition." The trail's end, we are led to believe, is a job. And not any job, for "Oregon's successful economy has created thousands of openings in well-paying, skilled positions for which there is a shortage of qualified Oregonians."

Unfortunately, however, there's not one of those well-paying, skilled positions for every Oregonian. According to the Oregon Occupational Employment and Wage Data for 1996—the latest available on the web—of the 1,121,500 wage earners, over half (52.8%) had employment in an occupation for which the median pay was less than $10.00 an hour, which amounts to about $20,000 a year. Almost one out of five (19.8%) were in occupations in which the median wage was less than $7.00 an hour. The occupation which employed the greatest number of people (54,180) was retail sales for which the median wage was $7.56 an hour; the second largest was office clerks (41,390) with a median wage of $8.99 an hour.

Nor are there a passel of well-paying, full-time jobs elsewhere in the States. In 1995, some 30% of all U.S. workers earned less than $7.25 an hour; more than 40% earned less than $9.25 an hour. Downsizing, out-sourcing, and the use of temporary help have radically changed the nature of the American workplace. By the mid-nineties, the country's largest employer was Manpower, Inc., a temporary-help agency. In the last 20 years, over 43 million jobs have disappeared. And what's replacing them? Economist Jeremy Rifkin reports in *The End of Work*: "In August, 1993, the federal government announced that nearly 1,230,000 jobs had been created in the United States in the first half of 1993. What they failed to say was that 728,000 of them—nearly 60 percent—were part time, for the most part in low wage service industries. In February 1993 alone, 90 percent of the 365,000 jobs created in the United States were part time, and most of them went to people who were in search of full-time employment. Increasingly, American workers are being forced to settle for dead-end jobs just to survive."

A dead end. That's where the trail leads for many of our students if we make a well-paying, skilled job the goal of the educational journey. To suggest to our students that a gratifying job awaits them if they stay on the educational trail is deceptive and defeating. If that's the goal, large numbers of students are doomed to failure before they take their first step on the journey.

To make the journey a rewarding experience for everyone, I suggest we focus on the trail, rather than the end of the trail. That we make the trail an end in itself. A trail that unfolds new vistas each step of the way. A trail where one experiences the satisfaction and confidence that comes from accessing and exploring the myriad talents latent in every human being. A trail that becomes so absorbing it becomes a lifetime occupation.

Rather than embarking on a plan to develop "a work force equal to any in the world by the year 2010"—with the underemployment and disappointment that is sure to follow if all are to be included—let's embark on a plan to provide an educational experience second to none in the world. An experience that profits everyone every step of the way, from preschool through postsecondary and on into the community at large, whatever the circumstances may be.

What's Missing?

October 30, 1998

A number of years ago, dissatisfied with the way I was teaching mathematics—filling the blackboard with proofs and procedures while students dutifully recorded everything I wrote—I was looking for other ways of conducting classes. Mathematics educators of the time were urging that mathematics be "taught from a discovery approach, an approach that encourages the learner to manipulate devices, to play mathematical games, to gather data, and to form his own conclusions."

Such practices seemed promising to me—at least, they took the students out of the passive role they assumed in a lecture setting. I visited a classroom where, I was told, I could see these methods in action. I came away with mixed feelings. The students were willing participants—they carried out the activities as instructed but, in the end, nothing much seemed to happen. The classroom milieu was pleasant, it had a sense of industry about it, but there was no electricity in the air—no lights were coming on. I wondered what was missing.

A few days later, reading Rollo May's *The Courage to Create*, I discovered what it was. There was no encounter between the students and the subject matter at hand. Everything had been laid out so nicely that the students were proceeding step by step, as if they were following a recipe. Everyone was comfortable—and nothing creative was happening.

According to May, creativity—the process of bringing something new into being—always entails an encounter; an encounter between a highly involved individual and some aspect of his or her world. And, May continues, this encounter brings with it an anxiety, "a temporary rootlessness, disorientation."

"That's it," I thought. The mathematics classroom should be a place where learners, whatever their age, encounter their mathematical world in a way that expands and enlarges it—that brings something new into being. My job, as the teacher, was to set the stage for this encounter, and to provide a safe and supportive environment if anxiety ensued.

I found the encounters I wanted can occur. They happen most frequently when I am able to frame a mathematical situation in some context that connects with my students' worlds and, at the same time, provides pathways to new territory.

The issue became: How does one bring about this encounter? It didn't occur when I demonstrated a procedure and asked the students to practice it twenty times. Or when I presented a flawless demonstration of some theorem. Or when I led them through a series of small steps designed to get them to arrive at a foregone conclusion. The students weren't encountering their mathematical worlds. They were being presented with some textbook version of the mathematical world and expected to quietly absorb it.

On the other hand, I found the encounters I wanted can occur. They happen most frequently when I am able to frame a mathematical situation in some context that connects with my students' worlds and, at the same time, provides pathways to new territory. Posing a problem that catches their interest and moves them into this new arena, and then leaving them to their own devices, precipitates the encounter. The boundaries of their knowledge are challenged; ways to extend them are conceived and explored.

The process does generate anxiety—or, disequilibrium, as we call it around here. Both the anxieties of students who are unsure of their thinking or afraid of being wrong, and one's own concerns that students become involved and learning takes place. But the anxiety vanishes in the light of the first "Aha!"

What Evidence Will You Accept?

December 2, 1998

I clearly remember when I first asked that question and what, unexpectedly, was the outcome.

I was directing a multiyear statewide project supported by a grant from the National Science Foundation. The funding went through a state educational agency and, since the grant funds were part of the agency's budget, the director of the agency had to appear before a legislative committee and defend the inclusion of the grant as part of the agency's annual budget. This meant testifying that the grant program was consistent with agency goals and was, indeed, accomplishing what it was designed to do.

The directorship of the agency changed hands midway through the project. The old director had been involved since the conception of the project and had no doubts of its legitimacy. The new director, however, was skeptical. He had inherited the project and knew little of its context and the situations it addressed. From time to time, he would ask me for evidence of the project's effectiveness. In response, I would report some of the results gathered by the project's evaluators. No matter what I reported, he found a way to discount it. One day, in desperation, I asked him, "What evidence will you accept?" The question was unanswered. He changed the subject and never raised the issue again. Later, at a legislative committee hearing, I was surprised to hear him cite, as evidence of the project's effectiveness, the same results he had discounted earlier.

Since then, I have used the question on a number of occasions. I find it particularly helpful in situations, like the one I mentioned, when whatever evidence one offers is discounted. Discounting is easy to do, and trying to respond is futile. No matter what one offers, the response comes back, "Yes, but what about...?" Those skilled at "yes-butting" can keep it up forever. Asking "What evidence will you accept?" helps break the cycle. Oftentimes, as happened the first time I chanced upon the question, there is no response or the subject is changed.

When questioned about what one is doing, especially if it runs counter to the questioner's practices or beliefs, I suspect there is no evidence one can offer that will lead to acceptance of what's being questioned. People, I find—myself included—don't make rational decisions about the validity of long-held positions. Instead they rationalize their position. Instead of starting with a clean slate and trying to collect all the data and information available about the issue at hand, and then draw whatever conclusions are warranted by the facts—as happens in a jury trial—people look for evidence to support their belief. Then they use that evidence, even if it requires a bit of distortion or invention, to provide an argument for their position. One way of maintaining their stance is to put anyone who challenges them on the defensive by asking them to justify their opposition, and then discredit any justification that's given.

Among educators, a common ploy by those who, for whatever reason, prefer the status quo, is to ask for the research or the test data that justifies what you are doing and, when research is cited or data is given, proceed to tell you what was wrong with the way the research was conducted, the tests administered, or how the situation at hand is different. Since their minds are already made up, there's no evidence they will accept that some other course is possible. Asking "What evidence will you accept?" helps make this clear. A stumbling reply reveals the situation, and one need not waste energy trying to hit upon a piece of evidence that will swing them to your side. On the other hand, if someone is really interested in what you are doing and wants to know what points to its success, asking the question focuses the discussion. One can either supply the requested evidence or, if it doesn't exist, say so. In either case, one avoids an interminable chain of "yes-buts."

It's also a question worth asking oneself. Before I challenge the effectiveness of someone else's educational practices, I had best ask myself what evidence I will accept. If there is none, I do well to keep quiet and ponder my own biases.

Influencing Instruction

January 8, 1999

Recently, while browsing J. W. A. Young's classic, *The Teaching of Mathematics in the Elementary and Secondary School*, published in 1907, I came across the following statement: "[Examinations] as a test of the pupil's attainments by some outside authority and in accordance with some outside standard…may be regarded as necessary evils and their influence upon instruction as bad…. In any system in which all or nearly all hinges upon the result of an examination of some outside authority, the examination is a fact, to which the teacher is compelled to bend his teaching, and no amount of theorizing will ever lead him to do otherwise. Fortunately, this extreme form of examination is by no means predominant in the United States."

I wondered what Young, professor of the pedagogy of mathematics education in the University of Chicago, would write today, when the "extreme form of examination" of which he speaks has indeed become predominant in the United States. Departments of education throughout the land have turned to statewide examinations as a means of assessing student achievement and, if certain political factions have their way, nationwide tests are imminent. Here in Oregon, "content standards" have been established and statewide assessment which aims to measure students' attainment of these standards has been inaugurated at grades 3, 5, 8, and 10. Further, students must meet the grade 10 state performance standards in order to receive a Certificate of Initial Mastery which, if all goes as planned, will be required for high school graduation by 2003.

Political issues may debilitate the whole process. While, on the one hand, the American public will decry a perceived erosion of standards, on the other hand, as long as society maintains that a high school diploma is necessary for every respectable adult endeavor, parents and other interest groups will insist that their daughters and sons graduate, whatever the circumstances. Already, scoring of the tests has been adjusted. After last year's assessment in which only 39% of tenth graders met the state's standard for writing and 31% met the math problem-solving standard, the scoring method has been changed so that under the new method, these percentages would have been 44% and 39%, respectively. The state school superintendent maintained that the new scoring method was not a lowering of standards but "a much more accurate indicator of student achievement" and a state school board member said, in a deft bit of logic, "I definitely don't see this as weakening standards. The same total number is being called for. It's just that how it's computed is different." Despite these protestations, I suspect the hue and cry over the low scores had something to do with it. And, I suspect, adjustments will continue until a societally respectable number of students meet the standards or, what is more likely, other ways than meeting state standards will be devised to certify the successful completion of a high school program.

Meanwhile, one wonders what the influence of all these examinations on instruction will be. Would Young, if alive today, still view it as "bad"? I suspect so. Even though the nature of the tests may have radically changed over the last century—and one can't fault the state assessment for its effort to emphasize conceptual knowledge, problem-solving proficiencies, and communication skills over rote learning—Young's basic premise still holds: teachers feel compelled to bend their teaching to the test. One would hardly expect otherwise, when their competency as teachers is judged on their students' scores. Most teachers, faced with this situation, won't focus on the quality of their mathematics instruction, they will focus on getting good test scores.

Unfortunately, getting good test scores doesn't depend on good mathematics instruction. Good math instruction will result in good test scores—that's certainly the case with the Oregon assessment—but to suggest to someone that their class's scores will increase if they change their instruction doesn't get an enthusiastic response, especially when there is a simpler, more direct way to accomplish the objective: teach to the test. There are tried and true methods for doing this: gather sample test questions and questions from previous exams, find out about the scoring rubric—Oregon has official scoring guides—and then spend time every week or, if necessary, every day, coaching your class on how to score well on the test. Class scores will go up, but that doesn't necessarily mean their understanding of mathematics is any greater. The statement that higher test scores means more meaningful knowledge is axiomatic at best, one can argue endlessly whether it is true or not.

So is the influence of state assessments on instruction bad? It is in that it diverts the focus of the instruction from the development of the students' mathematical abilities to the development of their test-taking abilities. It's also bad in that it takes tremendous resources, both time and money, to support the whole process. What little time and money teachers have these days for their own professional development is being gobbled up by the demands on teachers to acquaint themselves and prepare their students for the assessments.

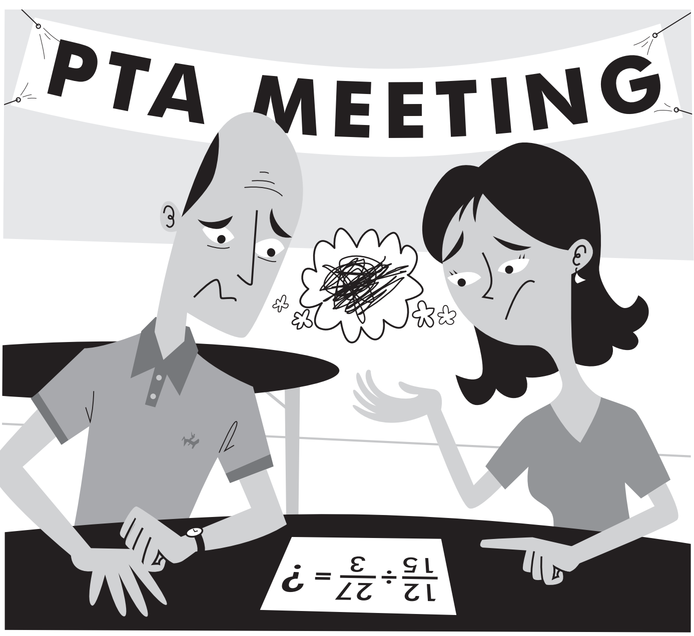
When it comes to instructional matters, we know teachers tend to teach the way they were taught and their notion of what mathematics is about is fixed by their own school experiences. For many teachers, as for other adults, this consisted of being shown one mathematical procedure after another, practicing them as one went along with little regard for underlying concepts so that mathematics becomes a collection of procedures, tending to the arcane and carried out in prescribed fashion.

But teachers also want the best for their students. Given the financial resources and a risk-free setting, teachers would welcome an opportunity to experience mathematics in a different way. The Oregon Department of Education's Office of Assessment and Evaluation reports that about 1000 Oregon classroom teachers gather for six days at 16 or more sites—earning $110 per day—to score the state mathematics performance assessment. Instead, suppose each year 1000 teachers were provided stipends to participate in an all-expense-paid, six-day workshop where they could deepen their knowledge of mathematics and experience for themselves the engaging and effective ways of teaching and learning mathematics envisioned in the Oregon standards. Which would have the most positive influence on instruction? I vote for the latter. I think Young would, too.

THE PTA DOES FRACTIONS

February 16, 1999

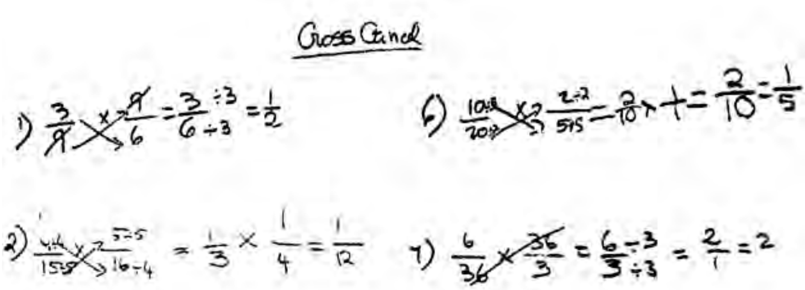
It was math night. We parents were listening to the fifth-grade teacher describe her approach to teaching fractions. Before showing us the algorithm she taught for dividing fractions, she asked us to divide two fractions using whatever method we were taught. Mass confusion ensued. As I looked about the room, I saw moms and dads conversing quizzically with one another. I heard fragments of hushed conversations: “What’s the rule?” “There’s something about inverting…” “Is this right?” No one I could see had any confidence in what they were doing.



"How much time and energy," I wondered, "was spent teaching these folks how to divide fractions? Here's a roomful of well-educated adults from a middle-class neighborhood in a university town. I suspect they have all been 'successful' at school. What went wrong?"

I was reminded of this incident by the morning paper, which brought me face-to-face with what went wrong, and is still going wrong, with the way fractions—and most other mathematics—is taught.

An article on charter schools included a two-column, three-inch photograph of Kevin, a sixth-grader, standing in front of a white board in his math class at ATOP (Alternative Thought Orientation Process), a school in Phoenix, Arizona, emphasizing the "basics." The photo shows Kevin, face buried in his hand, struggling, as the caption says, "for the answer in math class." Behind Kevin, one sees ten exercises in multiplying fractions under the heading "Cross Cancel." Kevin, with the help of his classmates we are told, has done nine of them. A replica of some of his work is shown below.



Cross canceling, it appears, has something to do with reducing fractions before multiplying them. I don't know how adept Kevin is at carrying out the process, but I'm willing to wager he hasn't any idea of why it works—other than the teacher said so. I suspect, as far as Kevin is concerned, it's just another magic trick among all those he's been taught in his math class. For

Kevin—and countless others—acquiring basic mathematical skills becomes mimicking magic tricks.

There's not just the magic of "cross-canceling," but there's also the magic of "inverting and multiplying," "moving the decimal," "cross-multiplying," "adding exponents," and on and on. Kevin and his classmates will learn these magic tricks a few at a time and gain enough proficiency with them to pass the test of the moment, but by the time they have children and are going to math night at the PTA, the magic tricks will have gotten all jumbled up so one no longer remembers quite how things are done or which trick makes aces drop out of sleeves or which pops rabbits out of hats. And so, magic night, oops, I mean math night, at the PTA becomes a mystical mess.

What's gone wrong? It's a misunderstanding of what's basic knowledge of mathematics. Somehow or other, the misconception has become ingrained in a large segment of the public—certainly in the parents and educators that established Kevin's school—that the basics in mathematics consists of rules for carrying out procedures; rules like cross-canceling that need to be memorized and practiced. To do this, students resort to rote learning which lacks meaning and context and provides no recourse once a single misstep is made or the slightest confusion occurs.

Basic long-term competency in mathematics isn't developed through memorization and drill on prescribed procedures. It's developed through the nurturing and enhancing of the number sense innate in all human beings [see Num•ber Sense/Numb•er Sense]. By providing students with a wide range of concrete experiences involving the variety of numbers and operations found in elementary mathematics, their number sense will develop and expand and they will create mental images for how things work—images that can be called upon whenever needed. They will have acquired the most basic competency of all: the ability to devise their own methods for dealing with a mathematical situation.

So when it's math night at the PTA, or any other time a mathematical problem arises, Mom and Dad will not be faced with extracting the right rule from that mass of rules that have become blurred and entangled over time. Rather, they will evoke their number sense and collection of mathematical models and images to construct their own process for resolving the situation at hand.

And young Kevin will not stand in front of the class, face buried in his hand, trying to remember and replicate by rote memory some prescribed procedure for multiplying fractions. Instead, with eyes wide open, he will describe to the class how he, using his well-developed number sense and his images of fractions and the arithmetical operations, figured out on his own how to multiply fractions.

THOSE TIMES TABLES

March 23, 1999

I was in the stands at a middle-school basketball game. The mother of one of my grandson's teammates had noticed my "Math and The Mind's Eye" sweatshirt and, during a lull in the game, commented about it. We chatted briefly about math until the game resumed. In our conversation she mentioned that her son was having trouble in math. I gave her our web address and a couple of days later I got an e-mail from her elaborating on her son's difficulties.

Her son, a sixth-grader, she wrote, had never "mastered" the timed tests in elementary school. Recently he had scored in the seventies on a three-minute timed multiplication test and the teacher had announced that all those who didn't score 85% on the next one—in two weeks—would have to go to "homework club" until they did.

She told her son, in true Nike-town fashion, it was time to "just do it." So, they devoted Monday of a three-day weekend to the task and her son passed the tests. "It's great to be past that hurdle," she wrote. Meanwhile, her fourth-grade daughter was struggling to up her score on a five-minute, 100-problem test from 68% to the teacher-mandated 95%. Mom looked for the "mental stumbling blocks" getting in her daughter's way. Finding those—the nines—she "showed her some relationships" and daughter did fine.

Mom also recounted some of her own experiences. "With four kids," she wrote, "I just haven't devoted my life to this drilling." She was also reluctant to have her children experience what she went through in fifth grade to pass "those tests," the scores of which "were posted on the wall for everyone to know where every student stood at any point in time"; creating, she added with a bit of wryness, "another nurturing exposure to Math."

She went on, "I've never been good at the pure 'memorization' game but succeeded with seeing or constructing relationships with material at hand as a way of latching it to the memory fibers of my brain."

I was reminded of my own family's stories about timed tests and multiplication facts. I remember a child in tears at the breakfast table because he got so nervous during timed tests on arithmetic facts he couldn't think, he said, even though he knew the answers.

I recall my youngest sister, after a game-playing session with her nephews, wishing she could do arithmetic as fast as they did and confessing that she had never learned to multiply. When I inquired what she meant, she told me that there were certain products she never could remember, like 8 × 7. She said she had to start with something she did remember, like 4 × 7 and count on by sevens until she got what she wanted. I told her she knew how to multiply; her system might take longer than her nephews' but that was fine as long as it worked for her. Unfortunately, she finished school with the belief that she has very little math ability.

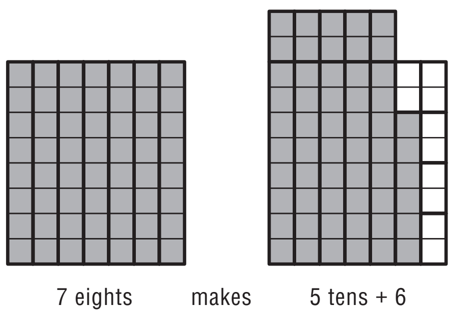
From what I gather, educators who insist that students have instant recall of the times tables believe that it's an essential basic skill, without which students will be hampered in their later mathematical development—especially when it comes to learning paper-and-pencil multiplication and division algorithms. But this confuses acquiring nonessential algorithmic skills with what is basic to all mathematics education: drawing out and developing the mathematical abilities that exist in every child, including the ability to devise one's own arithmetical procedures.

There is no question that being able to instantaneously recall arithmetical facts can be a convenience. However, one must weigh this convenience against the price paid in forcing rote memorization of facts and giving timed tests, especially when there are so many other ways of arriving at these facts, be it by counting, reading a chart, or punching a calculator. The principle costs are the negative emotions and beliefs that the struggle to memorize arouse, and the meaninglessness of rote learning.

As facts are stored in memory, so are the emotions evoked in acquiring them—and quite likely, from what is known about the brain, more vividly than the facts themselves. Also embedded in our psyche, are those erroneous beliefs about one's mathematical ability—such as my sister's—when too great an importance is placed on rapid recall of facts. Memorizing a page of multiplication facts is not something our brains are wired to do well, and for many of us requires considerable effort. It can be done in a verbatim fashion, like one memorizes a nonsense verse word by word. But, once in verbatim memory, it's also likely to carry about as much meaning as nonsense verse.

If one gives up the obsession with speed and the unwarranted emphasis on paper-and-pencil algorithms—if doing calculations quickly is the goal, use a calculator—and instead stress meaning and context, the multiplication facts will take care of themselves.

Most adults, not to mention school children, rattle off the sentence, "seven times eight equals fifty-six" with little attention to the meaning of the words they are saying. "Seven times eight" lends itself to several interpretations, depending on the model of multiplication one carries in one's head: some think of it as "seven eights," others as "seven taken eight times," still others as a 7 by 8 array. "Fifty-six" literally means "five tens and six." Saying "seven times eight is fifty-six" is simply saying, whatever one's interpretation of multiplication, that 7 × 8 can be rearranged to form 5 tens and 6.



Children shouldn't be expected to work on multiplication facts until they understand our numeration system is based on groupings of tens. (English-speaking students are at a disadvantage here. Other languages directly reflect the base ten nature of Arabic numerals. If, for example, our names for numbers were similar to the Chinese names, instead of saying "twelve," we would say "one ten two"; "twenty-three" would be "two ten three;" "fifty-six" would be "five ten six." Studies have shown, at the age of four, the Chinese child, on the average, counts to 40, while the American counts to 15. When given some unit cubes and bars of ten and asked to use them to represent 25, Chinese children select two bars of ten and 5 units, while at the same age, most American children count out 25 units. Perhaps, when introducing counting to schoolchildren, we should use more literal names at first: “one ten and one, one ten and two, …two tens, two tens and one, two tens and two” et cetera, and leave the standard “eleven, twelve, …twenty, twenty-one,” and so forth, till later.)

Once children understand the literal meaning of number names and how these names reflect the grouping-by-tens nature of our numeration system, using models based on their intuitive understanding of multiplication, they can develop their own times tables by converting products into groups of ten and recording the results. (Addition can be treated similarly. Most folks can imagine in their minds how a stack of 8 blocks and one of 7 can be converted to stacks of 10 and 5 by moving 2 squares from the stack of seven to the stack of eight, that is, 8 + 7 is one ten and five, or 15.)

In so doing, children will come to know the meaning behind the multiplication facts they are being asked to remember. They will have mental images, other than symbols, that convey these meanings. Games and other number activities in which number products are met in nonstressful settings will help them implant these facts in their memories. The recall may not always be instant, but, given the time to do so, they will have ways of recovering what's for the moment forgotten. And they won't be storing up strong emotions and negative messages that may not only block recall of a number fact but, as it has for my sister, lead to an aversion for all mathematical activity.

What's Basic?

May 7, 1999

What is a basic skill in mathematics seems to be an imponderable that defies description. On the one hand, some people use the phrase as if everyone knows what it means, as in an editorial that appeared recently in the Sunday *Oregonian* concerning the adoption of mathematics texts in Portland Public Schools. A subhead said the Portland schools were going to put "problem-solving and discovery before basic skills." The head left me wondering, "So what is a basic skill?" The editorial didn't provide an answer other than the parenthetical comment "number manipulation" that appeared after the term "basic skill."

On the other hand, one can find long lists of "basic mathematical skills." These are invariably open to debate. In an apparent attempt to find consensus among math educators on what ought to be on a list, the National Council of Teachers of Mathematics April issue of *Dialogues* includes a questionnaire containing 44 mathematical tasks and asks readers to check whether or not each task is a basic skill. To clarify what is meant by "basic," they offer the following definition, "Skills are usually called 'basic' when they 1) are deemed necessary for later mathematics or 2) they are deemed so important that everyone should learn and be tested on them."

I found there isn't any item on the list I was willing to check as "basic." The list, for the most part, is a collection of procedures—calculating this, graphing that, solving things in particular ways—and I am hard pressed to think of any mathematical procedure that I would say everyone should learn and be tested on. One could lead a meaningful and productive life with little knowledge of any one item on the list; and one could be quite adept at a variety of mathematical endeavors with hardly any knowledge of the listed items. On the other hand, to pass the required math courses in a typical school math program, one would need a number of the listed skills. Over the years I've noted that many lists of basic mathematical skills could be more aptly described as school survival skills—you need them to get through school but not for much else in life.

An inherent danger I see in most lists of basic skills is they encourage carrying out formulaic procedures at the expense of meaning and insight. The development of conceptual understanding and a problem-solving mentality has far greater applicability and obviates the need of formulas and procedures, other than those a student adopts on their own. For example, one of the proposed basic skills on the NCTM list is "using formulas to find the area and volume of common shapes." I've watched people with a formula knowledge of area go through all kinds of machinations trying to find the area of a common geoboard triangle like that shown in Figure 1, where the area of each of the squares shown is one.

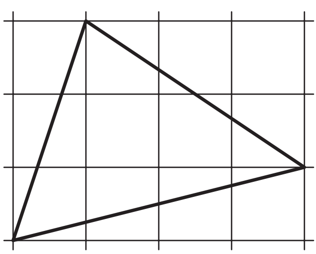


Figure 1

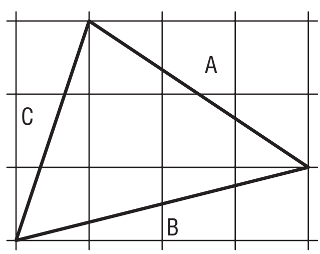


Figure 2

Finding a height and base of this triangle is challenging for most folk, while anyone who knows what area means and has a bit of ingenuity can readily find that the area is 5½ without the use of any formulas. (Encase the triangle in a rectangle of 12 squares as shown in Figure 2, and subtract away the areas of corners A, B, and C. Corner A is half of a rectangle of 6 squares and hence has area 3, similarly the areas of B and C are 2 and 1½, respectively. Hence, the area of the triangle is 12 – (3 + 2 + 1½).

Conceptually, finding the area of a plane figure is straightforward: One decides what the unit of measure will be (one inch, one centimeter, one mile, or any other length of one's choosing) and then determines how many unit squares (i.e., squares whose side is the unit length—square inches, square centimeters, etc.) fit into the figure at hand. For rectangles whose sides are multiples of the unit length (as in Figure 3), one sees the area is the product of its dimensions.

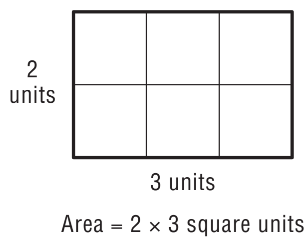


Figure 3

For other rectangles (e.g., one whose dimensions are √2 and 1.769), one makes the axiomatic agreement that their area, too, is the product of their dimensions. Given this and a supply of figures, students, working individually and with one another, will devise their own methods for finding areas, including the development of formulas they find useful. They also will have developed a conceptual understanding of area, as well as ownership of a variety of methods for determining it that will serve them far better than a knowledge of area that's no more than a collection of memorized formulas.

Another negative aspect of lists of basic skills is the impression they give that mathematics is a hierarchy of techniques and manipulations, devoid of any plot, rather than a cohesive, developing body of knowledge with a rich history—a vibrant subject that can be found in all kinds of human endeavors, from the mundane to the exotic. Perhaps, we should abandon our efforts to compile such lists and concentrate instead on unfolding the story of numbers and shapes and how we measure things. Any school version of this story ought to begin with the intuitive knowledge the child brings to the classroom, but, other than that, there are many ways to tell the story. So rather than fragmenting mathematics into a bunch of tasks to be mastered, how about setting a course or, as we say these days, establish a curriculum, through which we can guide our students as they relive this fascinating story for themselves?

The Big Lie

June 9, 1999

A hallway conversation between sessions at a recent National Council of Teachers of Mathematics convention centered on the role of math in everyday life. An engineer-turned-science-writer expressed her surprise at discovering how inconsequential math was in the lives of most writers in the university town in which she lived. From what she observed, they got along just fine without paying any attention to mathematical matters. The conversation drifted elsewhere and at the time I didn't think to ask her why she found this surprising.

All of us, I believe, know people who lead satisfying lives and have little use, if not an aversion, for anything mathematical. Besides my own acquaintances, I can name any number of persons famous enough to be listed in the biography section of Webster's, who, according to their biographers, fared poorly in school mathematics and had little regard for it: humorist George Ade, novelist Ellen Glasgow, journalist Edgar Guest, publisher Randolph Hearst, playwright William Inge, poet Vachel Lindsay, diplomat Henry Cabot Lodge, historian William Prescott, to name a few.

So why is it we feel surprised when we find a covey of math-avoiders who are perfectly content in their math-less world? The reason I suggest is that it contradicts a message drilled into us in grade school from many different sources—home, society at large, and especially school: you must learn mathematics because you'll need it sometime. Messages heard repeatedly in childhood from authority figures become part of one's belief system—lying dormant in one's psyche until some event in adult life brings it to a conscious level. It wasn't until my own children were in school that I became aware of the pervasiveness of that message and how I had never thought, till then, to challenge it.

I remember our youngest, a third-grader at the time, announcing at the dinner table that they were doing something new in math at school, but he couldn't remember its name. I asked him to describe what they were doing. He said something about rubber bands and boards with nails in them and I said, "Geometry?" He said, "Yes, that's it. What's geometry good for?" Not ready to make a case for the value of geometry to a third-grader and knowing art was one of his favorite subjects, I deflected his question with one of my own, "Well, what's art good for?" "Oh, that's fun," he said. "Was what you did today in geometry fun?" I asked. He replied that it was. "That's why you did it," I said. The answer seemed to satisfy him, because he didn't push the matter further.

Only a few days later, while engaged in a chore with my middle-school son, I asked him what he liked about math. His answer was immediate, "I like magic squares. What are they good for?" This time I was a bit quicker on the uptake. "To think about," I said. He said, with apparent satisfaction, "Oh."

We are misleading students and stymieing their interests, I decided, when we make future utility the motive for studying math. And so I formulated what, in my own mind, I dubbed The Big Lie of School Mathematics, namely, "You must study this because you'll need it sometime."

No matter what the mathematical topic, if I tell my students—or give them the impression—that it is something they will need to know at some point in their life after school, I'm almost certainly lying to someone in the class. For some topics, like dividing by a fraction, I'm lying to almost everyone in the class. (Some tell me they are doing this when they halve a recipe, but in this case they are not dividing by a fraction—they are dividing by 2.) Furthermore, telling a student they should study mathematics because they'll need it sometime provides a reluctant student with a great opportunity to discount your admonition. The scenario goes something like this: You introduce a topic. The student asks, "When am I going to need this?" You say, "When you do such and so." The student says, "I'm never going to do such and so." You say, "You'll also need it to do this and that." "But I'm never going to do this and that." And so it keeps going, as long as you last. In the end, the student wins—they are never going to do any of those things, so there's no need to study math.

But, most importantly, attempting to motivate the study of math on the basis of some possible future use has a negative impact on those students—and this is most of them—who are quite willing to study math for the satisfaction it gives them at the moment. Telling them The Big Lie is telling them they are studying math for the wrong reason and leads them to question the aptness of their efforts.

As far as my own teaching is concerned, I find it best to be straightforward whenever a student questions when they are going to use the topic at hand. My answer is, "Maybe never, but that's not the point." The point of studying mathematics is not to learn the mathematics one will someday need. There is no way of foretelling what that might be—indeed, the mathematics they need may not be invented yet. The point is that, whenever the need or interest to pursue a mathematical topic arises, one is confident and capable of doing that. With that as a goal, the particular mathematics one studies isn't as important as the process of studying mathematics. My hope is that the material covered in a course is appropriate for my students' stage of development, is as interesting to them as it is to me, and builds their mathematical competence and confidence.

Once I adopted this stance and no longer risked lying to my students about the future utility of what we were studying, the classroom atmosphere became much more relaxed. If a question arose about future application, it was out of a genuine curiosity on the part of a student, and it was acceptable to not have an answer to that question. A student might find a topic difficult or dull, but its utility wasn't challenged. And, best of all, it freed me and my students to study mathematics for its own sake.

On Reforming, Deforming, and Transforming

July 16, 1999

A few weeks ago I was at an end-of-the-school-year gathering of teachers, their spouses, and friends. The conversation turned to the Oregon version of the educational reform movement.

The teachers agreed that the aura of their classrooms was changing, especially in those grades (3, 5, 8, and 10) in which state-mandated assessments were taking place. Their students are being tested to see if they are attaining "benchmark expectations" in meeting "content standards" adopted by the state department of education. Since the public and, especially, politicians view the results as measures of teachers' effectiveness, the teachers felt compelled to make preparing students for these tests their number one priority. The tenseness of testing permeated their classrooms, replacing the comfortable feeling of students engaged in learning. And, everyone agreed, their enthusiasm for teaching was diminished.

The state's assessment program is sophisticated—complicated may be a better word—requiring much more than filling bubbles with No. 2 lead, although there's some of that, too. On the problem-solving portion of the math tests, students are graded on five criteria: conceptual understanding, processes and strategies, verification, communication, and accuracy. (Because of confusion about how one gives evidence they verified their work, the verification score is not used for "decisions about students" but "will inform the field test"—whatever that means.) In order for students to obtain the new, much-touted-but-of-unknown-significance Certificate of Initial Mastery, a student must achieve state-established scores on the tests as well as on 64 "work samples" in areas and grade-levels designated by the state. The "completed, scored student work" is to be "kept together in any fashion, from a portfolio, to a file folder or other system, as determined at the local level." This is just for the Certificate of Initial Mastery; the Certificate of Advanced Mastery is yet to come.

Teachers in a local school estimate that if they did all the preparation, testing, and assessment suggested by the state department they would spend a third of the school year on testing and "work samples." One fifth grade teacher reported that his 28 students collectively generated some 200 work samples this past year. With each of these graded according to 5 or 6 criteria, his students generated more than 1000 separate grades on these samples alone. In some school districts, student portfolios are passed along with the student from teacher to teacher. Supposedly teachers will study them to acquaint themselves with the student's level of achievement. But this is more information than a teacher can digest in a reaonable amount of time. Most teachers will learn more in a couple of weeks of observation than they will glean from perusing portfolios.

Teachers aren't the only apprehensive ones. Reader response to articles in the local paper about the new look in Oregon education are largely negative. Expecting something more, one writes, "The only reform-related product to reach the classroom is the assessment portion." Another chimes in, "Nothing but tests have materialized." The parent of a high-school freshman observes, "We have proficiency tests in everything but spitting." Parents are asking that their children be excused from taking the tests. The parent of an about-to-be third grader writes that his child has decided he won't take the test: "[the child] felt, as I did, that it would be a waste of time to take the CIM test just to find out that he is reading above grade level and is very good at math. We already know these things. So why should he take the test? On test days we'll do something enjoyable." Students wonder, "Why all the testing?"

"When you try to tell them it's important," says the director of instruction in a local district, "they say, 'For what?'" Apparently, there's no good answer. The Oregon Educational Act for the 21st Century isn't reforming education, it's deforming it—into an assessment quagmire.

Except for bogging down the system with assessment, Oregon's reform act has fostered no fundamental change in educational practices. A few things have been redone. New "content standards" and "benchmarks" have been established. But the standards are simply another listing of what somebody thinks everybody should know. An attempt is being made to replace diplomas with "certificates of mastery," but these, like diplomas, are earned by fulfilling a list of content requirements and passing a bevy of tests. The only essential difference is that requirements are being set at the state rather than the local level, which is not surprising given the increase in state funding of education, the decrease in local funding, and the high correlation between funding and control. Little, if anything, is being done to address the problems endemic to mathematics education: rote learning, debilitating instructional practices, elitism, math anxiety and abhorrence—to name a few. If it's generated any excitement about teaching or learning math, I haven't heard about it.

That's not to say exciting things aren't happening in mathematics classrooms. I was reminded of that by a message from a teacher that had an entirely different tone from the conversation I had heard a few days earlier. The teacher talked of the aversion to mathematics she developed in her school days and how, despite this, she had been "turned on" to teaching math the last two years. She was finding ways of making math interesting for her students. They "love it" and she loves it, especially "that sparkle in their eyes."

What, I inquired, brought about this transformation from someone with an aversion to school mathematics to someone who was excited about teaching it? Her response had much in common with that of other teachers who have made a similar journey: A teacher somewhere along the way who helped them see that mathematics wasn't a collection of incomprehensible rules and procedures and enabled them to make sense out of what they were expected to learn; a chance to experience instructional strategies that bring math to life and honor the learners' insights and intuitions, supportive colleagues who are also excited about teaching math, and the freedom and confidence to find their own way in the classroom. A scenario far different from that produced by mandating what's to be taught and what constitutes learning.

If we are serious about changing the course of mathematics education, I suggest we stop re-forming lists and requirements, give up our preoccupation with testing and, instead, spend our resources transforming mathematics classrooms. Transforming them from places where students, under threat of failure, survive by rote learning of prescribed rules and procedures—which only adds to the disdain of mathematics abroad in the land—to places like those described above. Such classrooms do exist—created not only by those teachers who have always been on friendly terms with things mathematical, but also by teachers, like my correspondent, who have overcome their own adverse experiences with school mathematics.

Such transformation doesn't happen by edict. It's more likely to happen one classroom at a time. The process to help teachers transform their classrooms isn't complicated—it's outlined above. Carrying out the process may not be easy; it requires time, patience, and commitment. But once accomplished, one doesn't need an elaborate testing program to determine if students are learning—you can tell by the sparkle in their eyes.

What I Learned from Rusty

August 17, 1999

When I was in school, I decided that an answer to a mathematical question wasn't complete unless I could explain how I got it. I had plenty of evidence to support this belief: Docked points on papers, with "Show your work!" written alongside, and the inevitable "How did you get that answer?" when called on to recite in class. Clearly, an answer by itself wasn't enough. It didn't matter if the answer was obvious to me, I had to make it obvious to the teacher, too. Sometimes, given the way the teacher thought, or if I wasn't exactly sure where my idea came from, it was better to keep quiet. When doing school math, knowing the answer wasn't enough.

I carried this belief into my own classrooms, passing on to my students the message ingrained in me by my teachers: correct answers aren't sufficient; you must also explain how you arrived at them.

And then I met Rusty.

Rusty was a fifth grader. I was involved in project SEED. I don't remember what the acronym stood for, but it was an organized effort to get professional mathematicians and scientists into elementary classrooms, especially in low-income areas. So each day, for one school year, I left my office at the university and drove to the other side of the tracks where I spent 45 minutes doing mathematics with Rusty's class. They still had their regular arithmetic period outside the time I spent with them, so I was free to roam over a variety of topics.

Rusty's desk was in the back of the classroom and, despite classroom distractions and his apparent inattention, Rusty, I discovered, was following whatever I was presenting. As I posed problems for the class to work on and roamed about checking on the students' progress, Rusty was among the first to arrive at an answer. This surprised me until I learned that his way of operating gave little indication of his involvement. And whenever I asked him how he had arrived at his answer, his usual response was a shrug. Pushing for a further explanation elicited nothing more than some variation of "I just know" or "That's what it has to be."

I was struck by Rusty's confidence in his answers and how unfazed he was by my queries. My university students—better versed in the nuances of professors' comments—might well have become suspicious that there was something amiss in their thinking or that I had asked them a trick question.

I puzzled about the course I should take. I could have followed my usual path and kept after Rusty to describe how he was arriving at his answers, suggesting to him that unless he did so, his answers were suspect. But I was convinced that Rusty had no doubts about his answers. I knew they were correct—what I didn't know was how he arrived at them and asking him about his thinking was futile. So I decided to accept Rusty's answers without question. If I had doubts about their authenticity, rather than questioning him about how he arrived at his conclusions, I would change the parameters of the problem. If Rusty adapted his answer accordingly, that would be evidence enough for me that he knew what was going on.

How I decided on this course of action is, to some extent, as mysterious to me as how Rusty arrived at his conclusions. Somehow it dawned on me that there was a difference between knowing something and explaining how one comes to know it; one could know something and not have the verbal skills or the conscious awareness of one's mental processes to be able to explain how that knowledge was acquired. That seems obvious to me now, but at the time knowing the answer to a problem in school mathematics and being able to explain how one reached it were so interwoven by years of external and internal messages that separating them required a mental shake-up.

Also, I was struck by the contrast between Rusty's quiet confidence and the trepidation of some of my university students. I had students ask me how to solve a problem only to discover they had reached a correct solution but believed it invalid because they hadn't used a prescribed school method. They didn't trust their natural mathematical insights and intuition. I decided I didn't want that for Rusty—I suspected his solutions sprang from a keen, intuitive number sense. I didn't want him to lose trust in or abandon it, and I didn't want him to believe his thinking was suspect or unacceptable because he was unwilling or unable to describe it.

I questioned why so much emphasis was placed on showing work and explaining thinking. If one develops the ability and confidence to deal with whatever mathematics arises in one's life, what need is there to explain how one does that? I had no doubt that Rusty would do just fine dealing with whatever mathematics came into his life. His verbal skills might be lacking, but that was another matter. I didn't want to see his mathematical intuition get discounted or, worse yet, destroyed.

As a result of my experience with Rusty, I quit discounting students' results if they couldn't explain how they arrived at them. If I doubted the validity of their methods, I would do what I did with Rusty—change the setting and see if they still got correct results; if not, I would simply report to them that something in their thinking had led them astray. I still asked students to reflect on their thinking. Describing, and listening to others describe, mathematical thought processes can add to one's insight. But thought processes can be elusive and, even if captured, may be difficult to describe in words. But that doesn't mean the quality of one's thinking is inferior or the result of that thinking is somehow inferior. As I learned from Rusty, there's a difference between doing mathematics well and describing how one does it.

A Question About Algebra

September 20, 1999

Last month I did an afternoon's workshop on teaching algebraic thinking at a conference of adult educators. Midway through the workshop someone asked if I personally knew anyone who used algebra in their job. I couldn't think of anyone at the moment. Neither could anyone else. Someone thought there were some engineers or physicists who did, but they didn't know when or how. When queried, nobody in the room could recall ever using algebra in any part of their lives outside the classroom.

"If in a group of 30 adults nobody uses algebra and doesn't know anyone who does it can't be very important; so why," the question came, "do we try to teach algebra to everyone?" I confessed I had no ready answer to that question, other than it's required to survive school.

The Oregon Department of Education graduation standards as well as Oregon University System admission standards require a year of high school algebra. The reason as near as I can figure out is that one must take first-year algebra in order to take second-year algebra in order to take trigonometry in order to take calculus, the capstone course. Given the vast number of high school freshmen that are squeezed into this algebra-to-calculus pipeline and the few college students that come out the other end, it hardly seems worth the effort. Especially if all one has to show for it is the standard first-year algebra course in which one learns to manipulate symbols according to prescribed rules that are, at best, dimly understood.

At worst the result is boredom or confusion, and a distaste for all things algebraic. Those for whom this happens are in famous company. "I despised algebra," General Dwight Eisenhower recalls in *At Ease: Stories I*

Tell to Friends. "I could see no profit in substituting complex expressions for routine terms and the job of simplifying long, difficult equations bored me. I by no means distinguished myself." In *Dreams, Memories and Reflections*, Carl Jung tells of his terror as he sat watching his algebra teacher at work: "He would scribble a few letters on the blackboard. I had no idea where he got them and why he did it—the only reason I could see was that it enabled him to bring the procedure to what he felt was a satisfactory conclusion. I was so intimidated by my incomprehension that I dared not ask any questions. Mathematics classes became sheer terror and torture to me."

At the other end of spectrum are those of us who learned the rules and found no difficulty in manipulating symbols and getting things to turn out right, rewarded by the distinction one gained from getting good grades in math. We weren't doing anything that nowadays couldn't be done more efficiently and accurately by an electronic symbol manipulator. We were simply slower versions of machines, programmed by our teachers to carry out procedures without worrying about meaning. What I gained from the experience is questionable, other than a false impression of what mathematics was about. (I remember being somewhat chagrined when I discovered—after committing myself to majoring in math and deciding that I wanted to be a mathematics professor—that mathematics was something other than mastering evermore complicated algorithmic procedures. But then I discovered it was a much more creative and absorbing subject than I had ever imagined.)

For the most part, things haven't changed much, other than the size of the textbooks. The contemporary 700-page text in use in a local high school is filled with boxes of definitions, formulas, and techniques and pages of worked out examples, telling the student what to write down and how to think, followed by pages and pages of exercises to practice what they've been told—I counted over 265 elementary factoring exercises, over 300 exercises concerning the arithmetic of rational expressions and some 260 exercises manipulating radicals. One can understand why the course breeds boredom and aversion, not to mention confusion, if what one is told to do and how to think makes no sense.

Then there are the so-called applications, all those word problems that are supposed to show how useful algebra is in everyday life. In reality, all of the problems in a first-year algebra book can be solved without using algebra at all; some number sense and a good sketch will do. And the problems are all contrived, witnessed by the fact that a group of thirty adults can't recall a single application of algebra in their lives outside of school.

It's not that studying algebra need be a worthless endeavor, or a difficult or boring one. First, one must understand, that much of what one studies in algebra, as in any other mathematical topic, may never be encountered again and isn't necessary to lead a successful and fulfilling life. One might argue that everyone ought to know how to deal with simple formulas and equations and not panic when they see an *x*, but that's not the point. The point in studying algebra, or any other branch of mathematics, is developing one's mathematical talents so one might deal confidently and capably with whatever mathematical situations arise in one's life, vocationally or avocationally. That doesn't mean one will study in school all the mathematics they someday may want to know—some of it may not be invented yet. Rather, it means that one develops mathematical maturity to pursue it when it does arise. In one sense, it doesn't matter what mathematics one studies in school, as long as one studies mathematics.

I can't predict what mathematics my students will want to know in the future, but I can help them discover that mathematics is not an arbitrary and capricious world; that learning mathematics is not a matter of mastering rules, but sharpening intuitions and developing understanding. Once that begins to happen, questions about the utility of what's being studied disappear, and the satisfaction and sense of accomplishment that come with developing one's mathematical potential take over.

Reaching that point in an algebra class isn't accomplished by wading through 700-page textbooks laden with rules and procedures for manipulating mathematical symbols. In algebra, like any other mathematical topic, I find that insight and understanding is most easily built by examining physical settings from which the mathematical ideas and procedures are naturally drawn. Tile patterns provide such a setting that is relatively easy to deal with in a classroom. (See, for example, *Picturing Algebra* in the *Math and the Mind's Eye* section of the MLC catalog.) As students examine tile patterns, and their extensions, they are lead to handling algebraic expressions and solving equations in natural ways that inherently make sense to them, without the need for prescribing procedures and repetitious drill.

Of course, if one has always presented algebra in the standard textbook fashion, this new approach takes some getting used to. The week after the workshop with adult educators, I spent three days with a group of teachers going through some of the *Mind's Eye* algebra activities. At first, as is generally the case, I sensed the resistance of those who had invested years in teaching paper-and-pencil textbook techniques. As teachers allowed themselves to experience algebra in a different way, the mood changed. "I was mentally against the tile," one algebra teacher told me. But then, by the end of the second day, she found working with tile made more sense to her than the textbook routines she was accustomed to using. She was seeing how things worked. "I understood more in two days," she exclaimed with excitement, "than I had in ten years of algebraic manipulations!"

There are better ways of developing algebraic literacy than slogging through 700 pages of rote learning and deadening drill.

The Christopher Columbus Stuff

October 25, 1999

"It's not that they don't know the Christopher Columbus stuff, they lack the common sense and judgment to be successful in the workplace." That, as reported in the local paper, was the response of an administrator in a local engineering firm to a reporter's query. The reporter was asking about the on-the-job performance of adults who had acquired high school educations as part of an effort to move welfare recipients into the workforce.

I was reminded of those programs which stress the "Christopher Columbus stuff" of mathematics: adding, subtracting, multiplying, and dividing whole numbers, common and decimal fractions, and signed numbers; calculating percents; evaluating geometric formulas; etc., etc. Procedures that one can learn by rote, test on successfully, and come away with hardly an iota of mathematical common sense or perception. Unfortunately, without the latter, the Columbus stuff isn't of much value—especially these days when there are machines that can do all of it more efficiently than human beings can.

One can argue about the worth of knowing the Columbus stuff and whether mathematical sense and insight are really important. Let's assume that it's all valuable and ask why it is we can get the former without the latter.

The problem, as I see it, is getting things backward—believing number sense and insight will emerge from mastering the Columbus stuff, rather than focusing on developing number sense and insight and getting the Columbus stuff as a byproduct.

The latter is the natural way to do things. Neuropsychologists working in the rapidly expanding field of mathematical cognition maintain that human beings are endowed at birth with an innate sense of numerosity and the capacity and inclination to develop mathematical procedures without formal education. Formal mathematics education, to be effective, should connect with and build upon this natural intuition, abandoning the rote learning of mathematical procedures. (See, e.g., Stanislas Dehaene's *The Number Sense* or the recently published *What Counts, How Every Brain is Hardwired for Math* by Brian Butterworth.)

One can learn and become quite skillful at carrying out a prescribed mathematical procedure without having any conceptual understanding of what's happening.

Teaching for mastery of the Columbus stuff, rather than enhancing mathematical sense, can have the opposite effect. One can learn and become quite skillful at carrying out a prescribed mathematical procedure without having any conceptual understanding of what's happening. If that's the case, the learner is at the mercy of, rather than in control of, the procedure. If called upon to adapt what's been learned to a different setting, the learner is at a loss on how to proceed and loses all confidence in their mathematical ability. Or if what's being taught doesn't connect with the learner's innate mathematical knowledge, the learner may decide, like the young Winston Churchill, that mathematics is a "hopeless bog of nonsense." The result, rather than a feeling of competence and confidence in one's mathematical common sense, is math anxiety and avoidance.

One can understand the appeal of teaching the Columbus stuff. One can organize it neatly into little bits and pieces of so-called basic skills, demonstrate a skill and drill the students until "mastery" has been achieved, and then move on to the next bit. It's all very orderly—and quite objective if one ignores that someone has to decide which bits and pieces are to be included and what passes for mastery.

On the other hand, teaching with emphasis on developing students' innate mathematical sense and understanding isn't as clean cut, but it can be done. It means listening to students discuss their methods for approaching a mathematical situation rather than telling them your method. It means providing experiences that build mathematical intuition rather than exercises for drill and practice. It means allowing students to discover and deal with their false starts and misimpressions rather than rescuing them from their difficulties. Depending on what happens, you may have to change lesson plans in the middle of a class, and you are likely not to cover all the material you had in mind. But your students will be making sense out of mathematics.

If your goal is preparing students to be confident and successful users of mathematics, concentrating on the Columbus stuff is likely to land you oceans away from where you intended to be.

THE LIFE OF RILEY

December 1, 1999

The life of Riley is not without its bumps and bruises, especially if you are United States Secretary of Education Richard Riley. The latest missile headed Riley's way is an "open letter," made public through a paid advertisement in the *Washington Post*. The letter, signed by six mathematicians and endorsed by an additional 201 mathematicians and scientists, including four Nobel laureates, urges the United States Government to cease its promotion of ten school mathematics programs developed with the support of the National Science Foundation.

If you haven't seen the letter, it's reproduced at the end of this article. The stance taken by the signers is exemplified by a couple of excerpts. One bemoaning the "astonishing but true" fact that "the standard multiplication algorithm for numbers is not explained" in one of the programs. The other "that the standard algorithms of arithmetic are more than just 'ways to get the answer'—that is, they have theoretical and practical significance. For one thing," the statement continues, "all the algorithms of arithmetic are preparatory for algebra."

I'm astonished that a mathematician would talk of the "standard" multiplication algorithm, as if one existed or even ought to exist. I suspect what the "standard algorithm" really means is "the algorithm I learned in school." (I remember when I accidentally discovered, early in my teaching career, that what I thought was the standard algorithm for subtraction was totally unknown to my students—I was taught an algorithm that was based on adding equal quantities to the subtrahend and minuend; they were all taught to borrow. And it wasn't until then that I thought about the theoretical basis for the algorithm I had been using for years. It's a rare student that pays any attention to the theoretical base of an algorithm they've been taught—I certainly never did.) There are lots of ways to carry out a multiplication and one of the problems for adults who have been taught traditionally is that they confuse the algorithm they learned with the process. One of the consequences is that most adults are terrible at mental arithmetic, which is the most efficient way to compute in that it doesn't require any external tools. Ask an adult to multiply 25 × 36 in their head and they are likely to try to recreate mentally the "standard" paper-and-pencil algorithm they have been taught, rather than recognizing say, that 25 × 36 is the same as 25 × 4 × 9, or 100 × 9.

It also astonishes me that a mathematician would imply that the "standard" way to carry out a multidigit multiplication these days is using some paper-and-pencil algorithm. From what I observe of the world outside of school these days, the "standard" way to carry out such a calculation is with a calculator. I find it ironic that some 18 years ago Richard Anderson, then president of the Mathematical Association of American and professor of mathematics at Louisiana State University, stated that "calculators—fast, efficient, and nearly omnipresent" will eliminate the need for students to do laborious paper-and-pencil calculations, yet today it is members of his community that are resisting such a change. But then, Andersen foresaw such a resistance. "The arithmetic that people have studied tends to become the arithmetic they're attached to," he said. "If it was good enough for them, it's good enough for everyone."

All this emphasis on algorithms—in statements like those quoted above—I simply don't get. I sit here wondering of what great theoretical significance is a paper-and-pencil algorithm for long division—in what way is it preparatory for algebra? I can't think of a single arithmetical algorithm I know that I couldn't get along without. There's always another way to do it. Algorithms indeed are just ways to get the answer. They aren't the important stuff. The important stuff is understanding mathematical operations and relationships.

A stress on algorithms is the bane of the mathematics classroom. A classroom in which algorithms reign does more harm than good. It destroys the natural mathematical intuition and curiosity children bring to the classroom; it cultivates disinterest and dislike. It substitutes symbol pushing for mathematical understanding. Children don't need algorithms, they need models and images that convey mathematical operations and relationships; they need to have their mathematical intuition honored and developed. Then, perhaps with a helpful hint or two from the teacher, they will develop their own algorithms that make sense to them. Even if an algorithm has theoretical significance, the thought that students appreciate this is absurd. In every math classroom I've been in—from arithmetic to calculus—where the emphasis is on algorithms, all students want to know is how to use them; they don't want to be bothered with the theory. I didn't as a schoolboy and I doubt if any of my classmates did.

I spent 17 years as a professor of mathematics in a research university and at one time or another I have been a member of the American Mathematical Society and the Mathematics Association of America, as well as the National Council of Teachers of Mathematics. In my experience, the typical research mathematician or scientist, Nobel laureate notwithstanding, has about as much expertise concerning the teaching of arithmetic as a Pulitzer prize-winning novelist has about teaching reading. Arithmetic is part of the inherent nature of a university mathematician—they never give it any thought. They are part of that small segment of the American public who breezed their way through school mathematics without much conscious effort. They have little idea of what's involved, much less any experience, in helping fifth graders make sense of fractions, or teaching algebra to a bunch of recalcitrant high school students. I have found, however, that a lot of them are pretty good at grousing about the sorry state of school mathematics, especially in the coffee room after teaching a class that didn't go well because, according to them, the schools did such a lousy job of educating their students. (I remember telling a colleague of mine who was ranting about how his calculus class didn't remember some elementary trigonometric property that, if he were any kind of teacher at all, he could teach them what they didn't know in less time than it took him to gripe about it.)

For those educators who wonder how to react to the open letter, I suggest they ignore it and assess the new programs in light of what the real experts on the teaching and learning of school math have to say—those folks such as Stanislas Dehaene and Brian Butterworth who have studied math cognition extensively (see, e.g., Dehaene's *Number Sense: How the Mind Creates Mathematics* and Butterworth's *What Counts: How Every Brain is Hardwired for Math*).

As for those research mathematicians and scientists who don't like what's going on in school mathematics, I suggest they hold their voices unless they're willing to commit time and energy to doing those things that earn them the right to be critics—such things as reading and reflecting on the effective teaching and learning of elementary mathematics, visiting schools and trying their hand at teaching fifth-graders, becoming involved in mathematics courses for prospective teachers, developing and testing precalculus curriculum, teaching algebra to adults who didn't get it the first time around. Otherwise, I suggest they quit heckling Riley and get back to their blackboards and test tubes.

An Open Letter to United States Secretary of Education, Richard Riley

Dear Secretary Riley:

In early October of 1999, the United States Department of Education endorsed ten K-12 mathematics programs by describing them as "exemplary" or "promising." There are five programs in each category.

The "exemplary" programs announced by the Department of Education are:

- Cognitive Tutor Algebra
- College Preparatory Mathematics (CPM)
- Connected Mathematics Program (CMP)
- Core-Plus Mathematics Project
- Interactive Mathematics Program (IMP)

The "promising" programs are:

- Everyday Mathematics
- MathLand
- Middle-school Mathematics through Applications Project (MMAP)
- Number Power
- The University of Chicago School Mathematics Project (UCSMP)

These mathematics programs are listed and described on the government web site: http://www.enc.org/ed/exemplary/

The Expert Panel that made the final decisions did not include active research mathematicians. Expert Panel members originally included former NSF Assistant Director, Luther Williams, and former President of the National Council of Teachers of Mathematics, Jack Price. A list of current Expert

Panel members is given at: http://www.ed.gov/offices/OERI/ORAD/KAD/expert_panel/mathmemb.html

It is not likely that the mainstream views of practicing mathematicians and scientists were shared by those who designed the criteria for selection of "exemplary" and "promising" mathematics curricula. For example, the strong views about arithmetic algorithms expressed by one of the Expert Panel members, Steven Leinwand, are not widely held within the mathematics and scientific communities. In an article entitled, "It's Time To Abandon Computational Algorithms," published February 9, 1994, in *Education Week* on the Web, he wrote:

> *"It's time to recognize that, for many students, real mathematical power, on the one hand, and facility with multidigit, pencil-and-paper computational algorithms, on the other, are mutually exclusive. In fact, it's time to acknowledge that continuing to teach these skills to our students is not only unnecessary, but counterproductive and downright dangerous." (http://www.edweek.org/ew/1994/20lein.h13)*

In sharp contrast, a committee of the American Mathematical Society (AMS), formed for the purpose of representing the views of the AMS to the National Council of Teachers of Mathematics, published a report which stressed the mathematical significance of the arithmetic algorithms, as well as addressing other mathematical issues. This report, published in the February 1998 issue of the Notices of the American Mathematical Society, includes the statement:

> *"We would like to emphasize that the standard algorithms of arithmetic are more than just 'ways to get the answer'—that is, they have theoretical as well as practical significance. For one thing, all the algorithms of arithmetic are preparatory for algebra, since there*

> *are (again, not by accident, but by virtue of the construction of the decimal system) strong analogies between arithmetic of ordinary numbers and arithmetic of polynomials."*

Even before the endorsements by the Department of Education were announced, mathematicians and scientists from leading universities had already expressed opposition to several of the programs listed above and had pointed out serious mathematical shortcomings in them. The following criticisms, while not exhaustive, illustrate the level of opposition to the Department of Education's recommended mathematics programs by respected scholars:

Richard Askey, John Bascom Professor of Mathematics at the University of Wisconsin at Madison and a member of the National Academy of Sciences, pointed out in his paper, "Good Intentions are not Enough" that the grades 6–8 mathematics curriculum Connected Mathematics Program entirely omits the important topic of division of fractions. Professor Askey's paper was presented at the "Conference on Curriculum Wars: Alternative Approaches to Reading and Mathematics" held at Harvard University, October 21 and 22, 1999. His paper also identifies other serious mathematical deficiencies of CMP.

R. James Milgram, professor of mathematics at Stanford University, is the author of "An Evaluation of CMP," "A Preliminary Analysis of SAT-I Mathematics Data for IMP Schools in California," and "Outcomes Analysis for Core Plus Students at Andover High School: One Year Later." This latter paper is based on a statistical survey undertaken by Gregory Bachelis, professor of mathematics at Wayne State University. Each of these papers identifies serious shortcomings in the mathematics programs: CMP, Core-Plus, and IMP. Professor Milgram's papers are posted at: ftp://math.stanford.edu/pub/papers/milgram/

Martin Scharlemann, while chairman of the Department of Mathematics at the University of California at Santa Barbara, wrote an open letter deeply critical of the K–6 curriculum MathLand, identified as "promising" by the U.S. Department of Education. In his letter, Professor Scharlemann explains that the standard multiplication algorithm for numbers is not explained in MathLand. Specifically he states, "Astonishing but true—MathLand does not even mention to its students the standard method of doing multiplication." The letter is posted at: http://mathematicallycorrect.com/ml1.htm

Betty Tsang, research physicist at Michigan State University, has posted detailed criticisms of the Connected Mathematics Project on her web site at: http://www.nscl.msu.edu/~tsang/CMP/cmp.html

Hung-Hsi Wu, professor of mathematics at the University of California at Berkeley, has written a general critique of these recent curricula ("The mathematics education reform: Why you should be concerned and what you can do," American Mathematical Monthly 104(1997), 946-954) and a detailed review of one of the "exemplary" curricula, IMP ("Review of Interactive Mathematics Program (IMP) at Berkeley High School", http://www.math.berkeley.edu/~wu). He is concerned about the general lack of careful attention to mathematical substance in the newer offerings.

While we do not necessarily agree with each of the criticisms of the programs described above, given the serious nature of these criticisms by credible scholars, we believe that it is premature for the United States Government to recommend these ten mathematics programs to schools throughout the nation. We respectfully urge you to withdraw the entire list of "exemplary" and "promising" mathematics curricula, for further consideration, and to announce that withdrawal to the public. We further urge you to include well-respected mathematicians in any future evaluation of mathematics curricula conducted by the U.S. Department of Education. Until such a review has been made, we recommend that school districts not take the words "exemplary" and "promising" in their dictionary meanings, and exercise caution in choosing mathematics programs.

Sincerely,

David Klein
Professor of Mathematics
California State University, Northridge

Richard Askey
John Bascom Professor of Mathematics
University of Wisconsin at Madison

R. James Milgram
Professor of Mathematics
Stanford University

Hung-Hsi Wu
Professor of Mathematics
University of California, Berkeley

Martin Scharlemann
Professor of Mathematics
University of California, Santa Barbara

Professor Betty Tsang
National Superconducting Cyclotron Laboratory
Michigan State University

Endorsed by an additional 200 mathematicians and scientists.

Not Fit for Human Consumption

January 31, 2000

A couple of things happened last month that were a thousand miles apart, literally and figuratively.

One was a half-page article in a neighborhood weekly that caught my eye while I was at a niece's home in a suburb of Phoenix. "Life as a Mathematics Teacher" proclaimed the headline. Since that was a topic I knew quite a bit about, I was curious what the author, a local community college teacher, had to say. He described a life far different from the one I lead, although, I must confess, it was a life I once embarked on. The author, explaining to all those who wonder why in the world anyone would want to be a math teacher, had this to say: "The reason I do what I do can probably be summed up by paraphrasing the Clinton campaign slogan of 1992: 'It's the logic, stupid!' Those of us whose lives revolve around mathematics," he goes on in a sweeping generalization that might lead one to question the logic involved, "are dedicated to seeing things through to the end....We try to eliminate unsolved mysteries, and what better way to do that then with the unassailable laws of mathematics?" While doctors, artists, and politicians are concerned with other things "we mathematicians blissfully conclude that *x* unquestionably equals 3. While the world searches in vain for heroes and role models, we have our ready-made icon at whose feet we bow. His name is Mr. Spock. Yes, we practitioners of the art of numbers are the True Believers."

Meanwhile, a thousand miles away back home in Portland, according to my daughter-in-law who witnessed the event, a member of the cast of a local improvisational theater company asked the audience to name the subject in school they hated the most. The chorus of "Mathematics" drowned out whatever other offerings there were.

So there you have it. The math teacher extolling his emotionless, calculating, humanoid version of mathematics. The math students—quirky, emotional, intuitive, human beings—yelling out, "We hate this stuff!" For all the communication that's going on, the teacher and the students might as well be on different planets. And yet, ironically, they are both saying the same thing: "Mathematics is not fit for human consumption."

I sympathize with the math teacher's point of view. It's close to the view of mathematics I held when I decided to become a math major. It was my freshman year in college. Life at my family home was in a turmoil and had been for several years. Not only had the Second World War been raging, which generally disrupted life, but for several years my father, a conservative Lutheran pastor, had been engaged in his own personal war, attempting to defend himself against clerical charges of heresy and civil charges of extortion. Whereas the latter turned out to be baseless, the former, although many thought baseless also, was upheld by the governing body of the church. As a result, shortly before I left for college, my father's position was terminated and we were evicted from the parsonage. For the moment, there was no income and no home. My whole world had unwound in a zany, irrational course of events.

Math was a refuge from all this. At least, math as I knew it from my high school days. It was a nice predictable world where things behaved in a sensible, logical fashion. And I was good at it. So I became a math major to put some sense and order into my life—and I suspect many a math teacher has done the same.

Then, as I became involved in upper-division and graduate mathematics, I discovered that mathematics wasn't as sensible and orderly as I thought. I learned the distinction between truth and validity—that truth was an elusive quality, more a matter of faith than of mathematics. When I deduced that *x* equals 3 it was not an unquestionable truth but only a valid statement within the parameters of the system I was working in; what these parameters were—the assumptions I made, the type of logic I used—were matters of choice. I learned about such technical matters as consistency and completeness and many-valued logics and—the final coup-de-grace to my unadulterated view of pure mathematics—the discoveries of the logician Kurt Godel: there are undecidable propositions in mathematics. There are statements which one can not establish whether or not they hold, and it isn't simply a matter of having overlooked a critical axiomatic property—before one could arrive at a set of properties sufficient to establish whether or not every statement in the system is the logical consequent of these properties, one would have introduced a contradiction into the system. Thus, it turns out in mathematics, like everything else in life, one has to live either with open questions or contradictions.

I also learned as I progressed in my studies and was expected to provide my own solutions to problems and my own proofs for theorems and do original research, that logic alone was inadequate, something else was needed beyond the realm of logic—call it what you will, insight, intuition, revelation; ideas that come out of the blue that lead one in directions never before imagined. One can use logic to test their validity, but logic does not provide them.

Learning this about mathematics was at first upsetting; mathematics was not that orderly, well-governed, logical world I was seeking. But then I discovered that mathematics was a much more exciting world than I had imagined—a world of infinite possibilities and variety. And a much more human world; a world that was more than machine; a world where creativity and imagination were free to roam; where there was room for wonderment and what ifs and emotional responses. A world that includes logic, but in its proper role: a means for establishing the validity of our creative thought. Being proficient at math was much more than becoming skilled at carrying out routines in Spock-like fashion.

As long as math teachers view Spock as the paragon of mathematical virtue and devalue the wonderful vagaries of the human mind and the rich panoply of human emotion, classroom mathematics will be a sterile and mechanical subject, more fit for humanoids than humans.

Assessing the Assessment

March 6, 2000

The Oregon statewide assessments, mandated by the legislature in the 1991 Oregon Education Act, are up and running. And so is the opposition. An article in the local paper tells of a group of parents who are actively opposing the tests. They say they "are full of flaws and are a waste of money," while negatively affecting the quality of educational programs by cutting into classroom instructional time and draining funds from such things as field trips and counseling programs.

On the other hand, state department officials maintain that the assessment efforts improve education by providing a comprehensive record of a student's performance, determining how well a school's curriculum is preparing students, and aiding local and state leaders in setting educational policy.

As for the math assessment, the department has developed an elaborate problem-solving scoring guide and trained scorers in its use. The problems in the assessment are intended to cover those topics listed in the state's math standards. For each problem, in addition to a score based on accuracy, the student receives a score for each of four "dimensions": conceptual understanding, process and strategies, verification, and communication.

Is the assessment actually measuring these things? A look at the sampler of student solutions and the scores may cause one to wonder. A grade 8 problem under the heading "Algebraic Relationships" states that a certain disk jockey charges a fee of $150 plus $2 per person to provide music at a dance, while a second disk jockey charges $250 plus $1 per person. The student is to "show" how many persons would need to attend the dance to make the fees the same.

In the first place, one might question whether this problem has anything to do with algebraic relationships. The first deejay initially charges \$100 less then the second, but charges \$1 more per person. So if a hundred people show up, his fee will be the same as the second deejay. Does one need any algebra to figure that out? And what about the four dimensions?

A student who simply showed that the two fees were the same for 100 persons got 16 points out of a possible 29. A second student first obtained a graphical solution by graphing costs against number of people for each fee structure and spotting where the two graphs intersected, then wrote an equation that was solved by painstakingly showing every textbook step of the solution, and finally wrote a page of explanation about what they did. That student got 28 out of 29. (They lost a point on communication, which was judged to be "thoroughly developed," which is worth 5 points rather than the 6 garnered for "enhanced.")

Looking at these responses and others in the sampler, one might conclude that the assessment, rather than fostering concept learning and problem solving, breeds redundancy, verbosity, textbook tedium, and that stock in trade of the accomplished test-taker: blathering (to put it politely).

So, who's to tell? Are the assessments accomplishing something of value, or are they a waste of time and money? We could demand that our state legislature carry through on their agenda of accountability and order an assessment of the assessment to determine if it's doing what the taxpayers can rightfully expect it to do. And when, after ten years of development, the assessment of the assessment is finally administered and the argument rages whether or not it's worth its salt, why, the legislature can order an assessment of the assessment of the assessment. And on and on—creating an assessors' nirvana.

Or we can do something else. We can recognize that any assessment ultimately involves judgment and we can return the assessment of student accomplishment back to the persons most qualified to make that judgment: classroom teachers and the students themselves, with the support and encouragement of the administration.

In schools where there is mutual trust and respect—between and among students, teachers, and administrators—assessment becomes a continual, nonthreatening, everyday activity. Where there is no stigma attached to not knowing, students, who know better than anyone else what they grasp and don't grasp, have no reason to withhold that information and willingly share it with their teachers. Teachers, based on this information and their own observations, plan activities and assign tasks which help students clear up their misconceptions, deepen their understandings, and strengthen their skills. Administrators support the decisions of the teachers and provide ways for them to hone their pedagogy and increase their knowledge of the subjects they teach.

That doesn't mean things are perfect. A teacher may err in their judgment; a student may dissemble when queried about their knowledge; an administrator may find it expedient to appease an irate parent rather than support a teacher. But all-in-all, in such a setting, education is well served. The professionalism of teachers is honored; students value learning over test-taking; administrators focus on the needs of the community rather than the demands of the state department. And resources aren't squandered on elaborate assessment schemes that promise much more than they deliver.

Who Gets What?

May 5, 2000

I came across a couple of statements a while back that disturbed me. The statements expressed similar notions of what education ought to be about. They came from disparate sources and I wondered if they expressed a common viewpoint.

One statement was made by the vice chairman of the Oregon Board of Education in an article discussing the state's attempt to come to some definition of "acceptable performance in reading and math." He said, "Our intent is to get all students to the same standard." While across the nation, a mathematics professor in an Ivy League college, commenting on K–12 education, states, "Teaching is the art of getting the students to learn the subject matter."

I know how formidable and oftentimes futile it is to get my spouse or kids to do something I want, even when I am communicating clearly and—at least to my way of thinking—have the best interest of everyone at heart. How am I supposed to get students to do what I want? Bribery, cajolery, duplicity, and coercion may help, but using such tactics in the classroom turns education into manipulation.

To assert that teaching is getting students to learn, or to attain certain standards, places educators in an untenable position—it assumes that students will always do what teachers want.

Perhaps I misconstrued the statements. I looked up "get" in my dictionary. It lists 14 different meanings, each with variations, when "get" is used as a transitive verb. Substituting these meanings for "get," I ended up with such statements as "Teaching is the art of delivering the students to learn," "Teaching is the art of causing the students to learn," "Teaching is the art of subjecting the students to learn," "Teaching is the art of irritating the students to learn," "Teaching is the art of achieving as a result of military action." None of these eased my apprehension.

To teach is to profess, to make known. To educate is to educe, to draw out. The successful teacher/educator lays out subject manner so it is accessible to the student, drawing out their existing knowledge and providing ways for the student to expand it. Whether or not this happens, that is, whether or not learning takes place, requires the consent of the student. If the student refuses to consent, no matter what the teacher does, there is no learning. To assert that teaching is getting students to learn, or to attain certain standards, places educators in an untenable position—it assumes that students will always do what teachers want.

That doesn't mean that educators have no responsibility for students' learning. Whereas learning will not take place without the consent of the student, the converse is not necessarily the case. The consent of the student doesn't guarantee learning. The student may want to learn and, indeed, put considerable efforts into their attempt, and still not succeed. No doubt, there is a myriad of reasons for this, not the least of which is ineffective instruction.

The quality of instruction is something that educators can rightfully be expected to address. Instead of focusing on getting students to learn which, ultimately, only the students themselves can do, let's focus on getting effective instruction to students. Instead of attempting to get all students to the same standards which, given one recalcitrant student, is only possible if there are no standards at all, let's get our teaching to a standard that supports learning.

How do we do that in the mathematics classroom? Here are some suggestions: Provide meaning and context so that mathematics doesn't become a litany of rules for manipulating symbols. Connect material to the students' existing knowledge. Honor the students' innate number sense. Conduct classroom activities that aim to develop intuition and insight. Foster an atmosphere in which students can explore and conjecture without censure. Value ideas that don't work along with those that do—examining ideas that don't work can be a more powerful learning situation than carrying out ideas that do. Make assignments that stimulate learning and are devoid of busy work. Show enthusiasm and interest in what you are teaching. You can add to the list.

Every teacher I know wants every student they know to learn. But wanting students to learn and getting students to learn are two different things. Students are human beings with minds of their own. As such, they can foil any attempt to teach them anything. No one can teach someone something they are unwilling to learn. To believe that teachers can get students to learn is to believe that teachers are omnipotent. I believed that when I was in the second grade, but I don't anymore.

Problem Solving

June 16, 2000

Every now and then math makes the headlines of the local paper, as it did a couple of weeks ago. "Answers to math puzzlers add up to millions" announced the banner at the top of the page. The story concerned a prize of $1,000,000 being offered by the Clay Mathematics Institute for a solution to any one of seven mathematics problems, considered by mathematicians to be among the most renown unresolved mathematical questions. The list of problems can be found on the internet at www.claymath.org. This site is just one of many where unsolved mathematical problems of current interest may be found.

For the average person, accessing one of these sites won't be very informative, mathematically speaking. The mathematics vocabularies and references to previous results are only intelligible to someone with a working knowledge of the mathematics from which the problem has emerged, although there is an introduction to each of the Clay Math Institute's prize problems that gives the general reader some sense of what the problem is about. Regardless of one's mathematical background, a visit to one of these sites does dispel the notion that mathematics is a dead subject. Quite to the contrary, one sees that there are hundreds of questions that mathematicians are pursuing. And there's no end in sight, each advance in knowledge leads to new theories and conjectures to explore.

That's what mathematics is about. Making conjectures, seeking relationships, validating theories, searching for solutions, verifying results, communicating findings—in short, problem solving. To do mathematics is to solve problems.

Problem solving—at least the phrase—has always been part of school math. Every math textbook series claims to emphasize it and every list of standards gives it special attention. Here in Oregon, the Department of Education's performance assessment in mathematics is a "problem-solving" test. Passing this test is a requirement for the Certificate of Initial Mastery—concocted as part of the Oregon Educational Act for the 21st Century in hopes of convincing the world that a tenth-grade Oregon education really means something.

However, in contrast to the professional world of mathematics, school mathematics isn't synonymous with problem solving. In school, problem solving is likely to be considered just another topic, along with adding fractions, multiplying decimals, or finding perimeters. And so teachers attempt to teach it like any other topic—which, for many, means "rule and rote": Here is the procedure to use and here are problems to practice on; you will be expected to remember it and pass a test to show that you have mastered it. Thus, problem solving is reduced to an algorithmic process, typically a list of "strategies" to try until one finds one that works.

What I find unfortunate is that the list of strategies almost always begins with "guess and check" and "make a table" and, what's more, almost all of the "problems" assigned yield to these approaches. Thus, students come to equate "problem solving" with using routine methods that neither develop creativity, provide insight, nor are particularly effective in more complicated situations. It's not likely that one is going to, say, find a formula for the sum of the squares of the first *k* integers by guessing and checking or making a table.

The emphasis placed on these rudimentary methods creates mindsets that are hard to overcome. What happens is similar to what happens when students are taught particular paper-and-pencil algorithms for doing arithmetic. Their capacity for doing mental arithmetic diminishes since paper-and-pencil algorithms don't work well for mental calculations—people who have been drilled on them have difficulty imagining other ways of performing computations. Similarly, as I have found in my teaching, students whose repertoire for solving problems is guess-and-check and make-a-table have difficulty shedding these methods for more creative and productive approaches.

I suspect our students would be better problem solvers if we would quit treating problem solving as just another mathematics topic to be taught, but rather regard all mathematics as problem solving, and teach it accordingly. Most school math is ancient history—the mathematics that occur in the curriculum are answers to mathematical questions that were posed years, or even centuries, ago. But these questions are new to our students.

Mathematics doesn't have to be taught as a cut-and-dried, here's-how-you-do-it subject. It can be taught in a reflective, inquisitive mode. No matter what the topic, students' perceptions and suggestions can be explored, tested, and refined. If every topic were introduced as a problem to be investigated rather than a process to be mastered, there would be no need to treat problem solving as a separate topic with its own set of rules and procedures. Students' ability to solve problems would evolve naturally, hand in hand with their mathematical knowledge and sophistication.

WHY EDUCATION?

July 31, 2000

Physicist Fred Raabe heads the Laser Interferometer Gravitational-Wave Observatory, briefly called LIGO. Under construction at Hanford Nuclear Reservation, LIGO, when completed, will search for gravity waves. Although many physicists believe they exist, none have ever been detected. LIGO hopes to change that.

Commenting on the usefulness of such an endeavor to a local reporter, Raab said, “If you ask me for a practical application for gravity-wave research, I can’t think of one. Come back in 100 years and I’ll tell you what the practical application was.” He pointed out that the pioneers working in quantum mechanics and special relativity 100 years ago had no inkling of the modern technology that would evolve from their work, “I’ll guarantee you that if you go back and talk to the guys who were doing that work, they never dreamed of any of that stuff.”

Raab’s comments, it strikes me, need little changing to provide an appropriate answer for that 13-year-old in middle school—and any other student—who wonders, “What good to me is all this stuff I’m supposed to be learning?” The truthful answer: “Nobody knows. In 50 years you may know. What you learn now is likely to impact your life in ways neither you nor I imagine. Some of the particular things you learn in school you may never encounter again, but there’s no way of knowing which these will be. All in all, becoming educated, through whatever means, will enhance the quality of your life.”

That answer, accurate as it may be, is not likely to satisfy many teenagers. The trouble, I think, is the answer runs counter to that which the existing culture provides. The popular view expressed in the media, political chambers, and many educational agencies is that the purpose of education is to get a good job—one that ensures financial security—ignoring the fact there aren't enough such jobs to go around. The situation in mathematics is especially woeful, where many teachers attempt to justify learning mathematics because of its utility in later life, ignoring the fact that most "real-life applications" of mathematics cited in textbooks exist only in the minds of the authors and many of the topics encountered in school math are infrequently encountered in life outside school.

Rather than promoting education as a preparation for a future many students won't ever realize, I suggest we view education as an end in itself. That we as a culture value and support education whatever the future holds. That we recognize that becoming educated is as natural a human pursuit as learning to walk and learning to talk. That developing one's innate intellectual capacities, talents, and interests is as much a part of the process of becoming a mature human being as developing physically.

Given that stance, the focus of school becomes educating for now, not training for the future. The school curriculum becomes driven, not by someone's list of what children need to know to "succeed," but by what educes and develops children's existing abilities. It accommodates their interests and honors their instincts, intuitions, and insights. Engaged in such a curriculum, children will be so absorbed that, like the scientist searching for gravitational waves, future utility is of no consequence. And fifty years later, if they do happen to reflect on the value of what they learned in school, I suspect they will be surprised at all the ways it enhanced their lives, whatever their economic circumstances.

For the Fun of It

September 15, 2000

"Few pleasures equal the joy of the mind when it's being put to creative use."–Lewis Lapham

Dick Feynman, a Nobel prize winner, tells of the time while on the Cornell faculty that physics became drudgery for him. He used to enjoy doing physics and now it was beginning to disgust him. He wondered why. He decided that he once did physics because it was interesting and amusing to play with, and not because someone else thought that what he was doing was important, or that it was advancing the state of nuclear physics. He resolved to recapture his playfulness; to "play with physics…without worrying about any importance whatsoever."

A while later he was in the cafeteria when a prankster threw a plate in the air. Feynman noticed the Cornell medallion on the plate going round faster than the plate was wobbling. He was struck by what he observed and decided to see if he could figure out what was going on. He established that the rate of rotation was twice the wobble rate; that is, the plate rotated twice for every time it wobbled up and down. He told a colleague what he had discovered. His colleague found it interesting but questioned its importance and wondered why Feynman was doing it. "There's no importance whatsoever," Feynman replied, "I'm just doing it for the fun of it."

Current programs fostering math reform are criticized by some for their emphasis on making math fun. As one critic, for whom enjoying math class is of no import, maintains: "Math is…hard work, requires discipline and lots of practice." Which, without enjoyment, strikes me as a recipe for the affect that afflicted Feynman: drudgery.

To me, the position that enjoying math is of no consequence to the learner is untenable. Hardly anyone would quibble with the statement that, given their druthers, a person will spend their time and energy doing something they enjoy rather than a task that brings no pleasure. Also, learning something well brings a sense of satisfaction—good feelings, if you will—especially if one values what it is one has learned. If a student isn't feeling good about mathematics, they haven't learned much or they are not valuing what it is they have learned. Neither bodes well for a future in which mathematics plays a significant role. The vast hordes of mathophobes and math avoiders abroad in our land didn't get that way because they were feeling good about math. Stanislas Dehaene, who has studied math cognition extensively, is

convinced that children of equal abilities may become excellent or hopeless at mathematics depending on their love or hatred of the subject.

One can slog one's way through a math class, get a passing grade and not enjoy it at all. But it's questionable whether one has accomplished anything beyond meeting a math requirement. It's also true that one can have a lot of fun in math class and not learn anything. But that's not the point. The point is that feeling good about math is—in mathematical parlance—a necessary, but not sufficient condition, that effective learning is taking place. It's a consequence of learning and not a guarantee for learning. Rather than fretting about programs that seek to make math fun, one ought to be concerned about mathematics classrooms in which nobody is having any fun.

For many folk, learning math requires diligence, but it doesn't have to be drudgery. Enjoying what one's doing, doesn't mean one isn't working hard. The more students find pleasure and satisfaction in their learning, the more industrious and successful they become. Thus, the classroom environment ought to be conducive to having fun. While I know of no way to create a setting which guarantees fun for everyone—what's fun is a subjective matter—one can provide a setting which doesn't preclude it.

A first step is to remember that students are human beings and not calculating machines. Human beings have an innate intuition for numbers and space; they are capable of introspection, relish creativity, and have emotions which are linked to all that befalls them. Instruction that builds and informs students' intuitive knowledge, connects with and extends their existing understanding, allows them to exercise their creativity, and provides interesting problems for their consideration, has a much greater chance of leading to a pleasurable—and, in the long run, more productive—learning experience than any amount of drill on facts and procedures learned by rote.

Despite his colleague's skepticism, Feynman continued working on his wobbles. One thing led to another and before long he was involved in his prize-winning work on the motion of elementary particles. "The whole business I got the Nobel Prize for," Feynman reports, "came from that piddling around with the wobbling plate." Just for the fun of it.

Math in the News

October 20, 2000

I ran across three mentions of school math in the news a few weeks ago. All incidental. All negative.

The first occurred in a story about Ira Glasser, who is retiring as executive director of the American Civil Liberties Union. The story said he "was an odd leader" for the ACLU in that he didn't have a background in law. Prior to his involvement with the ACLU, he was a mathematics professor. "It was good training," the story reports. "Of all the audiences he has faced, he says, none have been so hostile as college students in required freshman calculus."

Then, in Friday's sports page, a story about the upcoming Oregon-Wisconsin football game concerned the uncertainty of who would be playing for Wisconsin because of the possibility of players fulfilling suspensions. The Oregon coach had given up trying to figure out what personnel his team would face. The paper reports him saying: "I'm not smart enough to figure out the scenarios…11 players suspended for three games, 15 players for one.…It's like one of those math problems I hated in high school.…It's a waste of our energy to worry about it."

A couple of days later, a commentary on the Napster situation appeared in Sunday's business section. The author identifies the founder of Napster as "the sweet kid I suffered through high school calculus with."

I suspect most people breezed by these comments without pause. Perhaps a wry smile appeared as thoughts of their own less-than-pleasurable experiences in school math were triggered. A few, however, may have reacted as I did, identifying with the professor trying to teach calculus to defiant students who are there because it's required. I was reminded of all those pre-meds who, for the most part, didn't care a wit about calculus and yet wanted A's so they could get into med school. (I have a vivid memory of running into a former student. The first thing he said to me was, "You're the one who kept me from being a doctor." I had given him a D in calculus.)

Comments about hating math and references to hostile and struggling students in calculus classes are commonplace. But they attract little attention. The public doesn't fret about them as it does about the rank of U.S. students in international math assessments or the math scores on state-mandated tests. It should. If the goal of school math is math literate adults, then the reaction of adults to their school math experiences ought to be given as much attention as student scores on state and national tests.

Hatred and hostility are not hallmarks of literacy. And raising test scores isn't the cure. I suspect the current emphasis on high-stake tests and the accompanying move to stiffen math requirements—such as the "algebra for all" movement—will only intensify the ill will adults harbor against math.

The rationale for requirements warrants scrutiny. The prevalent attitude seems to be that school math ought to prepare one for every eventuality. If there is some chance, however remote, that someday one will encounter, say, a quadratic equation, then include it in the curriculum. Then there is the insidious practice of using math requirements to weed students out of programs, for example, requiring a full-blown calculus course for pre-meds—I never have understood what learning calculus has to do with practicing medicine.

When I was chair of a small college math department, to the surprise of my colleagues in other departments, I fought against establishing a college-wide math requirement. I had two reasons. First of all, I didn't want students in math classes who didn't want to be there and, secondly, the math department had all the students it could handle. I had no control over what other departments and programs required, and if a student questioned me about why they had to take a certain course, I told them to go ask their advisor. As far as I was concerned, they were free to drop the course. If they didn't want to be there, I didn't want them there.

It's not that I discouraged students from taking mathematics, but I had little success motivating recalcitrant students by trying to convince them that learning the math at hand was critical for their future well being. Wondering why, it occurred to me that I was lying to many of them. So I decided to tell the truth. I told them they may never have need of the particular mathematics being studied and, further, it was impossible to know what mathematics, if any, they might encounter in their lifetime.

Then I would tell them what I wanted for them: that they develop their mathematical aptitude and abilities so if, at any time in their lives, they had the need or desire to learn about a particular mathematical topic, they felt capable and confident of doing so. To accomplish this, to a large extent, it didn't matter what topics we studied as long as it furthered their mathematical development. Once I adopted this attitude, the hostile "What's this good for?" challenges largely disappeared.

It seems to me that we as a nation are gifted enough to be able to figure out an educational system in which hordes of students aren't forced to take courses that end up as negative experiences. As a start, we can adopt more modest goals for school math. Let's not attempt to prepare students for any possible use they might make of mathematics in their adult lives. We're not that omniscient.

In elementary school, let's aim at developing number and spatial sense, building on the intuitive knowledge students bring to school. Beyond that, let math be an elective subject. Have it there for those who like it—which, if taught properly, will be a surprising number—and those who have decided it will be important for what they wish to accomplish. If certain math topics arise in the pursuit of another subject, let students start that subject, see for themselves how math is encountered and then learn what's needed. And let's not expect this to happen in the first fourteen years of a person's life. Rather than forcing ninth-graders into algebra classes, let's make it possible for any person at any time they are ready and willing to do so, to learn as much algebra as they want or need to know. Maybe this requires changing the way we deliver education, but why stick with something that doesn't work?

As long as we continue to force-feed mathematics to our students, the news reports about math will continue to teem with hostility and hatred. We're not going to change this by cramming even more math down our students' throats.

Attending to the Subconscious

December 4, 2000

Most everyone has had the experience of trying unsuccessfully to recall a name or some other bit of information only later to have it unexpectedly come to mind. In a similar vein, there are those who have tackled a mathematical problem, found no solution, quit all conscious efforts to do so, and then later have a means of solution pop into their mind.

Several such experiences of the French mathematician Henri Poincarè are reported in Jacques Hadamard's little volume, *The Psychology of Invention in the Mathematical Field.* For a fortnight, Poincarè had been attempting to work out the properties of a certain collection of functions. He hadn't succeeded in doing so when he went on an excursion with a group of people and, he reports, "The incidents of travel made me forget my mathematical work.…we entered an omnibus to go some place or other. At the moment I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it," that the functions were simply another set of familiar functions in disguise. "I did not verify the idea; I should not have time, as, upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty."

Sometime later, Poincarè continues in his report, "I turned my attention to the study of some arithmetical questions apparently without much success and…I went to spend a few days at the seaside and thought of something else. One morning, walking on the bluff, the idea came to me, with just the same characteristics of brevity, suddenness and immediate certainty" that disposed of his questions. Poincarè reports a third, similar instance of "unexpectedly" and "unpreparedly" having a solution to a mathematical problem he had been investigating come to him while away on army duty.

Poincarè's experiences suggest that whereas one may consciously quit thinking about a problem, one's subconscious will continue the conscious efforts. Not only that, it can arrive at a solution and bring it to our conscious mind at an unexpected moment. No one knows exactly how it happens.

Andrew Wiles, the Princeton mathematician who provided a proof for Fermat's Theorem—something that had eluded mathematicians for 350 years—describes his problem-solving process in *Fermat's Enigma,* Simon Singh's story of Wiles's achievement: "Basically it's just a matter of thinking. Often you write something down to clarify your thoughts, but not necessarily. In particular when you've reached a real impasse, when there's a real problem you want to overcome, then the routine kind of mathematical thinking is of no use to you. Leading up to that kind of new idea there has to be a long period of tremendous focus on the problem without any distraction. You have to really think about nothing but that problem—just concentrate on it. Then you stop. Afterwards there seems to be a kind of period of relaxation during which the subconscious appears to take over, and it's during that time that some new insight comes." Where it comes from, Wiles says, is "a mystery."

It would be nice if it were otherwise; if there was a recipe we could give our students to get their subconscious to solve their problems. Lacking such a recipe, I tell my classes that solving a problem doesn't necessarily happen instantaneously and on demand; it can require simmering time. Then I give them the following instructions: "When you find yourself at a dead end—when you are no longer making progress towards the solution of a problem and are devoid of ideas, quit working at it. But as you quit, say to yourself something like, 'I'm going to quit working on this problem for the time being and when I come back to it, I will know more than I do now.' Then do something that diverts your mind. Above all, don't continue working on the problem until anger and frustration set in and you give yourself the message, 'To heck with it. I'm not going to waste any more time on this problem. I'll never get it anyway.'" My supposition is that the former message gives the subconscious permission to keep working on the problem while the latter message encourages it to drop the matter.

I've no hard evidence that heeding these instructions creates, on the whole, better problem solvers. I do know the procedure has worked for me on occasion, not just in mathematical problem solving but in other situations, ranging from figuring out a clue in a cryptic crossword to formulating funding proposals. It does lead students to an awareness of the role of the subconscious in creative thought and to attend to the circumstances of their moments of insight—the settings in which their "ahas" occur. And it helps students recognize that in problem solving, after a point, relaxing is more effective than working hard.

Four-Syllable Words

January 22, 2001

“My goodness,” I thought, “they think education is competition!” I had just read a comment made by the chairman of the Oregon State Board of Education while defending Oregon’s standards-based testing program: “We are not talking just about standards that make our young people competitive in Oregon or the United States, we are talking about standards that make them competitive globally.”

Competition—a contest between rivals—and a world-wide one at that. What a dismal, dour prospect. Schools becoming boot camps to prepare youth to come out on top—intranationally and internationally—in head-to-head strivings for superiority. Civil War and World War all at the same time, to gain those top test scores, to be ranked number one!

Perhaps I’m given to hyperbole. I suspect the State Board would think so. No one is advocating bloodshed. Just sweat and tears. To make sure our students are the best in the world, we must hold them to high standards, and to make sure they meet those high standards, we must give them high stake tests—tests that become the ultimate measure of success; tests that let all the world know who’s getting the job done. Thus bringing the global competition down to the local level; where school competes against school and classroom against classroom to gain the accolades of being first and avoid the stigmas of being last.

What’s the result? Winners and losers, of course. That’s to be expected if education becomes competition.

If we’re going to describe the educational process with a four-syllable word, there must be a better choice than “competition.” A word I like is "expedition"—an excursion undertaken for a specific purpose; in this case, comprehension of the world about and within us.

The expeditions I have in mind are not grandiose like those intending to scale Mt. Everest. They are more homespun and down-to-earth. Like volksmarches. A volksmarch, if you've never been on one, is a noncompetitive walk along a designated trail. The trails—I quote from the web page of the American Volkssport Association—"may be in cities, towns, forests, rural areas, anywhere there is a pleasant or interesting place to walk." They are selected "for safety, scenic interest, historic areas, natural beauty, and walkability" and are rated on a scale of 1 to 5 for difficulty.

You start at any time you like within a several hour range and you proceed along the course (or, in Latin, curriculum) at your own pace. You can walk by yourself or with a group. There are checkpoints along the way to monitor your progress. If you don't want credit, most walks are free. There is a small fee for credit, which consists of a stamp validating the event and the distance you went, as entered in a record book you keep. For an additional fee you can get an award such as a medallion, cup, or patch which commemorates the event.

Viewing education as competition evokes for me a tense, stressful, winner-take-all setting ill-suited for meaningful, long-term learning. On the other hand, seeing education as an everyday expedition—a volksmarch at a comfortable pace along a scenic trail with checkpoints to gauge one's progress—evokes an entirely different prospect. One that's educationally rich yet friendly and short on stress.

If the educational process is to be epitomized by a four-syllable word, I vote for "expedition." You may have another choice. There must be many that are better than "competition."

TAKING THOUGHT FOR THE MORROW

March 9, 2001

Does anyone care about today? A fifth grade teacher tells his class the reason he gives them lots of homework is to prepare them for middle school. A seventh grade algebra teacher tells his students that they have to write down every step because that's what the eighth grade teacher wants. A high school English teacher says her job is to prepare students for college. All this against a backdrop of state and national efforts to make schools the training grounds for corporate America. Students are bombarded with propaganda: "Stay in school now and there's a good job in your future." Education isn't for now, it's for some time down the road. Hang in there, kid, the payoff's coming!

But when? The fifth grade teacher tells you their job is to get you ready for middle school. The middle school algebra teacher tells you their job is to make sure you know everything the high school geometry teacher wants. The high school teacher tells you they're getting you ready for college and the college teacher tells you they're getting you ready for graduate school. And where are all those high tech jobs the politicians keep talking about? In Oregon, where technology has replaced lumber as the largest industry, the Employment Department reports less than 4% are in "high-tech manufacturing;" 54% are in "services" and "retail trade"—and agricultural jobs aren't included in their report.

If school is for a future that never happens, what's the point? Small wonder that students become disheartened and drop out. In the last ten years, the percentage of people in Oregon who have not completed high school has risen from 10% to 25%. In my cynical moments I say that's a move in the right direction: if we keep at it, maybe we can get the dropout rate to match the rate of dead-end jobs. We have a ways to go though: 25% is still a far cry from 77%, which, according to the University of Washington's Northwest Policy Center, is the percentage of jobs in Oregon that don't pay a living wage—enough for an adult and two children to live on.

However, the gap between reality and fantasies of the future is only one hazard faced by an educational system that sees its primary function as preparing students for a time that's yet to come. Far more deleterious is its effect on students.

For one thing, promoting education as the pathway to a future career feeds the message that one's value is measured by the job one has—a message that debilitates many adults when layoff time comes or when a job commensurable with one's education doesn't appear. I find it ironic that counselors are telling sixteen-year-old students to stay in school so they can get a good job while other counselors are telling terminated sixty-year-olds that their self-worth isn't dependent on their employment. If nothing else, we're keeping the counselors busy.

Worst of all, though, education that focuses on the future disparages our students. Rather than embracing them for whom they are at the moment, we cast them in some contrived future mold. We look past them, ignoring their presence—social, emotional, and intellectual—and design our instruction not for them, but for what we want them to be. Our zeal to prepare students for the future only serves to obliterate the present, including whatever sparks of enthusiasm still burn.

As paradoxical as it may seem, the best way to prepare for the future is to address the needs of today. On second thought, why is that so strange? The present is always with us, the future never is.

Testing the Logic

April 19, 2001

When George W. Bush speaks it's not always clear to me what he's saying. The following quote comes from his February 27, 2001, address to Congress: "Critics of testing contend it distracts from learning. They talk about 'teaching to the test.' But let's put that logic to the test. If you test a child on basic math and reading skills, and you are 'teaching to the test,' you are teaching math and reading. And that's the whole idea."

If you don't agree with his premises, no amount of logic compels you to accept his conclusions.

When he says "that logic," I wonder, "What logic?" Making a statement like "teaching to the test" doesn't entail any use of logic. What he seems to be doing is casting slurs upon those who oppose his testing program while attempting to offer a logical argument of his own to support it. An argument that runs something like this: If the teacher teaches to the test and if the test tests knowledge of basic math, then, ergo, the teacher is teaching basic math.

The logic is impeccable. But there's a hitch. Impeccable logic doesn't guarantee the truth of the conclusion one reaches. All that impeccable logic guarantees is that the conclusion of the argument is true provided the premises are true. If one bases an argument on a false premise, impeccable logic notwithstanding, the conclusion one reaches may well be false. (Here's an example of a logical argument which has a false premise and arrives at a false conclusion: All Texans are ten feet tall. President Bush is a Texan. Therefore, President Bush is ten feet tall.)

Whether or not one agrees with Bush's agenda for testing the "basics" isn't as much a matter of logic as it is belief. If you don't agree with his premises, no amount of logic compels you to accept his conclusions.

I, for one disagree with his premises. I don't believe it's possible to construct a test to be administered on a large scale that adequately measures knowledge of basic math. In the first place, I don't believe there is any consensus on what constitutes basic math, and secondly, I believe test scores are more a measure of test-taking ability than of knowledge.

During a half-century of teaching math, I've been engaged in lots of exercises to list the "basic" mathematical skills. Generally these attempts begin with some kind of litany about adding, multiplying, subtracting, and dividing whole numbers, fractions, decimals, etc. The discussion turns to what ought to be known about these things. Should it be algorithms for computing; if so, which ones? There are lots of algorithms, are any of them basic? Does one need to know formal mathematical definitions of all these operations and a formal list of rules governing their behavior? Or is it sufficient to have a good intuitive understanding—good "number sense"—something I recognize when I see it but have a hard time describing definitively. In the end, "basic math" takes on the aura of an undefined term—not an entirely unforeseen circumstance; undefined terms are at the root of any mathematical discourse.

In reality, when a test is composed that supposedly tests basic skills, what's basic doesn't determine the test but, conversely, the test determines what's basic. Put a test in the hands of teachers, tell them it's a test on the basics, and whatever is in the test becomes the basics. And that's what gets taught as the basics, regardless of its appropriateness or importance.

Also I, and many others, question whether any test given repeatedly on a large scale—even if versions change over the years—measures anything but the most superficial knowledge. Rote memorization and drill—solving lots of problems like those known to be on the test—carry the day on such exams, and require little in the way of profound understanding or working knowledge of the subject at hand.

The American public is never going to be convinced of the appropriateness of a nation-wide testing program on the basis of logic. As in any situation where there are strongly held and widely diverse opinions and beliefs, logic won't carry the day. One is not going to gain consensus on a common set of premises from which to proceed. So the president will never achieve acceptance of a nation-wide school testing program by logical argument. If he achieves it at all, it will either be by persuasion, at best, or, at worst, by demagoguery.

Dropping Out

June 12, 2001

The future for high school dropouts these days is dismal. Here in Oregon, according to the state superintendent of public instruction, dropouts are twice as likely to be unemployed as high school graduates and, if employed, earn 30% less than high school graduates; and they comprise 80% of the adult prison population.

It wasn't always that way. Three of my father's brothers, his sister and their spouses were dropouts, but that didn't prevent them from earning honest, albeit modest, livings. His brother-in-law and one of his brothers were maintenance workers for the local water district, another brother started a successful roofing business, and the third became harbormaster of a fishing port. In the eyes of my German-Russian immigrant grandparents, there wasn't much point to school once one learned to read and write and completed Lutheran confirmation instruction—unless one studied for the ministry, which is what my father did.

My mother, too, was a dropout. An oldest child, she left school after the eighth grade to help tend a brood of siblings, but in her lifetime she worked competently in a number of different positions: sales clerk, dental aide, laboratory assistant—acquiring whatever training was required on the job.

But there's a different social and economic climate today. More often than not, dropouts are viewed as social outcasts. And they are becoming economic outcasts also. Increasingly, employers are requiring a high school diploma, whether or not it's relevant. At the same time, jobs that pay more than a poverty-level wage are diminishing. (One 1999 study found that 77% of the jobs in Oregon did not pay the amount it takes a family of one adult and two children to meet their needs without assistance.)

So, what should be done? The popular answer is to eliminate dropouts as if, somehow, that will increase employment, boost the average wage, and reduce the prison population. It might, but I doubt it. The nation's unemployment rate and average wage aren't determined by the high school graduation rate. If everyone in the United States graduated from high school, the most likely effect is that more high school graduates would be unemployed, and the average wage earned by those graduates who are employed would drop. Also, dropping out of school and being imprisoned doesn't mean that one causes the other. They may both well derive from a common circumstance that has little to do with the amount of one's schooling. Attempting to reduce crime by eliminating dropouts may be as futile as trying to cure a disease by eliminating one of its symptoms, and have little effect on the prison population other than raising its educational level.

I suggest we take a more realistic approach. There will always be school dropouts, and those who might as well drop out. Instead of striving to eliminate dropouts, let us eliminate the barriers dropouts face in trying to make their way in today's society. That doesn't mean we quit encouraging and counseling youths to take advantage of whatever educational opportunities present themselves, but it does mean that we treat those equitably who find formal schooling untenable or unbearable.

To begin with, we could quit requiring high school diplomas for jobs for which they are not essential. True, the technological world of today is quite different from the world of my parents, but there are still plenty of jobs that can be capably performed by a literate person with a measure of common sense. And there are still many trades and occupations that can be learned on the job, indeed, may best be learned on the job without all the trappings of high school.

We could also attack the imbalances in our economic system so those who fill the less glamorous jobs in our society don't have to struggle with poverty-level wages. The disparateness in earnings between the highest paid and lowest paid employees in corporate America borders on the bizarre.

Changing the educational system isn't going to cure all the ills of America. Dropouts trapped in jobs paying poverty-level wages, unemployed, or imprisoned are symptoms of societal problems that are far broader than those that can be addressed by tinkering with school programs. But we are a resourceful nation; if we set our minds to it we can establish the conditions that enable one, as they did a generation ago, to drop out of school without being dropped out of society.

A Talk with the Mailman

September 10, 2001

I was out pruning the shrubs around the mailbox when the mailman arrived with Saturday's mail. As he handed it to me he asked,

"Are you a doctor?"

"Not the medical kind," I replied.

"What kind, then?"

"A Ph.D."

"What area?"

"Mathematics."

"I'm impressed. Do you teach?"

"I did."

"I wish I could have more of a conversation with you but I don't know much about math."

And then he told me his version of a story I've heard many times. A story that starts "I did fine in math until…"

In my mailman's case it was percentages. He said it just didn't make sense to him when he was told he should divide to find out what percent 18 is of 39. It had seemed to him that one ought to be multiplying. I acknowledged that percentages in school could be confusing and, switching from gardener to teacher, I suggested there were ways of thinking about percentages that might make more sense to him than what he remembered from school. He didn't pick up on my suggestion but rather—as if to assure me that he wasn't a mathematical incompetent—told me that as an adult he had figured out percentages and, without giving me an opportunity to find out how that had come about, continued on his way.

I can only surmise what confused my mailman. His reference to multiplying reminded me of what a colleague had mentioned a couple of days earlier: How he'd been put off by a teacher who told him, "In mathematics, *of* means *multiply,*" when he knew that to find something like ⅓ of 48, one divided 48 by 3. My colleague isn't the only schoolchild to get the "*of* means *multiply*" message. It's one I got and I'm sure countless others have gotten. I could imagine my mailman getting this message and then being told that to find what percent 18 is of 39, one divides. And I could imagine his reaction: "What do you mean? *Of* means *multiply*, and now you use the word and tell me to divide? I'm confused!"

I was struck by how vivid my mailman's recollections were. He could tell me the exact point, with an example, when mathematics became confusing to him. I've noticed that with other adults who, finding out I'm a math teacher, have told me their stories. There's a particular incident or topic they can identify that marked the beginning of their difficulties with math—a point at which mathematics stopped making sense. A point at which insight and intuitive understanding were overwhelmed by dicta—by authoritative pronouncements of propositions and procedures.

Mathematics by fiat doesn't work. When math is cast as a bunch of dictatorial rules to be followed regardless of understanding, it ultimately becomes a morass of confused and contradictory half-memorized, half-manufactured messages.

There are other ways to teach math. One doesn't have to dictate procedures at all. In my mailman's case, if the teacher had made clear what a percent is—through pictures, diagrams, and words—and then set the students to work collectively on a set of problems that built on their understanding, the students would have come up with appropriate ways of doing things. And the future mailman wouldn't have divided unless it made sense to him.

Of course, that's supposition on my part. I would have liked to continue my conversation with the mailman to get a better sense of what he remembers of his schooling in mathematics and what remnants of it remained in his adult life. If we really want to know how effective we've been as a nation in our mathematics instruction, we should be talking to the mailman, and a lot of other adults, rather than testing students at the end of grades 3, 5, and 8, or whenever, to see what they've crammed into their short-term memories.

It Doesn't Make Sense

October 29, 2001

It doesn't make sense. On the front page of Monday's *Oregonian* there's a story on the Certificate of Advanced Mastery, which the State Department of Education is pushing as a replacement for the high school diploma. The story, under the heading "Learning for 'real life'," says that the certificate will introduce teenagers to careers and, if successful, make school relevant by showing them the connection between what they learn in class and what they need to know to be successful on the job. Meanwhile in the business section, the headlines read "Jobless tally reaches the highest level in a decade" and "Laid-off workers prepare for a tough transition." In the latter article, seven people are interviewed who have just been laid off, all highly skilled, some with college degrees. So just what are we preparing students for? With Oregon in a recession and the nation heading that way, "real life" may well mean "life without a job."

It didn't make the headlines, but on the same Monday evening, our book club—old-timers for the most part—continued its discussion of *A Life Beyond, Finding Peace and Purpose in Midlife and Beyond*. The author, Sallirae Henderson, spent a number of years as a counselor in a retirement community in our neighborhood. She warns of the "vacuum in the psyche in which emptiness and despair thrive" if our self-worth is based on roles and careers no longer a part of our everyday lives. We put 16-year-olds in an educational system that tells them a job is the ultimate goal of life and then tell 60-year-olds not to believe it. It doesn't make sense.

As far as that goes, using the term "mastery" to describe the outcome of a secondary school education doesn't make much sense either. For those of us who have been around for awhile, mastery is not a term we would apply to many things in our lives, other than simple tasks like tying our shoes or routine procedures like solving linear equations. And certainly not to something we learned in our teenage years.

So what makes sense? Mostly, I think, a different kind of rhetoric—a different way of talking about education and what we expect from it. If we must give certificates, let's make them certificates of educational progress and initiative, rather than mastery. Mastery suggests the end, rather than the beginning, of an educational journey. Completing high school is but the first step in an adult's education which ought to be a life-long process that may or may not take place in formal settings.

Above all, in our public discourse, let's not equate education with job training. Education serves the much broader purposes of developing the vast and varied talents innate in every human being, and overcoming ignorance and prejudice, for the betterment of both individuals and society. Preparing for a job may be a part of that, but it's only a fraction of the total. An education ought to serve one equally well in all aspects of life, regardless of color, creed, age, gender, and employment status.

Thankfully, there are those who take a broader view. While talking about these matters at the book club, one of the mothers mentioned that her son, recently graduated from college, was selling cars. An acquaintance of hers, upon hearing that, exclaimed, "What, he's graduated from college and selling cars?" Before she could respond, a daughter interjected: "In our family we go to college to get an education and what we do for a job is up to us." It makes sense to me.

Everybody's Mad About Math

December 13, 2001

While looking through my files the other day I came across the results of a research project I undertook a number of years ago. I never published the results—my project protocol wouldn't meet the stringent conditions that publication requires—but the information I gathered points to an irrefutable conclusion: everybody's mad about math.

It's something I had suspected for a long time, and I wanted evidence to support my belief. Hence the study.

The design was straightforward. My copy of Webster's—at least the one I had at the time I concocted the study—listed ten different meanings of the word "mad." So I made up a checklist containing these ten meanings, in the order given in the dictionary, and gave it to students and other audiences and asked them to check all that applied to them. Here's the list:

Check each word that applies. When I do math, I become:

Insane	________	Enraptured	________
Senseless	________	Rabid	________
Illogical	________	Hilarious	________
Furious	________	Frantic	________
Angry	________	Wild	________

Since everyone who participated in the study found something to check, the obvious conclusion is that everybody's mad about math. But not in the same way. Here are the percentages of checks each item received.

Insane	10%	Enraptured	8%
Senseless	8%	Rabid	2%

Illogical	10%	Hilarious	2%
Furious	15%	Frantic	22%
Angry	18%	Wild	5%

Although my study may lack scientific stature, it does give us math purveyors a glimpse at the mood of our customers. A third of the responses indicate that math makes many simply downright mad: furious, angry. Over a quarter of the responses point to pleads of insanity—math makes people crazy mad: insane, illogical, senseless. Another third of the responses speaks of giddiness: in the presence of math, people turn foolishly and frenetically mad: rabid, hilarious, frantic, wild.

Only a small fraction of the responses indicate the madness of enthusiasm and desire—as in "I'm just mad about Mary (or Maury)"—only one in a dozen responses indicates the rapture of the math enthusiast. Which raises the question: Why is this response so paltry? You could conduct a follow-up study to research the matter. Unless, of course, you already know the answer.

College Football, the Postal Service and Bush-era Education

February 4, 2002

College football, the postal service, and education in the Bush era may seem like an odd trio, but they do have one thing in common: the creation of elaborate, technologically sophisticated mechanisms intended to overcome the foibles and inefficiencies of individual human judgment and effort. With a common result: tyrannical systems that aren't very successful, except in diminishing the human touch.

The Bowl Championship Series (BSC) poll is supposed to be a precise way of deciding who's the best in college football. Before the BCS, various polls of coaches, sportswriters, and other pundits would offer their choices for who's number one, often with some disagreement. So the BCS standings were contrived to make a definitive and objective selection based on a variety of measurable factors. At the end of the season a game between the two teams at the top of the standings would determine the champion. A formula was developed to give teams a numerical ranking based on various polls and computer rankings, the strength of schedule—a number contrived from "the cumulative won/loss records of the team's opponents and the cumulative records of the teams' opponents' opponents"—losses, and something called quality win points.

If you follow football, you know that many think the BSC number-crunching doesn't work. The nation's best two teams didn't play for this year's championship (in the interests of full disclosure, I am an Oregon

Duck fan). Most likely the formula will get tweaked again, but given the vagaries of athletic contests, it's unlikely any formula will be derived that gives better results than the judgment of "experts," no matter how arcanely these judgments are made. But as long as it's the BCS standings that count, whatever their shortcomings, that's what teams will compete against. Forget about the other team—let's not be satisfied with simply a win. Let's do what gets us the most points in the BCS standings, even if that means running up the score against some hapless opponent.

While football fans were debating the BCS ratings, residents of Columbia County, a rural Oregon county of some 40,000 residents, were wondering what was going on. The checks were in the mail but they weren't getting delivered, at least not to the intended recipient. That's why the county was sending out delinquent tax notices. It turns out the folk in Columbia County are accustomed to addressing mail intended for county offices simply to Columbia County Courthouse, without a street address, and the local post office had no problem delivering it down the street. As a matter of fact, the County Courthouse had no street address. And that appears to be the problem because now, you see, the mail doesn't get sorted locally—it gets sent to Portland to get sorted before it's sent back to be delivered down the street. Apparently the automatic sorter, fooled by the lack of a street address, routed a bunch of the mail intended for the courthouse to Colombia, our South American neighbor.

The problem is being addressed. The U.S. Postal Service office in Washington, D.C., has been notified so it can take care of the software glitch that is whisking domestic mail around the world. And Columbia County has assigned the courthouse a street address for future mailings. Its residents will learn to address their mail so the mail-sorter in the metropolis can read it, and in so doing, a bit of the charm of small town living will disappear. It's hard to imagine what's gained—economically and otherwise—by sending the local mail away to be sorted. If human touch and interaction have any value, it's certainly a loss.

Continuing the trend toward large mechanistic systems, touted to be foolproof, is the recently unveiled Bush education act. In the interest of accountability, covered with a veneer of rhetoric about not leaving any child behind, the plan orders the annual testing of students to see if they're up to snuff on the standard school subjects, and if they're not, watch out! The administration doesn't want to intrude on states' rights, so the testing will be left to each state to develop and administer. But the feds might as well do it. The result will be the same.

Vast effort will go into developing, first, standards setting forth what everyone should learn and, second, standardized tests that cover this material in valid, comprehensive, and impartial ways. Once they are administered and graded, we will have foolproof, objective knowledge of what every student in America has been taught and, thus, who has done a good job of teaching and who hasn't. At least that's the theory.

But in practice, just as one plays the game to earn points in the BCS standings and addresses mail to satisfy automatic mail sorters, teachers will figure out how to get their students to perform well on the tests. Passing tests will be mistaken for getting an education. And the judgment of the individual teacher—and hence their professionalism—will be constrained.

The classroom teacher is in the best position to assess student progress and determine appropriate instructional programs. Those of us who have been involved in teacher education, know that teachers, given appropriate educational opportunities and reasonable resources and support, do that quite—even admirably—well. Rather than putting huge sums into attempting to develop a fail-safe educational system by devising elaborate testing mechanisms designed to hold teachers accountable, let's for a change be accountable to teachers. Let's see to it that they get the education, compensation, resources, and respect to provide a meaningful education in a setting where human interaction and individual identity are valued.

A Postlude: *Just as I finished writing this, the Oregon Department of Education released its 2002 School Report Card. The Report Card, mandated by the 1999 state legislature, is another example of a massive effort signifying next to nothing. The state department—having gathered data on attendance, dropout rates, and state tests and plugging it into a formula developed by their number crunchers that requires 23 pages of a technical manual to explain—have reported to the state that 1,098 of the 1,112 schools graded are "satisfactory" or higher, in fact, 55 percent are "strong" or "exceptional"; none are "unacceptable." The State Superintendent would like folk to think the high rankings are the result of his leadership, while critics say the report is propaganda. A deputy superintendent, reacting to criticism of the glowing results, says "it may be time to ratchet up a notch." So the formula will be manipulated until it gives results that better fit the public's perception of how things are, and more of the state's ever scarcer education dollars will get frittered away.*

THE EDUCATION/BUSINESS CONNECTION

March 15, 2002

It was a bit fortuitous, I thought. Ironic, too.

On the front page was a story concerning the Certificate of Initial Mastery (CIM), the centerpiece of Oregon's high school reform movement. Intended to dethrone the high school diploma, the CIM requires passing extensive state-administered tests and completing work assignments in core courses. However, most schools don't require it for graduation and only a fifth of last year's seniors earned one.

A handful of schools do require it and the chair of the English department of one of these schools commented on how the curriculum had been adjusted so that students could meet CIM requirements: "Now, maybe students read only two novels in the class, instead of four. But I can be confident every student who has the CIM can write a coherent paragraph when he needs to on the job."

And right there in the business section—would you believe it?— was an example of one of those paragraphs written on the job. The author was Kenneth Lay, Enron ex-chairman and the paragraph was a note sent to Donald Sanders, John Olson's boss. John Olson, a stock analyst who for years questioned the value of Enron stock. The note: "Don— John Olson has been wrong about Enron for over 10 years and is still wrong. But he is consistant. (sic). Ken"

Perhaps if Ken had a CIM he would have been a better speller. But it wasn't the misspelling I found ironic but rather that the paragraph was entwined in one of the greatest instances of culpability we've witnessed in this country, sullying not only the business world but all those who abet it: politicians, government agencies, accounting firms and, I wonder, our educational system, too?

I don't know how, if at all, the educational system fed the scheming avariciousness of the Enron gang, but it does seem that education is in danger of becoming the pawn of business and industry. Increasingly, our educational institutions are primarily viewed as the training ground of our nation's workforce. Education initiatives are passed with the avowed purpose of developing the world's most highly skilled workers. We cater to the interests of business in allocating resources and establishing programs. We entice students to stay in school by dangling glamorous high-tech jobs in front of them, despite the fact that most jobs out there are in service industries—many of them at poverty-level wages that don't really require much formal education.

Perhaps it's time that education—especially at the pre-college level—maintain an arm's length relationship with the business world. Let educators use their knowledge of children's abilities and interests, and how they learn, to establish the school curriculum, independently of business interests and influence. The curriculum may look much the same as it now does, but the rhetoric of education would be quite different. Rather than being led to believe that school was preparing them for some job that may never materialize, children would hear that school was designed to develop all the capabilities they possess as human beings—including their sense of morality—to enrich their lives and the lives of those around them.

In the process, the world of business and industry, and all other worlds of human enterprise, would find a cadre of well-educated, principled young people to assimilate into their workforce. If the world of business would entrust education to the educators and the world of education would delegate job training to business, both worlds would win.

Notes to Myself— Some Reflections on Teaching

April 29, 2002

A while back I ran across an article entitled "Notes to Myself" that I wrote for the September 1984 issue of The Oregon Mathematics Teacher. *At the time I had been teaching for over thirty years. Now it's been over fifty years since I taught my first class and I find that these notes have served me well. The article is reprinted here in its entirety.*

It's been over thirty years since I taught my first mathematics class. During that time I have evolved a list of notes to myself that serve as guidelines when I undertake a class. Mostly I carry the list in my head. From time to time I attempt to record it. Whenever I do, the list never comes out the same nor does it seem complete. However, there are four items which are in the forefront of my mind right now that have occurred on all my recent lists. These are: 1. Have a story to unfold. 2. Nourish insight. 3. Tell the truth. 4. Be open to change.

Have a story to unfold

Robert Davis, director of the Curriculum Laboratory at the University of Illinois, once wrote, "…most first-year algebra courses…—like the Manhattan 'phone directory—contain a great abundance of detail, but no clearly recognizable plot." That is not only true of algebra—school mathematics, in general, tends to be a collection of isolated topics without any apparent continuity or cohesiveness. The result is that, for many folks, the subject becomes boring and pointless, an unending sequence of procedures to be mastered for some purpose that apparently will be made clear in a future course that never seems to arrive.

For me, the antidote is to have a story line, so that each course has a beginning, an ending, and a plot, or is at least a collection of short stories, each with its own integrity. I don't find this easy. Story lines seem absent from most textbooks or, if the authors had one in mind, they aren't willing to divulge it. Recently, I was looking at the seventh grade book of a popular text series and wondered how I could create a story line to fit it. There didn't seem to be any major theme to the book, but some ideas did emerge. I thought about taking all the topics on fractions and combining them into the story of rational numbers—why they were invented, how one operates with them, why the operations are defined as they are, examples of the usefulness of rational numbers, and their limitations, for example, in representing some distances precisely. That could lead into the story of decimals and the real numbers. That's probably a story line I'd use—at least until I had a better idea.

Nourish insight

Psychologist Robert Sommer in his book *The Mind's Eye* maintains that the reason the "new math" failed is that it devalued imagery. Students were not developing images in their mind's eye that they could use in thinking about math. Literally, they had no insight. Without insight, one can learn paper-and-pencil procedures and how to correctly manipulate symbols, but it's difficult to solve problems, apply mathematics, and build conceptual knowledge. Now, if there's any point at all to mathematics education, it's developing these higher order abilities. (We can get machines to do the symbol-pushing for us.) Thus, I want to foster the growth of my students' mathematical insight.

In my view, sensory perception is a critical element for many people in the developing of insight. Hence, creating active learning experiences and using manipulatives, models, sketches, and anything else that provides sensory input becomes a critical part of the mathematics classroom. So this statement is a reminder to me to get out the blocks, not just for first-graders but for college students as well.

Tell the truth

A woman in a methods class I was teaching told me in class one day that her son was learning all about sets in school and she asked me when he would ever use that information. "Truthfully," I replied, "the answer may be 'never'." A guffaw of surprise swept through the class. I suspect the answer was unexpected. What was expected, I think, was some variant of what I call the "Big Lie" of school mathematics: "You have to learn this because you'll need it sometime." I'm familiar with it because I used to say it. Then one day it occurred to me that every time I said it I was probably lying to some member of the class, and sometimes, say if the topic was the division of fractions, I might be lying to almost everyone in the class. At any rate, I resolved never again to attempt to motivate the study of mathematics on the basis of future utility.

Life in the classroom has been much better since then. I no longer get involved in the game which begins with a student asking me what use he or she will ever make of the topic of the day, and I responding by listing all the circumstances I know in which the topic in question might be used. The student responds, "I'm never going to do that, so why do I need to know this?"—and I'm right back at square one. I find it much more satisfactory to be truthful and say, "Apart from school you may never use this." I believe I'm being honest when I tell students there are certain mathematical skills, like knowing the long division algorithm, that are simply school survival skills. They will need to know them to get out of the fifth grade, get into med school or whatever—and I have no rational explanation for that. I also tell students what I want for them: that, although I can't predict what mathematics may be useful for them to know in the future, I want them to feel confident and competent about learning whatever math they may want to learn at any time of their lives. I suspect much of the math that will prove useful to our students hasn't been invented yet.

Be open to change

I think most people get into routines that are comfortable and secure for them. I do. And when something comes along that threatens to disrupt that routine, I resist. I wonder what the resistance is about because I find, when I do open up to change, it's exciting and vitalizing. At least, that was my experience when I finally tried other teaching strategies besides lecturing. Of course, before I could even consider teaching by some other method, I had to be aware of some alternatives. And so this note to myself to be open to change carries with it the charge to be versed in alternatives, to have other options available when what I am doing isn't working or becomes trite or outmoded.

I believe this is especially important as the world of mathematics is drastically changed by the rapid advances in calculator and computer technology. With machines available to perform mechanical mathematics processes, human activity in mathematics becomes conceptual and not computational. This not only affects the content of courses, but also the manner in which they are taught. Conceptual mathematics doesn't lend itself to being taught by drill-and-practice methods. Thus change becomes imperative, and teachers become uncomfortable. Recently, I heard a teacher exclaim that if calculators were allowed in school, the whole fifth-grade curriculum would be destroyed. The teacher knew of no alternatives, he was unable to envision a mathematics curriculum with calculators, and how it might be taught. There are a lot of options, and knowing about them will help overcome the resistance to change.

These four notes to myself are among those which currently set the tone and strategies for my teaching. Perhaps they may be helpful to you as you think about your philosophy of teaching mathematics.

Math in the Lives of Two English Professors

June 24, 2002

They had much in common. Both were born in the 1860s and both died in the 1940s. Both attended Yale University as undergraduates and both, after receiving Ph.D. degrees in literature, taught at Yale. They became full professors within a year of one another and remained colleagues until their retirements. Both wrote autobiographies. Even their names were alliterative. But they differed vastly in one respect. William Lyon Phelps abhorred mathematics. Wilbur Lucius Cross relished it.

Mathematics, Phelps wrote in recounting his school experiences, "were the curse of my life at school and college, and had more to do with my unhappiness than any other one thing, and I bitterly regret the hours, days, weeks, months and years that I was forced to spend on this wholly unprofitable study." He promises to return to this subject later in his autobiography with "more venom." And he does.

While describing his college days at Yale, Phelps digresses to vent his rage at all things mathematical: "…for those who have no gift and no inclination, mathematics are worse than useless—they are injurious. They cast a blight on my childhood, youth, and adolescence. I was as incompetent to deal with them as a child to lift a safe. I studied mathematics because I was forced to do so.…After 'long division' nearly every hour spent on the subject was worse than wasted. The time would have been more profitably spent in manual labor, in athletics, or in sleep. These studies were a brake on my intellectual advances; a continuous discouragement and obstacle, the harder I worked, the less result I obtained. I bitterly regret the hours and days and weeks and months and years which might have been profitably employed on studies that would have stimulated my mind instead of stupefying it."

It's not only his own circumstances he deplores, but the tragic fate of "hundreds who were deprived of the advantage and privilege of a college education because of their inability to obtain a passing mark in mathematics. They were sacrificed year after year to this Moloch [an ancient deity worshiped by the sacrifice of children]."

Cross, on the other hand, found arithmetic easy and prided himself on his ability to make mental calculations. In college, he recalls, " I was almost equally interested in pure and applied mathematics. Euclid…fascinated me, not because it added anything new to my knowledge of geometry, but by the art portrayed by the old Greek mathematician in proving by a strict deductive method the truth of propositions which any one might see were true at a glance. It was like traveling over a beautiful road to the foreseen end of one's journey. Likewise, in a course in analytical geometry…, we played with the curves of algebraic equations which fell into strange and wonderful patterns, rivaling anything I have ever seen in the most fantastic designs of wallpaper. Though drawn in the first instance to higher mathematics by a kind of artistic sense, I maintained a secondary interest in mathematics as the foundation of science. The more difficult the problem, the more intense was my desire to attempt its solution."

Though each is treated in the other's autobiography as an esteemed colleague, no mention is made of any discussion between them about their polar views of mathematics. If there were, I suspect it did nothing to change these views, which in all likelihood were deep-seated, emotionally laden beliefs springing from their childhood experiences and the messages they received from the authorities in their lives.

Phelps says nothing of his mathematics teachers or classes. Whatever went on in school or at home, it's clear he heard some of the saws about mathematics that still hold sway today, e.g., some people simply do not have a mind for math—which he accepts and cites as the cause of his struggles with the subject—and, everyone should study mathematics because of its usefulness and the intellectual discipline it develops—with which he vehemently disagrees.

Cross offers some clues for the felicity he found in mathematics. He mentions a teacher to whom he owes "a lasting debt for the practice he gave me in mental arithmetic" and he cites a teacher in the college preparatory course he took in high school that "was one of the best teachers I have ever seen in action. He knew well how to keep his students steadily at work. And yet he was not a drillmaster. Rather, he assisted his students in laying good foundations in mathematics…."

Whatever the case may be, Phelps emerged from his schooldays seeing math as drudgery, entailing long hours of burdensome work for very little return and no satisfaction. While Cross emerged from his seeing math as an aesthetically pleasing and absorbing subject to which he willingly gave his time.

So what made the difference? According to Phelps—and Cross might have agreed if they had ever discussed the matter—it was their nature: some people are endowed with a "math mind" and some aren't; Cross was; Phelps wasn't. But I think not.

Rather I think it more likely that Cross was fortunate enough to have had teachers who, rather than drillmasters, were educators. They connected to the nascent, naturally curious, mathematician within Cross and nurtured it. Consequently, for Cross, math became a natural and intriguing subject which he willingly encountered.

As for Phelps, I suspect he was victimized by a mythology that still surrounds the teaching of mathematics: Math is a collection of procedures entrusted to authorities that pass them on to students regardless of their appropriateness or their significance to the learner and his inherent mathematical knowledge. Furthermore, if the student doesn't acquire mastery of these, despite prodigious effort, it's because the student doesn't have a math mind and not the fault of the instruction to which they were subjected. And further, the failure to master these procedures has dire consequences, since the mastery of them is critical for success in all but the most menial tasks, as well as being a measure of one's capacity for analytic thought. One can understand Phelps' anger when he came to realize he had been forced to struggle for years to master something that made no sense to him, only to find out in the end, the misery he experienced in doing so had been in vain. Math played no role in the considerable success and satisfaction he experienced in his profession.

The Cross/Phelps story has intrigued me ever since I stumbled upon it leafing though autobiographies. First, I find it interesting that a couple of close associates with so much in common could have such a divergent outlook on mathematics. Secondly, if I hadn't known it's a century-old story, I might have thought it took place yesterday, so familiar is the theme. Mathematics still catches the fancy of a few and repels a lot of others. And I continue to believe it's neither in the nature of either human beings or mathematics that this is inevitable. I cling to the belief that if math were taught in ways appropriate for human beings, there would be lots more Crosses in this world and the Phelpses would become all but extinct. Maybe next century.

A Postlude: *Cross gravitated into administration at Yale, becoming dean of the graduate school in 1916 and provost in 1922. He retired from Yale in 1930 and shortly thereafter was elected governor of Connecticut, an office he held for four two-year terms. Phelps retired from his professorship at Yale in 1933. A popular teacher and prolific author, he received a number of honorary doctorates, including those from Brown, Colgate, Syracuse, Rollins, and Yale. The William Lyon Phelps Foundation, headquartered in Huron City, Michigan, where Phelps maintained a summer home, is dedicated to his "writing, values, life and times."*

How To Make a Mathophobe

August 5, 2002

You're not likely to find it in the dictionary. It's a coined word, patterned after such words as Francophobe or Anglophobe, formed by the attachment to a descriptor of the combining form *-phobe,* meaning one that is averse to whatever has been named. Thus one gets *mathophobe*—someone who has a repugnance for or distaste of mathematics.

As anyone knows who has ever heard math discussed at a social gathering, there are lots of mathophobes. That perhaps is regrettable but, if one thinks about it, not surprising, because they are quite easy to make. Here's one recipe that works: Take a bright student. Force them to take math every year. Bore them with a long list of contrived rules. Remove all sensory experience. Stifle any creative urge.

The recipe worked in David's case. David is a high-school student whose score in his first attempt at the SAT was 1600, the highest possible. His achievement was reported in the local paper along with a thumbnail sketch. David, we learn from the article, will be a high-school senior this year, runs cross-country, and intends to go to college and major either in philosophy, literature, or psychology. We also learn that although David "has an aptitude for math, he detests it."

I have known other students who, when forced to do so, successfully slogged their way through a math class while resenting every minute of it. Several pre-med calculus students come to mind. Nonetheless, I was struck by the seeming incongruity of someone acing the math portion of the SAT while detesting math. I phoned David to find out what led to his strong feelings about math.

David and I chatted a while, and also exchanged e-mails. I think my recipe captures the essence of what soured David on math. He was required to take a math class, and he was running out of classes to take. So he took calculus. He found it boring and he resented all the time it required—time he would have preferred to spend on other school matters. Also he didn't find much in the course that seemed applicable to him.

I don't think anyone intended to turn David into a mathophobe. But it's readily done in today's educational climate. Excessive requirements and high-stake tests, coupled with an emphasis on training for a future that seems remote, create an atmosphere in which the recipe is easily followed.

Requiring students to take mathematics when they detest it is pointless. Human memory being what it is, what will be remembered will be the strong negative emotions and very little, if any, mathematics. Teaching our students all the mathematics they might ever need or want to know is impossible—nobody knows what that is (I suspect some of it hasn't been invented yet). Our students would be better served if we worked at maintaining their interest and developing their understanding rather than covering a vast array of topics. If our students are confident of their learning and not put off by mathematics, then if the need ever arises for them to learn more math, they can do so. David feels no need to know more math now, but if ever in his lifetime he does, his major hurdle in learning it will be his jaundiced view of the subject.

Much of the aversion to math created in our classrooms could be avoided. But first, the focus must change from covering material to connecting with students; from ploughing through textbooks to strengthening students' understandings; from pushing formulas to providing context; from grading tests to discussing ideas.

But in the end, what difference does it make? As long as we turn out students who ace the SATs and pass all those tests, who cares whether we produce one or a million mathophobes?

The Algebra Blues

October 8, 2002

Whenever I think about a conversation I had with my granddaughter, I get the blues. She was in a seventh grade advanced math class and they were studying algebra. I asked her how it was going. She showed me her homework.

Her solution of a simple equation like $2x + 10 = 16$ required 5 lines. First she had written down the equation. Next she had written "$2x + 10 - 10 = 16 - 10$, subtract 10 from both sides." Then, "$2x = 6$, collect terms." And so on.

I asked her why she wrote all that down; couldn't she figure out the solution in her head? She said, "Yes, it's 3, but I just can't write that down." I said, "What about writing $x = 3$ because $2(3) + 10 = 16$." She said, "No, I have to do it this way," pointing to her paper.

I asked her if she had any idea why the teacher wanted her to do it that way. She did: "Mr. X says that's what the eighth grade teacher wants and his job is to get us ready for the eighth grade."

I was disheartened. Mr. X was turning algebra into drudgery and destroying whatever number sense his students possessed. Besides that, he was putting the interests of the eighth grade teacher ahead of his students' welfare. I could only wish that Mr. X would undergo some marvelous transformation that would change him into a teacher like Mrs. Y.

In contrast to Mr. X, Mrs. Y puts her students' educational development ahead of their next teacher's expectations. For her, mathematics instruction isn't dictated by what the next teacher wants but by the present knowledge and understanding of her students. She believes addressing the latter will take care of the former and if it doesn't, then the next teacher needs to change their expectations.

For students in Mrs. Y's classes, algebra is not a collection of rules and procedures imposed on them by some authority. Rather, it becomes a way of communicating about and dealing with mathematical situations based on their own investigations, investigations she instigates that are designed to naturally lead the students into the topic at hand. Her students talk and write about their thinking. Clarity of expression, both oral and written, is valued, but students aren't forced to turn their work into ersatz axiomatic demonstrations.

Teachers like Mrs. Y are more concerned about their students' current mathematical development than what some future teacher fancies they ought to know. In their classrooms, algebra is not a set of rules and procedures of mysterious origin to be imposed upon their students, but a subject that evolves naturally from a set of experiences. Their rooms are alive with activities that evoke algebraic concepts and procedures. Algebra is not imposed on their students, but it is drawn from their observations and discussions. Students' insights and intuitions aren't smothered by forcing them to use the teachers' or textbook's way; they are encouraged to use methods that are based on their own understandings and insights. Instead of being taught there is only one way to carry out an algebraic procedure, students are encouraged to find alternate ways of proceeding. They aren't castigated if their methods don't work, or are based on misconceptions. Rather, their efforts are valued, and ferreting out whatever goes awry becomes a learning experience for the whole class. In such a teacher's classroom, algebra is lively and vibrant.

But in my granddaughter's mind, it's tedious and banal. And should there be more Mr. Xs than Mrs. Ys, there must be hordes of other students with the same frame of mind. What a melancholy thought.

Manipulatives and Metaphors

December 9, 2002

"Manipulatives," the saying goes, "are a bridge to the abstract." Presumably, I suppose, from the concrete. But, given the image the word "bridge" conjures in my mind, the metaphor seems wanting to me.

"Bridge" does suggest a connection, and I would agree, that the concrete and the abstract ought to be connected. But, "bridge" also suggests to me a sense of separation and departure. A bridge may indeed take me to some new place, but once I'm there the bridge has served its purpose; it becomes a thing of the past and may as well be forgotten. Thus, in the bridge metaphor, manipulatives provide a means of escaping the concrete to get to the abstract.

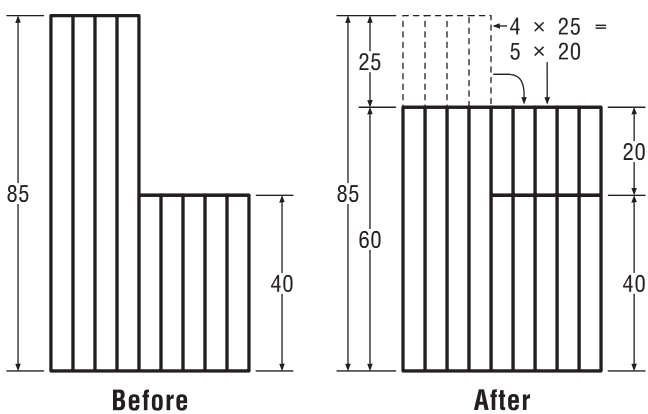
For me, the role of manipulatives is not to move one from the concrete to the abstract, but to provide a concrete support for the abstract. A metaphor that is more appealing to me is that of the studs in a wall that provide support for the structure. As the structure is being erected, the studs are in full view, however as the structure gets built, they become covered over and are no longer exposed to our view. However, they still remain there, continuing to support the structure, invisible to the naked eye, yet still visible in the mind's eye.

Take, for example, the notion of arithmetic average. One way that I've found effective for introducing this notion to students is to have students erect some stacks of blocks and then, without changing the number of stacks, level them off. Thus, if they create 5 stacks, containing 3, 8, 10, 4, and 5 blocks respectively, when leveled off, each of the 5 stacks will contain 6 blocks. I tell them that, mathematically speaking, one says the average of 3, 8, 10, 4, and 5 is 6. (The numbers in this example were chosen so that things work out nicely—in most instances, one finds that one has to cut some blocks into parts to make the stacks all the same height, which leads to a nice discussion about fractions.)

As students become familiar with this model for finding averages, they determine that one way to find the height of the leveled off stacks is to determine how many blocks there are all together and divide that total by the number of stacks, that is, they discover that the average of a set of numbers is the sum of that set of numbers divided by how many numbers there are, which is the usual textbook definition. At this point, one could leave the blocks behind and cross the bridge to the abstract and henceforth deal with averages in the abstract world of pure number. But to do so, loses all the power of the image of leveling off stacks of blocks which gives much more insight into how averages behave then the abstract definition does.

For one thing, the "leveling off" image can be quite useful when computing. It's much easier to average 93, 89, 95, and 94 by leveling off these numbers than by adding them up and dividing by 4. Thinking of these numbers as stacks of blocks, moving 1 block off the last stack and 2 off of the second last stack, and putting these three blocks on the second stacks gives us stacks of 93, 92, 93, and 93. So we have 4 stacks of 92 with 3 extra blocks to be divided among the 4 stacks. Hence the average is 92¾, and very little arithmetic was required to determine this. For another example, consider this typical school problem about averages: Helen has grades of 83, 75, and 90 on three math exams; what score must she get on tomorrow's exam if she wants an average of 85 on the 4 exams? What Helen wants are 4 "stacks" that level off to 85. The first stack, 83, needs 2 more, the next stack needs 10 more, while the third stack has 5 to spare. Hence the first 3 stacks need a total of 7 more to level off at 85. Hence the last stack needs 7 more than 85, or 92, for all 4 stacks to level off at 85.

The image can also be adapted as the curriculum progresses. (If the examples that follow come from territory that is foreign to you, skip this and the next paragraph. You can get the point without following the details in the examples.) All those mixture problems encountered in algebra courses are nothing but averaging problems in disguise. For example, to find the amount of a 40% sugar solution that must be added to an 85% sugar solution to create 1800 ml of a 60% solution, think leveling: stacks of 40 and stacks of 85 are to be leveled off to get stacks of 60. Since 85 is 25 more than 60, 4 stacks of 85 provides an excess of 100 over 4 stacks of 60. These 100, spread over 5 stacks of 40 will bring these stacks to 60. So 5 stacks of 40 and 4 stacks of 80 will provide 9 stacks of 60. (The figure may be helpful.) Thus to get 1800 stacks of 60, take 1000 stacks of 40 and 800 stacks of 80. In terms of the original language of the problem, one needs 1000 ml of the 40% solution.



When one gets to calculus, the average value of a function over an interval can be thought of as the value of the ordinates of the graph of the function once they've been "leveled off." For positive-valued functions, this is the height of a rectangle whose base is the length of the interval and area is that of the region in the interval which lies between the graph of the function and the x-axis.

You may not have followed the mathematical details involved in the previous paragraphs, but that's not essential for understanding the point I'm trying to make, namely, manipulating blocks in elementary school to investigate averages provides images that carry through calculus, helping to clarify ever more complicated concepts. That's what a good manipulative can, and should, do: provide images that not only clarify the concept at hand but can be adapted to provide new images as the concept is extended to new situations.

Rather than viewing manipulatives as something one leaves behind to wander the world of abstract mathematics, I prefer thinking of them as the structural framework that provides support and coherence for mathematical constructs.

Playing by the Rules

February 13, 2003

The Oregon Department of Education released its annual school report card last week. The report, as mandated by the state legislature in the massive Oregon Educational Act for the 21st Century, is to contain information on "student performance, student behavior, and school characteristics." In addition the report must include data on some 16 areas that are specifically named, including, among others, attendance rates, school safety, dropout rates, facilities for distance learning, and local bond levy election results.

The Act further requires the Board of Education to "adopt, by rule, criteria for grading schools," which grades shall "include classifications for exceptional performance, strong performance, satisfactory performance, low performance and unacceptable performance." Each school's grade must be included in the report.

And that's what education has become: a game. Lots of high-achieving students have long recognized its game-like tendencies, figuring out, teacher by teacher, what the rules were for getting an A.

In 2002, 50 schools were graded exceptional, 563 strong, and 485 satisfactory, while only 14 schools were low, and none were unacceptable. With some 56% of the schools being rated strong or exceptional and none flunking, the Board was criticized for having too soft rules, so this year things were toughened up. While the number of exceptional schools almost doubled to 91—mostly elementary schools whose assessment scores went up when some exams were cut—the number of strong schools dropped by 164 to 399, the number of satisfactory schools increased to 557, the number of low schools reached an all-time high of 28, and 7 schools were actually designated unacceptable—a first in the brief history of the report cards.

One principal, bemoaning his school's drop from "strong" to "satisfactory" lamented, "Here we are working hard to improve, and then the rules change." But he was ready to staunchly move forward: "…the rules changed for everyone, and it's still a level playing field.…We are going to refocus and continue with targeted improvement." Ah, yes, that's the spirit! You can win this game!

And that's what education has become: a game. Lots of high-achieving students have long recognized its game-like tendencies, figuring out, teacher by teacher, what the rules were for getting an A. But now the playing field is broader and the stakes are higher. It's a school against school statewide competition, and with the Bush administration's recently released new game with the alluring title No Child Left Behind, everything's in place for the national championship!

It may take a while for administrators to figure out how to successfully play this new game, but they will. They're a resourceful bunch. Here at home, in only a few years—notwithstanding some fiddling with the rules which have caused some minor setbacks— administrators have figured out how to win at the Oregon game. Here's how the principal and vice-principals of a middle school that got an "exceptional" rating have done it: They visit three to five classrooms a day. They give teachers "pointers" and pore over test results. They send home frequent progress reports and enlist parents' help when their child's performance slips. Students who aren't "on target" to pass state reading tests have to give up their electives and are sentenced—

that's my word, not theirs—to a specialized lab to "study reading" which, I suppose, is their way of saying "prep for the test."

There's no avoiding playing the national game. The consequences of losing at it can be troublesome: being publicly labeled a "low-performing school" and—taking language from the U.S. Department of Education's websites—facing "real consequences" that ultimately entail "corrective action and restructuring measures" designed to get your school "back on course," not to mention losing students to a better school, "along with the portion of their annual budget typically associated with those students."

Administrators will figure out how to avoid this opprobrium. They will make sure their teachers get plenty of pointers on what must be done to ensure their students are playing this new game properly. Meanwhile, the teachers will wonder how their profession metamorphosed into game monitoring. And the students? Like pawns on a chess board, they will press onward dealing with every test the game makers have placed in their way until either they are overwhelmed by opposing forces, which isn't supposed to happen if the game is played correctly, or they attain the goal: a finely embossed certificate which announces to all that they have completed the game successfully. That, again, is my description of the certificate. The game makers hail it as the mark of a quality education.

Other Nooks & Crannies

Folk Math

One of my first realizations that something was awry in school mathematics was the chasm that existed between people's everyday encounters with mathematics and what they learned in school. "Folk Math" addresses the nature of that gap and offers some suggestions for narrowing it.

"Folk Math" first appeared in the November 1976 issue of The Math Learning Center Report *(later the* Continuum*), published and distributed with the support of a National Science Foundation Dissemination grant. It was reprinted in the February 1977 issue of* Instructor *magazine and in the December 1980 issue of* Mathematics Teaching*, a publication of Great Britain's Association of Teachers of Mathematics.*

Consider the following question from the "consumer mathematics section of the first National Assessment of Educational Progress, the nation-wide testing program that has surveyed the "educational attainments" of more than 90,000 Americans: "A parking lot charges 35 cents for the first hour and 25 cents for each additional hour or fraction of an hour. For a car parked from 10:45 in the morning until 3:05 in the afternoon, how much money should be charged?"

The question was answered correctly by only 47 percent of the 34,000 17-year-olds tested, a result widely cited as an example of Americans' poor mathematical skills. But does the "parking-lot" exercise have any validity? Does it actually measure ability to handle real parking-lot arithmetic?

Obviously, in a parking lot the problem of figuring one's bill is never so clearly or explicitly stated. One is not handed a paper on which all necessary data are neatly arranged. Instead, information must be gathered from a variety of sources—a sign, a wristwatch, a parking-lot attendant. Paper

and pencil are seldom available for doing computations. One is unlikely to go through the laborious arithmetical algorithms or procedures taught in schools and used in tests. One is more likely to do some quick mental figuring.

And few people compute an exact bill, even mentally. It is easier and more efficient to figure an approximate answer: "I've parked here less than six hours and the rates are slightly higher than 25 cents an hour on average, so my bill shouldn't be more than $1.50." Unfortunately, approximation is little taught in schools, and most people do not feel comfortable enough with numbers to try it. Most people might have a very rough notion of what the bill should be, intuitively or based on prior experience. They rely on the parking attendant to compute it. The attendant, of course, almost certainly relies on some mechanical or electronic computation device.

MR. BROWN'S GARDEN SPRAY

There may be some mathematics done in parking lots, but probably very little of the sort measured by the National Assessment. The situations are radically different. People do parking-lot arithmetic in parking lots using methods appropriate to parking-lots, not in classrooms using paper-and-pencil methods.

But testers are not alone in producing specious measures of "real-world" mathematical skills. The average elementary mathematics textbook is full of "story problems" like the following: "Mr. Brown made 3 gallons of garden spray. He put the spray into bottles holding 1 quart each. How many bottles did he fill?" It is hard to imagine anyone facing such a problem outside a classroom. Certainly few elementary students have ever manufactured garden spray, or even witnessed adults doing what Mr. Brown was purported to do.

Yet such "problems" give the appearance of being from the "real-world." Supposedly they are intended to relate school experiences to life outside school. But they have little in common with that life. They are school problems, coated with a thin veneer of "real-world" associations. The mathematics involved in solving them is school mathematics, of little use anywhere but in school.

Much school mathematics consists of abstract exercises unrelated to anything outside school. This is why school math is disliked and rejected by many. Schoolchildren recognize that school math is not a part of the world outside school, the world most important to most people.

Yet if school is to be preparatory for life outside school, the school world ought to be as much like the nonschool world as possible. In particular, young people in classrooms ought to do mathematics as it is done by folk in other parts of the world. School math ought to emulate folk math.

Math Folks Do

Woody Guthrie defined folk music as "music that folks sing." In that same way, folk math is math that folks do. Like folklore, folk math is largely ignored by the purveyors of academic culture—professors and teachers—yet it is the repository of much useful and ingenious popular wisdom. Folk math is the way people handle the math-related problems arising in everyday life. Folk math consists of a wide and probably infinite variety of problem-solving strategies and computation techniques that people use. I believe the first goal of mathematics education should be to assist students to cultivate and enlarge their inherent affinities and abilities for folk math.

Folk math is the way people handle the math-related problems arising in everyday life.

Attempts have been made in recent years to teach "real-world" mathematics, a phrase betraying the curious notion that school is somehow unreal. These attempts usually have failed, producing jumbles of "story problems" like "Mr. Brown made 3 gallons of garden spray…" Such "real-world" mathematics describes no place I have ever been or wanted to be. Such curricula only serve to make school math seem more meaningless and absurd.

Surveys "in the field" to determine what mathematics is used in various occupations have produced lists of topics that read much like the table of contents of an arithmetic text: addition, subtraction, multiplication, division, fractions and decimals, ratio, proportion, and percentages. Teachers conclude they are already teaching those things, and return to the security of the text.

What is overlooked is *how* and *why* mathematics is done outside school. School math and folk math range over much the same mathematical topics.

But folk mathematicians—that is, all of us when we're facing a math-related problem in everyday life—do mathematics for reasons and with methods different from those commonly involved in school math.

SOME EXAMPLES

I've heard a college math teacher tell me how she became aware of the vast differences between school mathematics and the mathematics in her kitchen. She noted that in halving a recipe that called for ⅓ cup shortening, she simply filled a ⅓ cup measure until it looked half full. In school this would be presented as a "story problem" with the correct answer "(⅓) ÷ 2 = ⅙." Solving the problem would require paper and pencil, certain reading and writing skills, and the ability to divide fractions. But in the kitchen paper and pencil are seldom handy. Neither the problem nor the solution is written. And dividing fractions is not really necessary.

I remember eavesdropping on a friend of mine, a building contractor, as he related his train of thought in computing 85 percent of 26. "Ten percent of 26 is 2.6, and half of that is 1.3," he said. "So that's 3.9, and 3.9 from 26 is—let's see, 4 from 26 is 22—22.1 is 85 percent of 26." He computed 15% of 26 and then subtracted, using some slick mental arithmetic in the process. I was struck by what I heard, knowing that the method of finding percentages taught in school involves a complicated algorithm requiring paper and pencil. I asked him whether his school experience had anything to do with how he handled the percentage problem. "Didn't you know? I quit school in the sixth grade to help out on the farm," he said.

I once watched a crew of workmen replace the metal gutters and downspouts in an old three-story building. With only a few metal-working tools and measuring tapes, they quickly cut, bent, and fitted the gutters and downspouts, probably unaware of how nicely they were dealing with three-dimensional space. I wondered what would happen if a crew of mathematics educators wrote a textbook on mathematics for gutter- and downspout-installers. I imagined all the standard school geometry and trigonometry the textbook would contain, and how the crew I was watching would see no relationship between what they were doing and the contents of the text. Such are the differences between school math and folk math.

The differences don't only exist in the adult world. Watch children playing *Monopoly*. One lands on Pacific Avenue and owes rental on two houses. "That's $390," demands the owner. A $500 bill is offered, correct change made, and the game proceeds. But translate the same problem into school mathematics. "Mr. Jones sent Acme Realty a check for $500. However, he owed them only $390. How much would be refunded?" The same children scurry for pencil and paper, ask a flurry of questions, worry over correct procedures, and hurry to get on to other things.

Consider an incident the father of a third-grader related to me. His son brought home a teacher-made drill sheet on subtraction. The child had completed all the exercises, save one which asked for the difference 8 – 13. That one the child had crossed out. The father knew his son had computed such differences in trinominoes, a game in which it is possible to "go in the hole." The father asked the boy if he knew the answer.

"Yes," he said, "it's –5."

"Well, why did you cross that exercise out?" the father asked.

"In school, we can't do that problem, so the teacher said to cross it out."

Further discussion revealed that since negative numbers had not yet been introduced in class, the teacher had said that one can't subtract a larger number from a smaller number. Hearing this, the father's older son, a sixth grader, asked, "Why do teachers lie?"

For the third-grader, school math had already become different from folk math. Knowledge and skills acquired outside school no longer seemed to apply inside, a most confusing development. To the sixth grader, teachers were no longer trustworthy. The older boy had begun to realize that school math is less authentic and reliable than folk math.

Some Generalizations

Some of the general differences between school math and folk math are clear. One is that school math is largely paper-and-pencil mathematics, while folk mathematics is not. Folk mathematicians rely more on mental computations and estimations and on algorithms that lend themselves to mental use. When computation becomes too difficult or complicated to perform mentally, more and more folk mathematicians are turning to calculators and computers. In folk math, paper and pencil are a last resort. Yet they are the mainstay of school math.

Another difference is in the way problems are formulated. In school almost all problems are presented to students preformulated and accompanied by the requisite data. For folk outside school, problems are seldom clearly defined to begin with, and the information necessary for solving them must be actively sought from a variety of sources. While talking with a trainer of apprentice electricians, I realized that an industrial electrician is much more likely to be asked "What's the problem?" than be told "Do this problem." Yet technical mathematics courses for electricians are filled with the latter statement, whereas the former question seldom, if ever, occurs.

And the problems themselves differ between school math and folk math. Many so-called "problems" in school math are nothing more than computation exercises. They focus on correct procedures for pushing symbols around on paper. Folk mathematicians compute too, but for them it is not an end in itself. Pages of long division exercises are not part of folk math. Problems in folk math deal with what it will cost, how long it will take, what the score is, how much is needed. Problems in folk math deal with a part of one's world in a mathematical way.

Some Solutions

How can school math be made more like folk math? School math should provide schoolchildren the opportunity to deal with the mathematics in their own environments in the same way proficient folk mathematicians do. Schoolchildren should be encouraged to formulate, attempt to solve, and communicate their discoveries about mathematical questions arising in their classrooms, their play yards, their homes. All are rich with questions to explore: How many tile are in the ceiling? How big is the playground? How old are you—in seconds? Children should be encouraged to develop their own solutions and ways of computing, building on their previous knowledge. Schools should be mathematically rich environments providing many opportunities to develop and exercise mathematical talent.

In this setting, the role of the teacher is to bring mathematical questions to the attention of students, encourage them to seek answers, and talk with them about possible solutions while allowing them to grope, err, and discover for themselves.

All this is terribly idealistic and difficult to achieve. But teachers, administrators, curriculum developers, and especially those of us who call ourselves mathematics educators should be seeking ways of making school math more like folk math. There may be something inherent in schools, in the constraints and demands placed upon them, that will prevent school math and folk math from ever being the same. But the gap between the two need not be a chasm.

Some Suggestions

A few suggestions follow that I believe are feasible for any teacher to try.

One is to extend math activities beyond textbook and drill work-sheets. Develop activities out of what's going on around school and in other subject areas. Encourage students to look for what is mathematical in their environment, and ask them to formulate math problems related to their discoveries.

Second, stress mental arithmetic and ways of computing that lend themselves to mental use. Good folk mathematicians are good at mental computation and estimation. Other computation tools may not always be available, but folk mathematicians always carry their brains with them.

Third, use games and puzzles to develop skills and friendliness with numbers. Many parents and educators feel that games and puzzles are out of place in school and should be permitted only when regular lessons are finished. I shared this view until I realized how much of my own early mathematical development and that of my children was enhanced by playing games. Outside school, little folk mathematicians use numbers mainly, perhaps solely, to play games. And if school math is to emulate their folk math, games should be included.

Fourth, take advantage of the innate fascination and aesthetic appeal of mathematics. Often students are urged to learn mathematics because it will be "useful" someday. Often that is a lie, for no one can predict precisely what mathematical skills any one child will need twenty or thirty years from now. Such urging fails to motivate most children anyway, since for them the future, beyond the next school holiday, is a blur. But most people can find immediate enjoyment in some aspects of mathematics, and schools should not overlook that appeal. In fact, I think it reasonable to include in the curriculum math activities whose major value is simply that they are fun to think about and do.

Fifth, accept and use the electronic calculator the way folk mathematicians do. Folk mathematicians use the calculator as it was intended, as an efficient, economical machine for performing calculations. I don't see folk mathematicians using the calculator as a device for checking paper-and-pencil computations, or as a toy for playing games. Good folk mathematicians use calculators at will. They make mental estimates as a check, and they don't use the calculator when mental computation is more efficient. When they use the calculator, it is as a replacement for paper and pencil.

Finally, refuse to let standardized tests determine the curriculum. Folk math skills can no more be measured by paper-and-pencil multiple-choice tests than can the ability to play the violin. I believe that good folk mathematicians can do well at these tests, and at school math generally. But many persons who are good at school math are poor at folk math. They have had their folk math abilities stifled and blunted by school math. They are unable to do simple addition or subtraction without resort to paper and pencil. They are enslaved to the slow and awkward procedures learned in school.

On Knowing and Not Knowing

"On Knowing and Not Knowing" grew out of my reading of the literature on the brain hemispheres that was prevalent at the time. The theory on the varying roles of the two cerebral hemispheres provided an explanation for the dissonance between peoples' working knowledge of mathematics and their school knowledge of mathematics, a phenomenon broached in the previous article.

"On Knowing and Not Knowing" first appeared in the Fall 1978 issue of Continuum. *It was reprinted in the September 1979 issue of* Mathematics in Michigan *and, in part, in the Fall 1978 issue of* Arizona Teachers of Mathematics.

Many people understand mathematics, yet don't understand that they understand. They know, but they don't know. It is this, I believe, that inspires much of the needless apprehension and unease many people feel towards mathematics.

Let me explain by means of a couple of examples.

A student once asked me to help her with an arithmetic problem based on an advertisement. A dress was offered for sale at 20 percent off the list price. The sale price was \$60. The problem was to find the list price.

I asked this student to tell me her difficulty in solving the problem. I was surprised when she promptly told me the solution and explained how she found it.

"The sale price of \$60 was 80 percent of the list price," she said. "So 20 percent of the list price was one-fourth of \$60, or \$15. Adding \$15 to the sale price of \$60 gives the list price of \$75."

"That's right," I said, a little baffled. "What is it you want to know?"

"How do you do the problem?"

"I don't understand. You just did the problem."

"No, I didn't," she insisted. "What's the formula?"

As we talked further, the realization dawned on me: She believed she didn't know how to do the problem unless she plugged numbers into a formula. She had correctly solved the problem, but couldn't believe she had. She knew, but thought she didn't.

Another example. Once when I was cashing a check, the bank teller noticed that it was issued by the Oregon Mathematics Education Council.

"Oh, mathematics," she said with a grimace. I asked about her reaction. "Mathematics was my worst subject," she said. "I never was any good at it."

She paused, then was struck by the incongruity: a bank teller bad at mathematics. She quickly tried to reassure me that she was a capable teller and actually quite good at mathematics.

She told me that she didn't know mathematics, and that she did. Almost in the same breath. "I know, but I don't know."

I observe this phenomenon often. I see it in well-educated people who handle the mathematics of their daily lives without difficulty. Yet they disclaim any mathematical ability, vowing it was their worst subject in school. I hear about this phenomenon from parents who tell me their children are hesitant and unsure when they do school arithmetic assignments. Yet in playing board games at home, these same children handle the necessary arithmetic with aplomb and dispatch.

Observations like these have led me to conclude that when people say they don't know mathematics, though they demonstrably do, they're referring to some special kind of math—to school math. For these people, the math they learned (or failed to learn) in school is completely unrelated to the math they use in their everyday lives—the math I've dubbed "folk math."

I want to suggest some reasons many people see little or no connection between these two brands of mathematics. My conjectures are based on recent research into how people learn, particularly into how the two hemispheres of the brain play different roles in the process of learning.

The human cerebrum—the large front portion of the brain, the seat of conscious mental activity—is divided into halves or hemispheres. The two hemispheres, the right and the left, are joined only by a band of nerve fibers known as the corpus callosum.

Over the past century, psychiatrists and psychologists have noticed that when one of the cerebral hemispheres is damaged, or when the corpus callosum is severed, a person's mental abilities will change in fairly predictable ways. Damage to the left brain is likely to impair the ability to speak and write. Damage to the right brain may affect spatial reasoning, the ability to sketch objects or physically manipulate them.

The corpus callosum has been severed surgically to control seizures in some persons suffering from severe epilepsy. In these persons, the two halves of the brain appear to work in isolation, the right brain guiding the left side of the body, the left brain guiding the right side. Such persons are able to perform visual, spatial tasks only with their left eye and hand, tied to the right brain. Conversely, they can name or give verbal description only to those objects seen or touched with the right eye or hand, tied to the left brain.

Thus the left brain seems predominantly involved with rational, analytic, sequential, logical tasks. It arranges words into sentences, does mathematical computations, and puts quantitative measures on phenomena. It dissects knowledge into sequential bits and pieces.

The right brain appears to deal with information in intuitive, global, relational ways. It recognizes shapes and faces, is responsible for spatial orientation, and has creative flashes and insights. It forms overall impressions or "gestalts" from diverse bits and pieces of data.

How do the two hemispheres relate in an undamaged brain?

"In most ordinary activities," says Stanford psychologist Robert Ornstein, "we simply alternate between the two modes, selecting the appropriate one and inhibiting the other." Ornstein's research into the patterns of brain activity suggests that the complementary working of the two hemispheres, and of their two modes of thought, underlies our most creative accomplishments. Bob Samples, a humanistic psychologist, argues that our psychological health depends on our attaining an equilibrium between the two modes of thought, on not allowing one mode always to dominate.

Many U.S. schoolchildren can perform basic arithmetic as taught in school, but cannot apply that knowledge to simulations of "real-world" mathematical problems.

This theory about the two modes of thought and how they work together appeals to me. It provides a framework into which my own experiences, observations, beliefs, and hunches about the teaching and learning of mathematics fit nicely. And perhaps this theory explains the phenomenon of "I know, but I don't know." I have an idea why many people find little connection between school math and folk math.

I suspect these people do their folk math using both hemispheres, both modes of thought, in a complementary way. They shift back and forth between hemispheres as needed. They possess a good, intuitive, right-brain sense of how numbers work. They translate that sense into appropriately analytic, symbolic, left-brain computations. The problem is solved.

But for these people, school math is not intuitively sensible. In school, right-brain thinking seems not to apply. School math instead is a sequence of abstract, isolated, linear tasks, tasks momentarily memorized, tasks whose meaning or significance is never directly experienced and never fully understood. School math is for the left brain only.

These people—they may constitute a majority of humankind—have two divergent understandings of mathematics, two concepts that are at odds. Their left-brain knowledge of math, mostly acquired in school, doesn't fit and perhaps doesn't even connect with their right-brain knowledge, acquired mostly outside school.

This may be why, if the National Assessment of Educational Progress can be believed, many U.S. schoolchildren can perform basic arithmetic as taught in school, but cannot apply that knowledge to simulations of "real-world" mathematical problems. This also may be why many people find mathematics in school an ugly, painful, frightening thing, to be avoided at any cost.

How does this happen? Why this gap between school and life?

Not long ago I visited a first-grade classroom during arithmetic time. The teacher asked someone to write the numeral "five" on the chalkboard. A little girl volunteered. She began writing it legitimately, but not precisely in the way taught in class. I thought she did the task in an acceptable way, but the teacher told her she was wrong, without explanation. Another child was called on, who used the "right" method and received the teacher's approval.

I wonder about that little girl. I'm sure she since has learned the standard school method, which is fine. But I wonder what else she has learned. Her method appeared as efficient and reliable as the teacher's, and apparently it had emerged from her own experience, her own sense of how numbers work. Yet she was told her method is illegitimate, that it doesn't apply in school. She was told—or at least her right brain was told—that school math is a miscellany of arbitrary, capricious rules, rules that don't make sense.

The teacher was not at fault. Teachers have long been taught that mathematics—and almost everything else—is most easily learned in little bits and pieces. This is the one belief shared by proponents of various educational fads, from "competency-based education" to "individualized instruction." The smaller the bits, the easier learning is thought to be. Mathematics is thus reduced to a vast number of rules for pushing symbols about with pencil and paper.

But the parts are less than the whole. Teachers and students become so immersed in procedures that they lose sight of what is important. The ability to analyze problems, to grasp relationships, to intuit possible solutions—all are forgotten as students and teachers slog through a rigid hierarchy of largely meaningless skills.

School math thus becomes a wholly left-brain activity, clean and logical, but sterile, abstract, and uncreative. Students come to regard math as painful, and teachers learn to tolerate it as dull.

Outside school, confronted by an actual, practical problem requiring some mathematical thinking, teachers and students alike may find themselves helpless, unable to perform the right-brain task of "seeing" how bits of school math might be combined into a solution. Their ability to cope with living is diminished.

The ability to analyze problems, to grasp relationships, to intuit possible solutions—all are forgotten as students and teachers slog through a rigid hierarchy of largely meaningless skills.

Or else they may solve the problem by shifting to their right brain, their folk math, developed through experience and maintained despite the almost constant efforts of educators to repress it. Then they are unaware that they are doing mathematics. They are unaware that the thinking involved in their folk math is actually closer to the well-springs of human knowledge than the school math they fear and despise.

I reflect on my own mathematical training. I've succeeded at lots of school math, which I pursued through to a doctorate. But much more of the math I studied I comprehended only in a left-brain way. I mastered logical connections, analytic methods, and technical language. I passed tests. But I had no intuition, no feeling, no "big picture." Only those mathematical topics that I messed with and mulled over, that I could envision graphically, that I had an intuitive sense for, a sense of the whole—only those topics did I truly understand, only those topics could I see in new or creative ways, only those topics could I make new discoveries about. I knew them in both left and right hemispheres. I knew them in my head and I knew them in my bones.

If making confident and creative use of mathematics depends on both right- and left-brain thinking, then school ought to encourage both. How might this be done?

First, school could honor intuitive and visual thinking as valid and useful accompaniments to rational and analytic thought. Appearances can be deceiving, but so too, unsuspected errors may lie hidden even in the most elegant deductive proof. In solving a problem, mathematical or otherwise, I find that mental images, intuitive hunches, and just plain guesses often suggest a likely strategy for finding a solution. Schools could be more concerned with providing children experiences likely to nurture their abilities to think.

Second, school could allow mathematics to emerge from students' own experience. Too much of school math is tied to some imagined future event, and introduced because someone thinks students will need it someday. Such prophecies often turn out badly. But in any case, children cannot intuitively understand mathematical situations or problems they have never encountered themselves. Schools could draw on children's mathematical experiences, and foster further experiences, as a basis for mathematical teaching.

Finally, school could encourage children to use the valid arithmetical methods they have figured out for themselves, based on their own experience. If these methods are slow or awkward, then quicker, more elegant ones may be offered as alternatives. Certainly children should not be made to think that a workable method is invalid merely because it is different. Above all, the importance and integrity of a child's own intuitive and reflective efforts to make sense of numbers and the world ought to be respected.

Why Competencies Can't Cope with Students' Needs

Among the fads of the Oregon Department of Education—and those of a lot of other states—competencies, with their emphasis on "survival skills," surfaced somewhere between behavioral objectives and benchmarks. Like so many movements in public education, they seemed directed more at meeting the authorities' thirst for assessment rather than the students' need for appropriate educational practices.

This article appeared in Continuum, A Math Learning Center Report*, Winter 1980. It was reprinted in the November/December 1980 issue of* Mathematics in Michigan *and the January 1981 issue of* Indiana Mathematics.

"Competency Testing Gets Good Marks," read a recent front-page headline in the Portland *Oregonian*. The headline might more accurately have been, "Competency Testing Gets a Shrug." The *Oregonian* reported on a survey of Oregon educators that found we, on the whole, believe the state's new minimum competency requirements for high school graduation have had a "modestly positive" impact. Yet the news story also noted that about half the teachers surveyed said the requirements have had no impact, or a negative one. The researchers were reported as suggesting that the quality of education in Oregon schools is "not deteriorating" as a result of the new requirements, and hence, "it would seem to follow from this that student achievement, although not necessarily enhanced, has not been negatively affected."

Intrigued by what seemed a rather backhanded endorsement of competency requirements, I obtained a copy of the report on the survey, conducted by two University of Oregon professors under contract to the Oregon Department of Education. The report explained that most administrators—

school-board chairers, superintendents, principals, and curriculum specialists—thought competencies have had a positive impact. But most teachers and counselors disagreed; 35 percent of the teachers thought competencies have had no effect, and 16 percent thought the effect has been "negative."

Furthermore, majorities of every group—from school-board chairers to teachers to counselors—said essentially that competency requirements are redundant, that course requirements could accomplish the same things. School-board chairers and superintendents weren't asked, but large majorities of every other group said they've had to "take significant amounts of time from other aspects of the school program" in order to put competency requirements into effect. Of those who said competencies have taken time from other things, most did *not* agree with the statement, "The investment has, on balance, been worth it in terms of educational outcomes."

The survey found that competency requirements apparently have not made graduating from high school any more difficult. Only six percent of the superintendents reported that their district had an increase in the number of seniors who failed to graduate in 1978, the first year the new requirements were mandatory. Based on the superintendents' responses, an average of 3.46 students per district failed to graduate because they did not meet competency requirements.

Finally, in interviews with 32 high school students, the researchers found that most did not think their district's competency requirements would improve their preparation for entry into adult life. The researchers reported that they "got the impression talking with the students that regulations and academic requirements of one sort or another were considered an integral part of going to school, and that these latest requirements were just another part of a familiar pattern." The students were unanimous about one thing: they were not involved in determining what competencies are required by their school.

And so, after many years and much money spent putting competency requirements into effect, not much has changed. Few educators show much enthusiasm for the requirements. Those who cope with the requirements in the classroom—teachers and students—have, for the most part, found them insignificant or counter-productive. Students do not believe the requirements have enhanced the quality or usefulness of their education.

What went wrong? It may be, as the survey report suggests, that the requirements are still new, and that attitudes toward them will change with time. But I doubt it. I'm almost certain that students' attitudes will not change. I say this because I believe that the rationale for minimum-competency requirements has a faulty foundation. The rationale, as it was developed in the early 1970s, was based in part on the work of Abraham Maslow, a humanistic psychologist. It is my belief that those who developed the minimum-competency idea fundamentally misunderstood Maslow—and human beings, as well.

In early documents, the Oregon Department of Education explained the need for minimum-competency requirements in terms of Maslow's "hierarchy of needs." Maslow noted that human growth follows a pattern, and that our first and most basic need is for survival. Once that need is met, we naturally go on to grow by meeting other needs, in a certain order: the need for security, then for love and belonging, then for self-esteem, and finally for what Maslow called self-actualization.

Citing Maslow's hierarchy, Oregon's minimum-competency *Administrative Guidelines* expressed concern "that we may be trying to provide young people with self-actualizing experience before meeting their survival needs. Both kinds of needs should be addressed, but self-actualization does not occur if survival is continually threatened." Competency requirements are needed, the *Guidelines* stated, "to identify what competencies are necessary for survival and assure that all students have these competencies." These "survival skills" then "can be defined by example with several hundred performance indicators," after each district has managed to "identify the basic skills and abilities that all students should acquire as a result of public schooling."

Another Department of Education report asked, "If the first level of need is survival, does it make sense to force a student to sit through the self-actualizing experience of Shakespeare and ignore their survival needs?" The report pragmatically declared that "real-life priorities must be recognized," and thus, "helping students to develop survival competencies is one of the primary tasks of schools in Oregon." The report asked educators to look to the future: "What competencies will be needed to survive during the remainder of this century? What kinds of competencies are required to cope with life as a citizen, wage earner, consumer, and life-long learner?"

The vision of education expressed in these documents is actually contrary to Maslow's own. Indeed, Maslow rejected the whole notion of imposing requirements on students, of presuming to tell them what they need, either to survive or to become self-actualized. "In the normal development of a healthy child," Maslow wrote in his *Toward a Psychology of Being*, "most of the time, if he is given a healthy free choice, he will choose what is good for his growth. This he does because it tastes good, feels good, gives pleasure or *delight*. This implies that he 'knows' better than anyone else what is good for him."

Adults should not try to direct a child's growth, Maslow said, "but make it possible for *him* to gratify his needs, make his own choices, i.e., let him *be*. It is necessary in order for children to grow well that adults have enough trust in them and the natural processes of growth, i.e., not interfere too much, not *make* them grow, or force them into predetermined designs, but rather *let* them grow and *help* them grow."

Maslow died several years ago, but I am sure he would cringe at what is being done in his name by the proponents of competency requirements. The proponents seem to have misunderstood what Maslow meant by survival needs. In Maslow's hierarchy, the basic survival needs are quite clear, quite pressing, and easily identified. Survival needs are what one needs at the moment in order to remain physically alive: primarily food, clothing, and shelter. Children and other human beings naturally seek these things out. Clarification by school districts concocting "several hundred performance indicators" is not required.

Contrary to what competency advocates seem to think, Maslow was not talking about needs at some future date—the 21st century or whenever. He meant needs of the present moment—the needs that children come to school with. Thus if the Oregon Department of Education truly believes that the survival of students is "continually threatened," then minimum competencies are unlikely to be of much help—better send an ambulance.

The creators of our competency requirements did hit upon one truth: in order to educate effectively, schools must meet students' needs. But what are those needs? In fact most American children come to school with their basic survival needs met—that is, they have the necessities for maintaining physical life. Most students, I believe, also come to school feeling physically safe and secure. They have their needs met at the first two levels of Maslow's hierarchy. Sadly, some children and adults do not have these needs met, but it has never been the main job of schools to meet them—that's what charities, social welfare workers, police, and public-health workers are for. Education can begin only after these basic needs are met, and most children who make it to school have had them met.

And so schools truly concerned about meeting children's needs should focus on the next steps in Maslow's hierarchy: on needs for love, belonging, esteem, and respect. Schools should seek to assist students in their growth, to "let them grow and help them grow," as Maslow says. Instead of attempting the impossible task of supplying students with everything they'll need in order to survive into the next century, schools should offer opportunities for students to explore those things that seem to *them* to meet their own needs.

Oregon's minimum-competency requirements work against this ideal, by encouraging schools to concern themselves with needs that don't exist—survival needs simply not felt by most students. Students' real, higher-level needs tend to get overlooked, ignored as "frills." One of those needs—the need to belong—cannot be satisfied so long as students feel they have no voice in the course of their education, as they are reported to feel in the University of Oregon survey. Oregon's competencies, allegedly intended to meet students' needs, actually work against the meeting of those needs, by further alienating students from school.

But if the truth were known, I suspect that competencies and other educational fads derive from the needs not of students, but of educators. Like students, educators need respect, esteem, and approval. And we've been getting rather little of those things, lately, from legislators, taxpayers, parents, and the press. We're told that students aren't learning the "basics," and that it's our fault. Feeling much abused, we've tried to regain public esteem by designing programs that we think will satisfy our critics. But we must appear selfless, so we rationalize that what we've tried to do for ourselves has actually been for the students. None of it will work, though; neither students' needs nor our own will be met. Public disillusion will set in once again, and we'll need to come up with another panacea.

Or else we could confront and try to meet students' real needs. We could try to create a classroom where students feel they belong, and where they feel they have our approval to explore and develop their talents. We could be less interested in predicting what skills students will need in the future, and more interested in challenging them with present uses for the skills they might feel a need for now. We could be less concerned with students' scores on standardized tests, and more concerned with how many say, "I like school," "I like English," "I like math." Then, perhaps, we ourselves could feel better about what we do—and so feel our own need for self-esteem satisfied.

Mathematics and Visual Thinking

In the literature on visual thinking, I found a paradigm—a philosophical and theoretical framework—for the teaching and learning of mathematics that makes sense to me. Efforts at incorporating visual thinking into mathematics classrooms led to the Math and the Mind's Eye project which received funding from the National Science Foundation over a six-year period beginning in 1984. Developing courses and writing curriculum materials for the Mind's Eye project, and its descendants, occupied me for 20 years, and continues to do so.

"Mathematics and Visual Thinking" stems from a talk titled "Math and the Mind's Eye," first given at the 1980 Northwest Mathematics Conference. The article reprinted here first appeared in the Spring 1985 issue of Washington Mathematics *and was reprinted in the September 1985 issue of* Oregon Mathematics Teacher.

School mathematics has never been kind to visual thinkers. Carl Jung, whose propensity for the world of dreams and other realms of visual thought led to major contributions in analytical psychology, had this to say about his mathematical training:

> *I felt a downright fear of the mathematics class. The teacher pretended that algebra was a perfectly natural affair, to be taken for granted, whereas I didn't even know what numbers really were. They were not flowers, not animals, not fossils; they were nothing that could be imagined, mere quantities that resulted from counting. To my confusion these quantities were now represented by letters, which signified sounds. …Why should numbers be expressed by sounds? …a, b, c, x, y, z, were not concrete and did not explain to me anything about the essence of numbers.…*

As we went on in mathematics I was able to get along, more or less, by copying out algebraic formulas whose meaning I did not understand, and by memorizing where a particular combination of letters had stood on the blackboard.... Thanks to my good visual memory, I contrived for a long while to swindle my way through mathematics.

Thus, Jung's visual gifts, rather than being used to provide insight about mathematics, were used to reproduce configurations he had seen on the blackboard, giving the pretense that he had some understanding of the matter. In reality, however, he had no idea where his algebra teacher got the letter he scribbled on the blackboard, nor why he did it. The young Jung was so intimidated by his incomprehension that he dared not ask any questions and ultimately, he reports, "Mathematics classes became sheer terror and torture to me." [1]

Other visual thinkers have fared better in mathematical matters, but as a result of their own determination and an abandonment of school methods. Freeman Dyson has this to say about his colleague Dick Feynman, who ultimately won a Nobel prize for his work in theoretical physics:

(Dick) said that he couldn't understand the official version of quantum mechanics which was taught in textbooks, and so he had to begin afresh from the beginning. This was a heroic enterprise.... At the end, he had a version of quantum mechanics he could understand. He then went on to calculate with his version of quantum mechanics how an electron could behave.... Dick could calculate these things a lot more accurately, and a lot more easily, than anybody else could. The calculation that I did... took me several months of work and several hundred sheets of paper. Dick could get the same answer calculating on a blackboard for half an hour....

We talked for many hours about his private version of physics, and I finally began to get the hang of it. The reason Dick's physics was *so hard to grasp was that he did not use equations. Since the time of Newton, the usual way of doing theoretical physics had been to begin by writing down some equations and then to work hard calculating solutions of the equations.…Dick just wrote down the solutions out of his head without ever writing down the equations. He had a physical picture of the way things happen, and the picture gave him the solutions directly, with a minimum of calculation. It was no wonder that people who had spent their lives solving equations were baffled by him. Their minds were analytical; his mind was pictorial. My own training…had been analytical. But as I listened to Dick and stared at the strange diagrams that he drew on the blackboard I gradually absorbed some of his pictorial imagination and began to feel at home in his version of the universe.* [2]

Feynman diagrams have become a standard mechanism for thinking about electron behavior. Sketches of them now appear in textbooks, helping physics students draw on their visual, as well as their analytic, faculties to assist them in learning.

Still others managed to translate the school version of things into their own imagery and used that to make sense out of school mathematics. Seymour Papert, Professor of Mathematics at MIT, describes his experiences as a child.

Before I was two years old I had developed an intense involvement with automobiles. The names of car parts made up a very substantial portion of my vocabulary: I was particularly proud of knowing about the parts of the transmission system, the gearbox, and most especially the differential. It was, of course, many years later before I understood how gears work; but once I did, playing with gears became a favorite pastime.…

I became adept at turning wheels in my head and at making chains of cause and effect.... I found particular pleasure in such systems as the differential gear....

I believe that working with differentials did more for my mathematical development than anything I was taught in elementary schools. Gears, serving as models, carried many otherwise abstract ideas into my head. I clearly remember two examples from school math. I saw multiplication tables as gears, and my first brush with equations in two variables (e.g., $3x + 4y = 10$) immediately evoked the differential. By the time I had made a mental gear model of the relation between x and y, figuring how many teeth each gear needed, the equation had become a comfortable friend. [3]

Papert, who also holds an appointment as professor of education, goes on to relate how his love affair with gears led to his formulation of what he considers "the fundamental fact about learning: Anything is easy if you can assimilate it to your collection of models. If you can't, anything can be painfully difficult.... What an individual can learn, and how he learns it, depends on what models he has available." [4] He suggests several reasons for the effectiveness of gears in helping him grasp mathematical ideas:

First, they were part of my natural "landscape," embedded in the culture around me. This made it possible for me to find them myself and relate to them in my own fashion. Second, gears were part of the world of adults around me and through them I could relate to these people. Third, I could use my body to think about the gears. I could feel how gears turn by imagining my body turning. This made it possible for me to draw on my "body knowledge" to think about gear systems. And finally, because, in a very real sense, the relationship between gears contains a great deal of mathematical information, *I could use the gears to think about formal systems…the gears served as an "object-to-think-with." I made them that for myself in my own development as a mathematician.* [5]

The need for models and images on which to hang one's mathematical thinking has been stressed by others. Robert Sommer, professor of psychology and environmental studies at the University of California, Davis, maintains that "new math failed because of its bias toward abstraction and its devaluation of imagery." [6] He claims that it tried to develop understanding at the expense of the senses. Sommer states:

A mathematical statement leaves the hearer cold when it evokes no images or associations. It is as if the words were uttered in a foreign language. Indeed, mathematics is often taught as if it were a foreign language, with only the most arbitrary connection between symbols and objects.... The problem is not the symbols themselves, but that our teaching of arithmetic detaches numbers from the stuff of life.... I have seen otherwise intelligent students turn in bizarre arithmetic solutions which they never would have considered acceptable if they had been using words instead of numbers. It was as if they were stringing together foreign terms according to some set of rules, without any idea what the words meant….

Emptying ideas of their sensuality does not produce meaningful learning or discovery, as some of [the new math's] proponents maintained, but mechanical and arbitrary learning. What must be criticized is not abstraction itself, which is too much a part of the human mind to be discarded, but abstraction at the expense of the senses. [7]

And that brings us to our goal: to bring school mathematics back to the senses. We are calling the process for doing this "visual thinking." This may be a misnomer because we have more than the sense of sight in mind.

A more appropriate name might be "sensual thinking," but this has connotations we want to avoid.

For our purposes, "visual thinking" shall mean at least three things: perceiving, imaging, and portraying. Perceiving is becoming informed through the senses: through sight, hearing, touch, taste, smell, and also through kinesthesia, the sensation of body movement and position. Imaging is experiencing a sense perception in our mind or body that, at the moment, is not a physical reality. Portraying is depicting a perception by a sketch, diagram, model, or other representation.

Whereas the previous quotations might suggest that some individuals are visual thinkers and others are not, everyone to a greater or lesser extent engages in visual thinking. Our dreams attest to that. And visual thinking, when nurtured and developed, can play a significant role in the development of mathematical understanding and in the creative and insightful use of mathematics in other areas.

There are those who claim visual thinking is primary and vital for all but the most routine and stereotyped thought processes. Robert McKim, in *Experiences in Visual Thinking*, quoting psychologists Jerome Bruner, Abraham Maslow, Ulric Neisser, and others, suggests that visual thinking is the primary thinking process. [8] It provides the content for the secondary process of rational, analytic, symbolic thought. This secondary process is vital also, for without it our thoughts would remain imprecise and incommunicable. However, to rely solely on this secondary mode of thought in teaching mathematics can lead to the situations described above in the quotations from Jung and Sommer. Large doses of visual thinking experiences are recommended for all learners.

In his book McKim stresses the importance of visual thinking in every field. He specifically mentions mathematics, an area in which one might suppose that symbolic thought is the dominant thinking mode. He mentions the study of Jacques Hadamard in which Hadamard concluded that the most creative mathematicians were visual thinkers. [9] Perhaps the most celebrated instance cited by Hadamard is that of Albert Einstein; a letter from him is reproduced in the appendices of Hadamard's book. Einstein writes, "The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined.…[These] elements are, in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage."

We may encounter few future Einsteins in our classrooms, but we can provide all our students with experiences in visual thinking that lead to increased understanding, enjoyment, and meaningful use of mathematics.

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Canal Vision

"Canal Vision" is the transcript of the keynote address delivered at the Washington State Mathematics Council Leadership Conference in Kennewick, Washington, on June 21, 1985. Kennewick was the site of a childhood walk along an empty irrigation canal that provides a metaphor for describing the failure to recognize and adapt to changes in the mathematical world. A slightly different version, with the title "Thirty Years Later," was the keynote at the Oregon Math Leaders Conference in McMinnville, Oregon, on August 21, 1985. This is the first time "Canal Vision" has appeared in print.

This evening I want to talk about change—especially with reference to the mathematics curriculum. Change is something we all know about and most of us try to avoid. But encountering change is a daily part of our lives—like my coming to Kennewick.

When I was talking with Barb Chamberlain about the location of this meeting she asked me if I was familiar with Kennewick. I said I was—about a half-century ago. The Kennewick I knew as a child has little in common with the Kennewick of today.

I grew up in southeastern Washington. My family moved from the Oregon coast to Walla Walla when I was in the second grade and left the year I graduated from high school.

My father was a Lutheran minister. One of his seminary classmates was a Lutheran minister here in Kennewick—and these two ministers' families, both removed from their relatives, became extended family for each other. We often were together on special occasions such as Thanksgiving—our family coming to Kennewick or the Kauth family making the 50-mile trip to Walla Walla.

In those days Walla Walla was the metropolis of southeastern Washington. In 1940, 18,000 people lived there; while 2000 people lived in Kennewick, across the river from Kennewick, Pasco had twice that population, while up the river, the hamlet of Richland had a population of 247. Thus the Tri-Cities, if you could call them that, had a total population of slightly over 6,000—and no wineries anywhere nearby—just a lot of sagebrush. Today, their total population is over 90,000, some 37,000 of whom live in Kennewick. The only similarity to the burg of 2000 I knew is the name.

Walla Walla, on the other hand, has stayed about the same size—it saw an influx of some 5000 during the World War II years, but other than that, its population growth has been modest. Its present population is about 26,000. In 1910, its population was 19,000.

Unlike the Tri-Cities area, I still recognize it. The Book-Nook and the Liberty Theater are still there—and so is Sharpstein Grade School. But there are changes—Sharpstein grade school has had some additions which have greatly changed its appearance; there's not a trace of the Wa-Hi I attended—except for the gym; the house we lived in is still there but it's no longer the Lutheran parsonage, that and the adjoining church building were sold years ago and a new church was built in the east end of town. So there have been changes, but they are not as conspicuous as those in the Tri-City area.

I can't recall the individual details of the two or three trips a year we made to Kennewick. Certain episodes stand out. One I remember took place on a cold, wintry day. I must have been about 12 years old. Alongside the parsonage where the Kauths lived, between the church property and the city park, was an irrigation canal. It was diked on each side and deep enough, as I recall, so that if I stood on the floor of the canal in the winter when it was empty, the top of the dike was over my head. On this particular day, we had had a noon meal at the Kauth house—it strikes me as being Thanksgiving dinner—after which, Ted Kauth and I headed out. Meanwhile, the weather worsened, the roads were getting icy and my father decided to leave sooner than I expected—wanting to drive back to Walla Walla before the roads became really treacherous. However, I was missing. My father decided to look for me, so he and Reverend Kauth took off in our family automobile thinking they would find their two youngsters along one of the streets in town—there weren't that many 45 years ago. While they did ultimately find us roaming a street in the downtown area doing whatever it is that young boys do—it had taken considerably longer than my father anticipated—how much longer I don't know other than it was sufficient time for him to develop a very irritable mood. When they did find us, there were the usual "Well, where have you been?" questions and the usual, innocent "Just wandering around" answers. "But we've driven all over town two or three times and this is the first time we've seen you." Well, Ted and I had spent most of the time walking along the bottom of the irrigation canal and couldn't be seen from the streets.

I tell you these boyhood recollections and facts about southeastern Washington partly because it's fun for me to reminisce, but also because they illustrate what I see happening to mathematics education.

The changes that have taken place in southeastern Washington in my lifetime are similar in some respects to the changes that have taken place in mathematics in my lifetime. Some aspects of mathematics have changed as radically and rapidly as the Tri-Cities area. New mathematical technologies and areas have emerged that were unthought of when Richland had a population of 247—just as I imagine those residents of Richland had no conception of the present city and the industries that support it. Other changes have been less dramatic and longer in the making—like the changes around Walla Walla—the gradual dismantling of the old and unworkable—like the disappearance of my old high school—to be replaced by that which better serves our purposes.

I want to talk about two changes in mathematics—one of each variety. That is, one change that has radically and obviously changed the mathematical landscape and one in which the observed changes have not been that conspicuous—but yet has changed the course of mathematics.

The first change is the revolution brought about by the advent of calculators and computers—which have irrevocably and dramatically changed the world of mathematical computations. The second, and not unrelated change, concerns our understanding of what mathematics is about and how one learns it.

What a marvel the calculator is. The thought of having all that computational power at one's fingertips would have been incredible 40 years ago. I wonder what it would have been like to show up in my high school physics class with a wrist calculator instead of a slide rule. Today, it wouldn't turn a head—we've gotten quite blasé about calculators. Having one is no big deal—we expect to see them (except perhaps in school) wherever there are people who have a modest number of calculations to make. Of course if you have lots of computations to make, you have a computer, and if you have lots and lots of computations to make, like the U.S. government, you have a supercomputer.

I read recently where the Cray supercomputer can do 4,000,000 "long" multiplications in a second. In an attempt to get some understanding of that statement, I decided to estimate how long it would take me to do that with paper and pencil. I don't know what is meant by a "long multiplication," but I timed myself and found out it takes me about 75 seconds to multiply a 5-digit number by a 4-digit number with reasonable care. At that rate, it would take me 300,000,000 million seconds to do 4,000,000 multiplications. That's 83,333 hours or 3,472 days or 9.5 years. If I limited myself to 8-hour workdays, 50 weeks a year, it would require about 42 years. That's a lifetime's work and a Cray supercomputer does it in 1 second.

The computer revolution doesn't just affect arithmetic. You can do all the algebraic manipulations in a second year high school algebra class and most all those in a beginning calculus class with a microcomputer and MuMath or some similar program that manipulates algebraic symbols. As a matter of fact, a computer has the capability of being programmed to do any procedure involving a finite number of symbols and a finite sequence of steps.

And what's more it's getting less and less expensive to take advantage of all this computational power. For a couple of hundred dollars one can buy a computer with more power than any computer of vacuum tube vintage. And the price of calculators has decreased even more dramatically than the population of the Tri-City area has increased. It took 50 years for that population to increase 15-fold from 5 to 75 thousand. The price of calculators decreased 15-fold in about 5 years. In 1978, I think it was, I paid $75 for a four-function calculator, today a calculator with similar functions and of superior quality costs about five dollars.

While the dramatic changes wrought by electronic computing devices are evident to all, there is another change that has taken place in the mathematical world over a longer period of time and, while of far-reaching consequence, is not nearly as obvious as the calculator and computer revolution. What I have in mind is the change in our understanding of the nature of mathematics and what's involved in becoming proficient in mathematics.

This change, as I see it, has its origins in a number of different developments that have taken place in the last 60 years. The most significant, from a mathematical standpoint, being the work of the German logician, Kurt Goedel. In 1931, he published the first of his results which shook the very foundations of mathematics. What Goedel proved in his so-called "incompleteness theorem" is that there does not exist an axiomatic system for arithmetic which is complete—that means there are arithmetical propositions which can't be proved and whose negations can't be proved either. And what's more, if one attempts to extend the system by adding more axioms so that more theorems can be proved, the system will become inconsistent, before it becomes complete. Basically, what this says is that you can't prove everything you think you ought to be able to. For example, consider the two statements: it is the case that such and such, and, it is not the case that such and such. It seems one ought to be able to prove one or the other. But Goedel showed that one can't always do this and attempting to add more axioms in order to do so will lead to a contradiction. In other words, in axiomatic mathematics, like everything else, one has either open questions or contradictions—there will always exist plausible mathematical statements for which no logical proof, as we know it, exists.

The consequences of this were devastating to early twentieth-century mathematicians like Hilbert and his colleagues. They had laid out a grand plan to reduce mathematics to the manipulation of symbols. What they intended to do was develop a set of symbolic axioms and a set of rules for operating with them so that in a finite number of steps one could determine whether or not any mathematical statement was a theorem. They would construct a ratiocinator—a mathematical reasoning machine. This machine could be used to learn all about everything. All one would have to do to determine a body of knowledge would be to mathematicize it by expressing its basic tenets as a set of mathematical axioms and then proceed to use the rules for operating the machine to deduce all the theorems that existed. Thus mathematics would become the fount of all knowledge, and deductive reasoning according to a prescribed logic would be the source whereby knowledge was obtained.

One consequence of this approach was that once an axiomatic system was established, the meaning of the symbols involved was unimportant. According to E.T. Ball in *Mathematics, Queen and Servant of Science*, Hilbert proposed that mathematicians forget about the "meanings" of their symbols and concentrate on the manipulation of these symbols according to explicit rules, as if one were moving chessmen. But Goedel had upset the apple cart. This approach left holes in our knowledge, there would always be an element of uncertainty involved.

The upheaval wrought in the mathematical world by this discovery was similar to that the physical world had experienced a quarter of a century earlier when relativity theory eradicated the notions of absolute space, absolute time, and the immutability of mass, while the Heisenberg Uncertainty Principle—the impossibility of determining both the position and momentum of a particle at a given instant—converted the laws of subatomic physics into statements of probabilities rather than certainties.

Thus, it might be expected in the mathematical world that one would have to discard the notion that there were absolute laws of logic and fixed axiomatic systems from which one could tell with certainty whether or not a mathematical statement was a theorem.

Meanwhile, on another front, biologists, neurologists, and others were delving into the mysteries of the mind. This reached a dramatic point in the 1960s in the brain-splitting surgery of Joseph Bogen and Roger Sperry, and their subsequent study of the effects of this surgery on those who underwent it. This thrust to the forefront the whole matter of right- and left-brain learning and again called to our attention what Goedel had already discovered in an entirely different way and what the poets and the painters had always known, namely that logical, deductive, symbolic thinking had its limitations—left-brained, rational, deductive processes aren't the only road to knowledge. They may not even be the best.

With this discovery, the eminence of deductive thought and logical inference has been called into question. Is such abstract thinking really the highest form of thought? Does one really understand a mathematical system if one knows how to properly manipulate its symbols—or is there more to understanding than that? What is meaningful knowledge and how is it obtained? How does learning take place? Thus, there is a growing interest in intuition, visual reasoning, and other modes of knowing as legitimate channels of knowledge and vital components of the learning processes.

Thus we've had two changes taking place. One in the world of objects, the other in the world of ideas. One sudden and revolutionary—the other gradual and evolutionary. One technological—the result of applied science; the other epistemological—concerned with the nature and ground of knowledge. Both of them have radically affected the mathematical world.

And while all these changes have been taking place, where have the educators been? I think I know where some of them have been. Walking along the bottom of the irrigation canal—oblivious to what's happening elsewhere—staying out of sight—unaware that the weather's changed.

How else do you explain this? Here's a museum piece: a page from *Ray's Arithmetical Key* (fig.1), containing solutions to the questions in *Practical*

Arithmetic written by Joseph Ray, M.D., professor of mathematics in Woodward College, and "entered according to the Act of Congress in the year 1845, in the clerk's office of the District Court of the United States, for the District of Ohio." I don't have a copy of Ray's *Arithmetic*, so I don't know how he explained the multiplication process to his readers. I suspect it was similar to the Rule for Multiplication found in E. E. White's *Intermediate Arithmetic* (fig.2), copyright 1876, which consists of 157 such articles with definitions and rules, and accompanying exercises in which one practices the rules.

Here is another example (fig.3). Only this is not a museum piece—it's from the 7th grade book in the new Heath series, copyrighted 1982. It's as if nothing has happened in the computational world for the last 100 years—yet we're in the midst of the most radical revolution in computational tools the world has seen since the invention of paper.

And so I ask myself, "Where are all the people who contribute to the existence of books like this?" I decided they must be walking along the bottom of irrigation canals—oblivious to what's happening in the world around them; not seeing very far to their right or left—victims of canal vision. Canal vision is very much like tunnel vision, except with tunnel vision, there's a light at the end of the tunnel. With canal vision, there's just more of the same.

Now 100 years ago, there might have been a reason for such books. There weren't many calculating devices around, other than paper and pencil, and operating paper and pencil required more effort on the part of the operator than today's computing machines. So the rules were more extensive—but note they were of the same nature as a set of rules for operating a calculator—you can understand the rules and use them without knowing anything but the multiplication tables, which one can learn by rote memory.

But what's the point of teaching those rules today? Our teaching the multiplication rules of 1876 is like drivers' education teaching students how to drive the conveyances of a hundred years ago!

I think one of the problems is that these rules have become identified with basic skills in mathematics—as well they might have been 100 years ago. But that's no longer the case, if it ever were. Something that one does not need to know to use mathematics effectively or be a successful mathematician cannot qualify as a basic skill in mathematics. You may need to know paper-and-pencil algorithms for multiplying 3-digit numbers by 3-digit numbers to get out of the seventh grade—but that doesn't mean it's a basic mathematical skill—that only means it's a school survival skill.

This raises one of the issues in mathematics education we have to face today, namely "What are the basic mathematical skills?" The difficulty seems to me that—like the rules for multiplication and the multiplication table—too many people have learned to answer this question by rote: The basic mathematical skills are knowing how, with paper and pencil, to add, subtract, multiply and divide whole numbers, signed numbers, decimals, and fractions. They have never thought seriously about their answer. I would like to suggest that we interrupt the discussion on what are the basic mathematical skills—which tends to result in the listing of topics—and focus first on the question of "What is a basic mathematical skill?" Is it something one needs to know to get out of school? To satisfy the parents of our students? Or is it something one needs to know to study advanced mathematics? Or to understand the mathematics in the world around us? Or what? I leave the question to you—I don't have the answer. Perhaps you can come up with one during your odd moments this weekend.

There's another aspect of the rules for multiplication that touches the other change we talked out. The rules are the grade school version of Hilbert's ratiocinator—the vision of mathematics as machine. To do mathematics, all one needs to know is the rules for operating the machine, not how the machine is put together. Putting the machine together, would be the task of Hilbert and other mathematical logicians. They play the role of the electronic engineers who design our calculators or the automotive engineers who design our cars. We don't have to know anything about the innards to operate them.

The Hilbertian point of view still dominates school mathematics—mathematics is a machine that manipulates symbols, so to learn to do mathematics one learns how to manipulate the symbols in machine-like fashion without regard to meaning. Let me illustrate what I am talking about.

There's a bright young graphic artist in our office who gets involved in our mathematical discussions. He took all the precalculus mathematics in high school and had a problem-solving course in college. Ted Nelson and I are preparing for a problem-solving course we're teaching this summer and have been collecting problems for it. I gave Jon, the artist, one of these problems to see how he would solve it. In one of his attempts, which would have led to a solution, he wrote down some relationships and arrived at a quadratic equation. "Oh," he said, "I've got to solve that. I've forgotten how to do that," and abandoned his attempt. The next day, I asked him if he was ready for a problem. He said, "Sure." "I'm thinking of two numbers," I said. "One is 8 more than the other and their product is 468. What are the numbers?" He grabbed a calculator and in a few seconds, he reported the numbers were 18 and 26. "Do you remember that quadratic equation you had yesterday?" I asked. "Yes." "Well," I said, "you just solved it." "Oh yeah," he said, with wonderment. I asked Jon how he arrived at a solution so quickly—he said he took the square root of 468, rounded it off and took four more and four less and it worked. Not a bad method, related in some ways to the completing the square procedure—and would serve as a good lead in to a discussion of that procedure.

Now you see, Jon's view of solving equations was the mathematics-as-machine view. Forget about meaning and operate on the symbols. The problem is he had forgotten how to operate the machine—as one is likely to do if you don't use it frequently—and he was stuck. On the other hand, once Jon was reminded of the meaning of the symbols, he had no difficulty in solving the problem.

If we want to develop creative problem-solvers and effective users of mathematics, I believe we need to break the rule-oriented, "this is how you do it" approach to mathematics.

Unfortunately, much of school math is cast in the machine mold. Its emphasis is training students to manipulate symbols mechanically, devoid of meaning. This mechanistic view of mathematics is the vestige of the Hilbert program—which we saw earlier was inadequate for describing mathematics, and I think is also inadequate as a basis for the teaching and learning of mathematics.

What ought to replace this mechanistic approach to mathematics? Many today would say problem solving. I have no quarrel with that, but I am fearful that canal vision may turn it into the same old thing. What I see happening in many approaches to problem solving is simply to add more rules. In addition to rules for multiplication and rules for solving quadratics, there will be rules for solving problems, patterned along the following: Here's a list of ten strategies and here are the rules for when and how to use them and, for each strategy, here is a list of problems upon which to practice that strategy. If that happens, we're no better off than before.

If we want to develop creative problem-solvers and effective users of mathematics, I believe we need to break the rule-oriented, "this is how you do it" approach to mathematics. Instead we should strive for the development of mathematical insight and intuition, which are the greatest assets in solving problems.

How does one develop these rather nebulous qualities? (I'm not even sure I can define them.) I have an idea about that. I suggest we look to the areas of architectural design, creative writing, and other fine arts for help. Now I don't know a lot about teaching in these areas. I do know, for example, that Robert McKim teaches a visual thinking course in the design division at Stanford University and his book *Experiences in Visual Thinking* was one of the seminal books for the Math and the Mind's Eye project in which I'm currently involved. I think visual thinking is an extremely fruitful avenue for the development of mathematical intuition and insight. I also think that critiquing and discussion of one another's work, such as happens, say, in a creative writing class, is a powerful and effective learning process that's almost nonexistent in mathematics and very prominent in other creative arts.

The purpose of such discussion and critique is not to judge the worth of one another's work but to hear how others thought about a problem and the strategies they used in tackling it, to find out and identify what worked well and what didn't, to see a diversity of approaches to the same problem, and to communicate one's insights to others.

I don't want to leave the impression in what I have said this evening that canal vision is universal among math educators. There are many for whom it was never a problem. Some that come to mind are J.W.A. Young, professor of the pedagogy of mathematics at the University of Chicago, who 80 years ago fought the machine view of mathematics, urging a laboratory approach to teaching math and arguing that the road to the abstract led first through the concrete; L.P. Benezet, superintendent of schools in Manchester, New Hampshire, who 50 years ago abandoned all formal instruction in math in the first 6 grades in favor of reading, reasoning, and reciting (reasoning and reciting meant solving problems and describing orally and in writing how one went about it) and found that the students "by avoiding the early drill on combinations, tables, and that sort of thing, had been able, in one year, to attain the level of accomplishment which the traditionally taught children had reached after three-and-one-half years of arithmetic drill"; and another was the late Bob Wirtz, the beanstick man, who was a pioneer in the current problem-solving movement. There are many others, but these are three who with lots of others, for the most part, classroom teachers, helped me overcome canal vision.

It took me a long time to get rid of my canal vision. I had a pretty narrow view what mathematics was about and how you taught it. It was a mechanistic view. At first, when I was a beginning college student, mathematics for me was a collection of procedures, adding fractions, factoring, solving equations, solving triangles, finding derivatives, and the like, and learning math was developing a skill at these procedures. During my upper division graduate years, the notion of "proof" was added to the list of procedures, and so I became versed in theoretical mathematics and math became proving theorems by deriving results by logical deductions from a set of axioms; the form of the argument, and not the content, was the utmost concern. The learning of mathematics for me was extended to include the construction, or largely reconstruction, and in many cases, memorization of proofs. High school geometry had prepared the way for this.

And then in graduate school two things happened. For one thing, I learned about Goedel's result and, for another, about the existence of computers. The first deeply affected me—it bothered me because one of the things that was appealing to me about mathematics was the certainty that surrounded it in my mind—and Goedel's results said that didn't exist in my version of math—there were open questions that would always remain. But it was also relieving to find that Hilbert's ratiocinator didn't exist and all knowledge couldn't be captured in a set of axioms and wrung out from them by a procedure called proof.

The other event I ignored. I spent the 1951–52 academic year as a graduate student at Princeton. One of the faculty members arranged for any graduate students in math who wished, to go with him to the Institute for Advanced Studies, which was a few miles away, to learn about their computer—it was one of the first few in existence and I suspect filled rooms with vacuum tubes. But I didn't bother to go find out. By the way, one fellow gradute student who did was Ralph Gomory who became director of research for IBM.

My next brush with computers came about 10 years later when I was chairman of the math department at Pacific Lutheran University. The head of the business department came to me and said we ought to get together in a joint effort to bring computing to the school. I wanted nothing to do with it and successfully fended him off. I was afraid to get out of the canal—I didn't know anything about computers, didn't want to learn anything, and I certainly didn't want them disrupting the math program.

It was about 10 years after that, when I was at the University of Oregon and computing became part of the math department, that I did do all the lab assignments in a BASIC course, but computers were still too expensive and inaccessible, I thought, to affect the average math classroom. Then came the handheld calculator and slowly I began to realize that here was the new slide rule—and, more than that, as they became cheaper and more widespread, the new computational tool of our society—it would replace paper and pencil as the most common household computational tool. It was this peek out of the canal that ultimately led me to realize that paper and pencil were not sacred. They were being replaced by better computational tools and there was little point in training paper-and-pencil multiplication experts in the age of Cray supercomputers—especially when it contributed little to basic mathematical understanding.

However, the only reason I was able to reach that conclusion was that my canal view of the teaching and learning of mathematics was finally changing. For a number of years, the teaching of mathematics for me was the teaching I had experienced. (Teaching how you have been taught is a good way of spreading canal vision.) In a beginning course one concentrated on procedures, introducing a few proofs that these procedures did what you wanted; in a more advanced course one concentrated on the theoretical side, mostly proofs with less attention given to procedures.

The changing point came for me when I began teaching mathematics to prospective elementary teachers at the freshman level. Normally at the freshman level, one would emphasize procedures—how to do things—but it seemed clear that an elementary teacher ought to know more than how to do it. They ought to understand the inner workings of mathematics. The typical way to provide that understanding was to show that everything one did was justified by deduction from a set of basic properties—so I tried that, I presented arithmetic as an axiomatic system. But that didn't work. In the first place, elementary teachers weren't math majors and most of them weren't interested in axiomatic methods; secondly, many of them found no real meaning in this approach—they used to think they knew what a number was and now they weren't sure—and it wasn't doing anything to minimize anxiety and avoidance. So taking a clue from the better elementary teachers I had watched during my visits while the Oregon System project was going on—I began to use objects, manipulatives they were called, to explain procedures and develop intuitions and insights that were at the root of the math that was being presented. It worked much better—students at least didn't hate the course and some, who had never understood before, were beginning to see why things worked as they did and life was much better all the way around—so I began to use these methods in other classes.

Meantime, other things happened: I wandered in to a section meeting at an NCTM regional meeting in which a psychologist was talking on what every elementary teacher ought to know about the brain and I became interested in right and left brain thinking; I became acquainted with gestalt psychology; I read about visual thinking and intuition; I got some glimpses into Native American culture and other ways of knowing. In short, I climbed out of the canal.

And it's been stimulating. Of course, outside the canal the wind blows every which way. Sometimes it's refreshing and sometimes one doesn't know which way to turn to get out of its blast. But it's exciting, challenging, and fun. After all, an irrigation canal is nothing but a humongous rut.

MULTIPLICATION OF SIMPLE NUMBERS. 17

(43)	(44)	(45)
2187	3489	1646
215	276	365
10935	20934	8230
2187	24423	9876
4374	6978	4938
Ans. 470205	*Ans.* 962964	*Ans.* 600790

(46)	(47)	(48)
8432	6874	2873
635	829	1823
42160	61866	8619
25296	13748	5746
50592	54992	22984
		2873
Ans. 5354320	*Ans.* 5698546	*Ans.* 5237479

(49)	(50)	(51)
4786	87603	83457
3497	9865	6835
33502	438015	417285
43074	525618	250371
19144	700824	667656
14358	788427	500742
Ans. 16736642	*Ans.* 864203595	*Ans.* 570428595

(52)	(53)	(54)
31624	126	823
7138	6	12
252992	*Ans.* 756 dollars.	*Ans.* 9876 dollars.
94872		
31624		
221368		
Ans. 225732112		

(55)	(56)	(57)
675	496	152
13	24	28
2025	1984	1216
675	992	304
Ans. 8775 cents.	*Ans.* 11904 cents.	*Ans.* 4256 miles.

Key.—2

Fig.1. A page from *Ray's Arithmetical Key*, published in 1845.

60 INTERMEDIATE ARITHMETIC.

The ***Multiplicand*** is the number taken or multiplied.

The ***Multiplier*** is the number denoting how many times the multiplicand is taken.

The ***Product*** is the number obtained by multiplying.

The multiplicand and multiplier are called the ***Factors*** of the product.

Art. 34. The *Sign of Multiplication* is ×, and is read *multiplied by*. When placed between two numbers, it shows that the number before it is to be multiplied by the number after it. Thus: 6 × 3 is read 6 *multiplied by* 3.

NOTE.—Since a change in the order of the factors does not change the product, 6 × 3 may also be read 6 *times* 3.

Art. 35. Multiplication is a short method of addition. The sum of 5 + 5 + 5 + 5 is the same as 4 times 5.

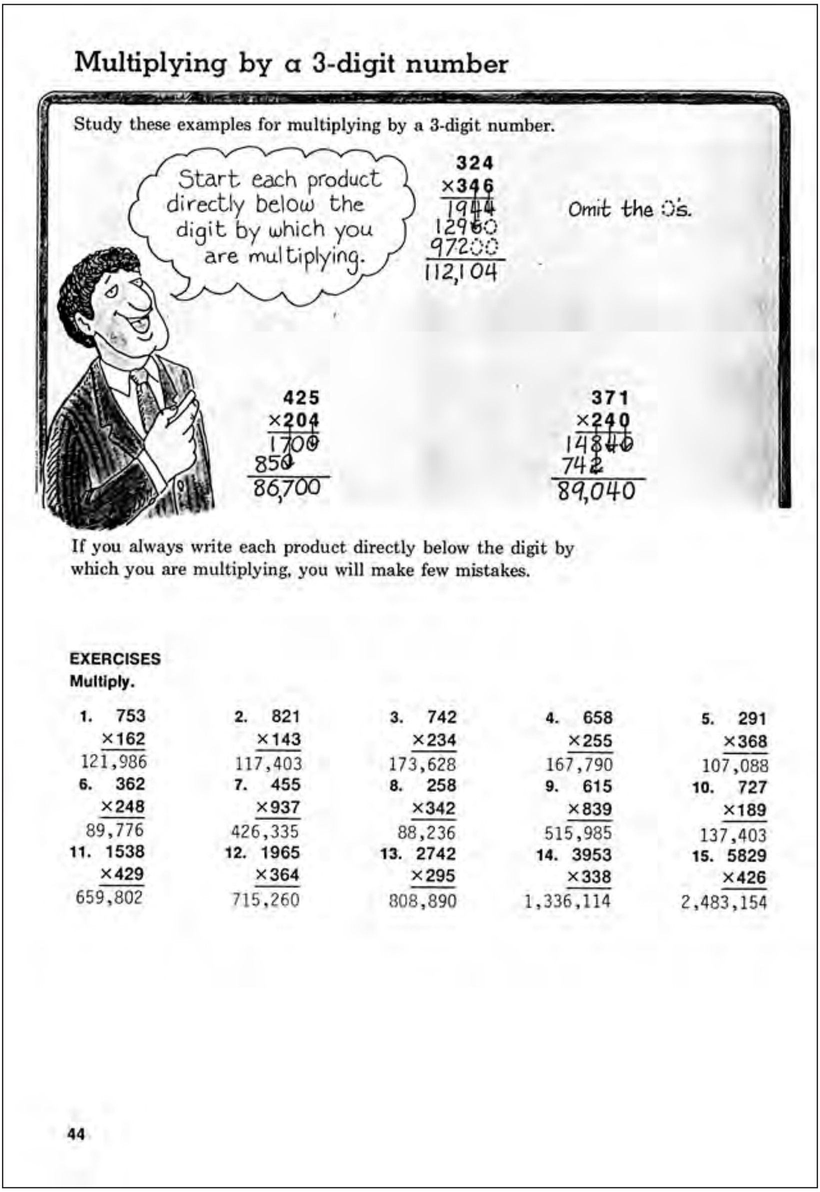
Art. 36. RULE FOR MULTIPLICATION.—1. *Write the multiplier under the multiplicand, placing units under units, tens under tens, etc.*

2. *When the multiplier consists of but one term, begin at the right and multiply successively each term of the multiplicand, writing the right-hand term of each result in the product and adding the left-hand term to the next result.*

3. *When the multiplier consists of more than one term, multiply the multiplicand successively by each significant term of the multiplier, writing the first term of each partial product under the term of the multiplier which produces it.*

4. *Add the partial products thus obtained, and the sum will be the true product.*

Fig.2. A page from E. E. White's *Intermediate Arithmetic*, published in 1876.



Multiplying by a 3-digit number

Study these examples for multiplying by a 3-digit number.

If you always write each product directly below the digit by which you are multiplying, you will make few mistakes.

EXERCISES

Multiply.

1. 753 ×162 = 121,986	2. 821 ×143 = 117,403	3. 742 ×234 = 173,628	4. 658 ×255 = 167,790	5. 291 ×368 = 107,088
6. 362 ×248 = 89,776	7. 455 ×937 = 426,335	8. 258 ×342 = 88,236	9. 615 ×839 = 515,985	10. 727 ×189 = 137,403
11. 1538 ×429 = 659,802	12. 1965 ×364 = 715,260	13. 2742 ×295 = 808,890	14. 3953 ×338 = 1,336,114	15. 5829 ×426 = 2,483,154

44

Fig.3. A page from Heath's seventh-grade text, published in 1982.

Paper-and-Pencil Skills "Impede" Math Progress

A picture which appeared in Education Week *prompted me to write a letter to the editor which was printed in the April 5, 1987, issue under the title "Sophisticated Technology for 'Antiquated Process.'" The letter follows.*

To the Editor:

The picture on page three of your March 18 edition, "Miss Brooks and Mr. Kotter, Move Aside," is an unwitting and ironic commentary on the unhappy state of the mathematics curriculum. The picture shows a teacher finding the product of 3,719 and 405 on the chalkboard while being televised for students to watch on "Homework Hot Line," a call-in television program. Everywhere in America, except in school, this answer would be found with a calculator or some other electronic computing device.

The irony is in the use of sophisticated technological equipment to produce a television show to teach an antiquated process that is better done with an inexpensive technological tool—that doesn't require a homework hot line to teach its use. The extravagance of such a television production is hard to justify. It also illustrates education's difficulty in dealing with technology—how we educators adapt technology to teach an outdated curriculum rather than adapting the curriculum to fit the existing technology.

As far as mathematics is concerned, a large part of the problem is the improper identification of "basic mathematical skills." A basic mathematical skill is a skill that is necessary to function mathematically, especially in the non-school parts of the world. Paper-and-pencil processes for doing multidigit arithmetic do not fit the criterion. One does not need to know them for computational purposes: as a matter of fact, compared with calculators, paper and pencil are cumbersome and inefficient.

Also, knowing these processes adds nothing of consequence to one's conceptual knowledge or mathematical insight. One can know how to multiply 3,719 by 405 on the chalkboard and not know anything about the concept of multiplication: its interpretation as repeated addition, the number of objects in an array or the area of a region, or circumstances in which it is an applicable operation.

It is true that one needs to know paper-and-pencil algorithms to perform satisfactorily in most school-mathematics programs. But that doesn't make them basic mathematical skills. That only means they are school survival skills.

There are educators who are struggling to provide a meaningful and appropriate mathematics curriculum that accepts and uses the computational tools of our day, and recognizes the distinction between basic conceptual knowledge and skills and algorithmic processes of the chalkboard.

I support their efforts and encourage your readers to join them. We cannot afford to waste our educational resources teaching a mathematics curriculum that stresses paper-and-pencil computational procedures. There is no individual or societal need, and absolutely no market, for long-division expertise.

Shortly after the letter appeared, Education Week*'s Commentary Editor invited me to write a article expanding on the views expressed in the letter. The result was "Paper-and-Pencil Skills 'Impede' Math Progress," which appeared as the back-page commentary in the April 15, 1987, issue of* Education Week.

"Breakthrough Made in Superconduction" declares the headline on the front page of the morning paper. The story reports the development of a new ceramic material that can carry electrical current with minute resistance, at temperatures warm enough to be of practical use. One possible outcome, the story reports, is that desk-sized supercomputers could "come down to the size of a football, and probably operate 10 times faster."

Statements like that overwhelm me. I struggle to grasp their meaning. Several years ago, when supercomputers were arriving on the scene, I read that one of them could do 4 million "long" multiplications in a second. In an attempt to give some relevance to that statement, I decided to figure out how long it would take me to do that many multiplications, using the old-fashioned paper-and-pencil method I learned in school. I'm sure, for the computer, a "long" multiplication was more than multiplying a five-digit number by a four-digit number, but that was "long" by my standards. I timed myself and found I could do such a multiplication, with reasonable care, in about 75 seconds. At that rate, it would take me 300 million seconds to do 4 million multiplications. That's 83,333 hours. If I limited myself to an eight-hour workday, five days a week, 50 weeks a year, it would take me about 42 years. That's a lifetime's work, and a supercomputer does it in one second! Now I'm told, the next generation of computers will do 10 lifetimes' work in a second.

So what's my reaction? It ranges from amazement and celebration to peevishness. The advances in computational tools I have used in my lifetime astound me—from the paper and pencil of school arithmetic, the slide rule of high-school physics, and the hand-cranked calculating machine of college statistics, to those marvelous tools, the solar-powered calculator and personal computer—and there's more to come. In my wildest schoolboy imagination, I would never have dreamed of possessing such powerful and tractable computing devices. But I do—and when I think of all the time and effort they save me and the possibilities they afford me, I revel in their use.

So what peeves me? Schoolchildren aren't allowed to join in the celebration. Despite all these magnificent advancements that have brought untold computational power to one's fingertips, school is "more like it's been than it's ever been before." As in the days of my youth, and my grandparents' youth, schoolchildren are drilled for hours to perform paper-and-pencil computations in machine-like fashion. There are no breakthroughs here. There are no superconductors that have managed to overcome the resistance that will allow any electricity to permeate the mathematics classroom.

Instead, there are staunch defenders of the status quo, those that insist on turning students into paper-and-pencil computing machines. For what purpose? It's ridiculous to expect there is a future for a paper-and-pencil machine that takes a lifetime to do what another machine can do in a moment. It's no longer a matter of economics when a hand-held calculator costs little more than a good mechanical pencil. And solar power has stilled those ominous voices that warned us about batteries going dead. So why insist on centering a mathematics curriculum on teaching students how to use outmoded computational tools?

The ostensible reason proffered by those resisting change is that students must learn the "basic mathematical skills." And, in order to do that, the use of electronic tools must be curtailed or even banned. As the author of one current textbook series explained in a national advertisement, "calculators should not be permitted in elementary schools, for this is the time and place for learning fundamental concepts and mastering paper-and-pencil skills. If students are permitted to use calculators too early, many of them will block and refuse to do the drudgery necessary to perfect the necessary paper-and-pencil skills."

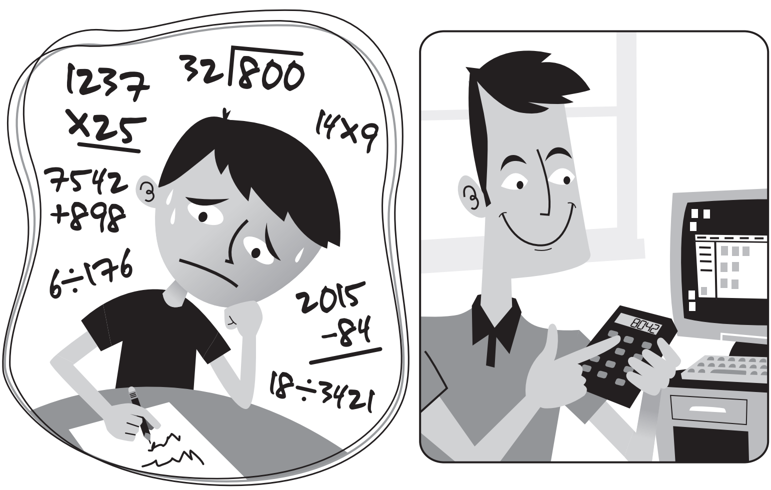
Such cries for skill drudgery come close to skullduggery. In the first place, paper-and-pencil skills are not "necessary." Secondly, anything that can remove drudgery from school mathematics programs ought to be extolled instead of condemned.

Designating a mathematical skill as "necessary" implies that it is needed to function mathematically. That is not the case with paper-and-pencil arithmetical skills; one can function mathematically quite well without them. You may object, pointing out that one can't get through school mathematics without them. That is very likely true, but that doesn't mean they are basic mathematical skills. That only means they are school survival skills.

To determine whether or not a mathematical skill is necessary, one ought to examine its essentialness in the nonschool parts of the world. Over the half-century I have been doing mathematics—as a schoolboy, as a college and graduate student, in any number of odd jobs that paid my way through school, as an industrial mathematician, as a university teacher and researcher, in everyday life, and just for fun—there is nothing I have done, apart from schoolwork, that today requires the use of paper-and-pencil arithmetical procedures. Calculators provide an economical and efficient way of doing computations I can't do in my head. And knowledge of these paper-and-pencil procedures does not provide me with mathematical insight of any significance.

There are times when I find these paper-and-pencil computational skills useful—although I can't remember the last time I used them for long division other than at school. Also, there are those who prefer these methods of computation. But even if these procedures are occasionally useful, or preferred by some, it does not follow that they are necessary skills. In my mathematical life, I can get along without them. Most adults do.

This doesn't only apply to the paper-and-pencil procedures of elementary-school arithmetic. It also applies to the paper-and-pencil procedures of high-school algebra, college calculus, and all other math courses. These days, any step-by-step procedure involving the manipulation of mathematical symbols, according to a fixed set of rules, can be done by a calculator or computer. Some procedures are simple enough that they are best done mentally or by hand, but any that require more than a modicum of time and energy to do manually are most economically done by machine. And, outside school, they are.



Thus, school math programs that center on the mastery of mechanical paper-and-pencil procedures are not necessary skills. They are vestiges of another age, when human beings, in conjunction with paper-and-pencil, were the computing machines of the day. To gear a math program to producing such machines does indeed reduce students to drudges.

I suspect the resistance to calculators in classrooms is not a tenacity for teaching basic skills, but rather an anxiety about what to do if existing programs are abandoned. I suspect many educators share the feelings of that fifth-grade teacher whose immediate response to the suggestion that he allow calculators to be freely used in his classroom was, "But that would destroy my whole program!" It would. However, once one sees the truth of that statement, lets the initial shock wear off, and asks what ought to happen next, one can envision a mathematics program that recognizes current technology, is economically feasible, and provides pertinent mathematics for purposeful students, without drudgery.

Such a program does not require that classrooms be equipped with the latest in electronic computing devices. Rather, it requires that the existence of these devices be recognized, and time and energy not be wasted teaching students paper-and-pencil procedures that, except in school, are done electronically. For most school computational purposes, inexpensive calculators will do. And, since calculators are easy to use, math programs need not devote much attention to computational skills.

Thus, computation plays a minor role in a pertinent math program. Such a program will emphasize meaning rather than symbolic manipulation. It will educe the mathematical creativity innate in every student; it will develop mathematical insight and intuition; it will stress cooperative problem-solving; and it will allow students to compute by whatever means they can—mentally, counting on their fingers, with an abacus, using pencil-and-paper, or punching the keys of a calculator. As students grow in mathematical maturity, they will find the computational methods that work best for them.

I recognize that there is a vast difference between listing characteristics of a math program that is appropriate for the electronic age and implementing such a program in the schools. Doing the latter is as exciting and challenging as searching for superconductors.

For the past several years, I have been involved with a coalition of school and college math educators who are working to instill the above characteristics into portions of the school mathematics program. The vehicle we have chosen is visual thinking—the use of sensory perception, models, sketches, and imagery to provide insight into mathematical concepts and bring meaning to mathematical symbolism. It's gratifying to watch general-mathematics students—who have become accustomed to perfunctory paper-and-pencil drill carried out with little meaning, mediocre success, and no interest—make contact with their mathematical instincts and come alive mathematically. It's rewarding to see math-anxious elementary teachers overcoming their doubts about ever understanding mathematics, or tackling an open-ended mathematical question that they cannot solve mechanically. And it's encouraging to know they no longer will pass on an apprehensive and distorted view of mathematics to their students.

There are other people, scattered throughout the country, engaged in similar activities. These are the people who celebrate calculators and computers for the computational power they bring to all students. They are in contact with their own mathematical spirit, ignite the mathematical spark in others, and know the essentials needed to nourish it. These are the people who, despite my peevishness, give me hope. It is their energy that can overcome the resistance of those who impede mathematical power with the drudgery of mechanical paper-and-pencil drill. It is their energy, conducted into classrooms that can electrify the mathematical potential inherent in every student.

Closing Address, Tanzania Math-Science Seminar

During each of the four summers 1993 through 1996 I was part of the faculty of a two-week math/science institute held in northern Tanzania in East Africa. Participants in the institute were teachers from secondary schools in the environs of Arusha and Moshi operated by the Evangelical Lutheran Church of Tanzania (ELCT). Because of budgetary constraints, only a handful of Tanzanian secondary schools are operated by the government. Most are operated by nongovernmental organizations, such as the ELCT, which charge modest fees, follow the government curriculum and confer diplomas based on student performance on government administerd exams.

In accordance with Tanzanian custom, the beginning and end of the seminar were marked by formal opening and closing ceremonies, including an address by a guest of honor. I was invited to be guest of honor at the closing ceremony of the last seminar I participated in July 1996. Following is my address to the seminar participants on that occasion.

I include this address because it relates to experiences of great importance to me, both professionally and personally. My African experience along with my experiences working with teachers of students from diverse cultural and ethnic backgrounds—native communities in Alaska, Indian reservations in the Pacific Northwest and Southwest, migrant workers in Oregon—give strong evidence of the existence of an innate mathematical spirit in all human beings. The address also alludes to the strong sense of vocation I have in my work as a mathematics educator.

Dear Mr. Nyiti, Mama Mary and fellow staff workers, guests, Mama Muta, chairperson of the seminar, and seminar participants.

It is a great honor for me that you have asked me to be the guest of honor at these closing ceremonies. It is but one of a large number of kindnesses you have shown me.

In my remarks this morning, I want to share with you an image of education that has proven to be very useful to me in my many years as an educator.

The image is that of a child as she or he progresses to adulthood—as she or he learns to walk, to talk, to feed oneself, to explore the world—on their way to becoming a self-sufficient adult.

So, within each of our students is this innate scientist and mathematician and, in my way of thinking, our role as science and math teachers is to nurture this infant scientist and mathematician just as good parents nurture their infant child.

Within each of us, you see, are many infants as we come into this world. There is an infant mathematician and scientist, just as there is an infant poet, playwright, artist, musician, theologian, philosopher, athlete—the capacity for any human endeavor. For we are made in the image of God—and, as such, we reflect all aspects of God's nature. Some of us may have more inclination towards one aspect than another—but, there is no aspect we do not have. We are not all Olympic athletes, but we can all participate in and enjoy the before-dinner volleyball game.

So, within each of our students is this innate scientist and mathematician and, in my way of thinking, our role as science and math teachers is to nurture this infant scientist and mathematician just as good parents nurture their infant child—they feed and clothe it, they encourage its development, they model appropriate behavior. Thus, for example, as the child learns to walk, the parent is there beside it—helping it to steady itself, to move forward as it strengthens its legs and develops its sense of balance. But the parent cannot walk for the child—nor can it learn for the child; one does not leave the child sitting on the floor watching as one walks across the room saying, "This is how you walk—now you do it." Nor, as the child takes its first tottering steps does one knock the child down and say, "No, that's not the way to do it, walk straight like this." No, one is patient, and lets the child develop its own strength and skills until it, too, one day can walk briskly and ably on its own. For you see, the capacity to walk is born into the child—we do not give it to the child—we do not somehow take our knowledge of walking and magically instill it in the child. But we are with the child, nurturing it and encouraging it, as it learns to walk on its own.

That, it seems to me, is what education is about—it is the process of drawing out, of educing the natural capacities within the learner. The role of the math and science teacher is to educe, to draw out, to nurture and encourage the innate scientist and mathematician that is in each of our students. To walk side-by-side with the learner as she or he develops their scientific and mathematical legs.

The learner will never gain any real understanding if it is not allowed to construct its own learning. Real understanding is not something the teacher, no matter how talented, can pour into the learner's head.

How do we do that? Not by trying to cram our knowledge into the learner's mind—not by telling the learner, "No that's not the way to do it," as they take their first tottering steps. No, the educator is patient, the educator allows the learner to totter—for without those first hesitant steps one will never learn to walk. The educator allows the learner to express their ideas, to do things the way that is natural for them, while providing them with experiences that help them clarify their ideas and gain new mathematical understanding, and modeling for them how a scientific or mathematical situation might be approached. The child will never learn to walk if we don't let it use its own legs. The learner will never gain any real understanding if it is not allowed to construct its own learning. Real understanding is not something the teacher, no matter how talented, can pour into the learner's head.

For a number of years now, the seminar staff have walked beside you as you have been developing your own understanding of math and science and a different way of teaching it—so you, in turn, can walk beside your students guiding their development. For myself, I have been with you for four seminars and it is my belief, that while there may be some among you still tottering a bit, there are many of you who are walking on your own. You have gained new perspectives about teaching math and science, have incorporated these into your consciousness and are incorporating them into your teaching. And, indeed, ultimately you must walk on your own, for no one else can do it for you—nor is there a need for anyone to do that, for you are perfectly capable of walking on your own.

And as you walk on your own, I see a shift in the focus of our work together. I know there are new and exciting developments taking place in the science community, especially the construction of well-equipped teaching laboratories. As for the math community, I am especially pleased that a constitution has been drafted and other steps taken towards the formation of the Tanzanian Visual Maths Association to continue the work beginning at this seminar.

I am also pleased that the constitution of the association allows for all to be members, Tanzanians and foreigners alike, so that I too may be a member and continue to walk with you, not as a mentor, but as a fellow member of the association—and, though, it is unlikely that I will come to Tanzania next year, it is just as unlikely that I will never come back again—your spirit will be walking beside me as I continue my work in the U.S., just as I hope my spirit will walk beside you as you continue your work here.

I want to end on a personal note. It was some four and a half years ago that Mama Mary called me one evening and asked me if I wanted to come to Africa. Now before that call, going to Africa was the furthest thought from my mind—but as Mary went on to tell me about the seminar, I developed a very strong sense that God was indeed inviting me, through Mary, to go to Africa. After I put down the phone and told Mama Maier about the conversation—she said to me, in her insightful way, "I think you will be going there more than once." She was right. I have come four times, and Mama Maier has come here twice.

I remember our first trip. I live in the Pacific Northwest of the United States alongside the Pacific Ocean. East Africa is a very remote place. It is an area neither Mama Maier nor I knew very much about, and we made that first long trip not knowing what to expect, although Mama Mary assured us that everything would be wonderful. It was a dark and rainy night when we arrived at Kilimanjaro airport. We crossed the tarmac into the airport, worked our way through customs, and went out into the reception area, having no idea where we were heading or what was going to happen next—and there was Angelista Tarimo with beautiful flower necklaces for each of us, and a sparkling smile and warm, "Karibu." From that moment on, Mama Maier and I have felt perfectly at home with you.

East Africa is still a long way from my home, but it is not remote in spirit. I come here and I feel most welcomed. I find myself at peace and quiet when I am among you. You are most gracious hosts and hostesses. I am certain that when I say that, I speak for all of the seminar staff who have come here from the U.S.

God bless all of you, my dear friends, and peace be with you as you stride into the future.

How the Mind Deals with Math

"How the Mind Deals With Math" draws on a small, but significant, part of the literature on how our minds process mathematical information and engage in creative thought to provide suggestions for teaching in ways that develop mathematical insight and expertise.

This article is a transcript of a talk given in San Francisco on April 22, 1999, at the Annual Meeting of the National Council of Teachers of Mathematics and again in Portland, Oregon, on October 7, 1999, at the Northwest Mathematics Conference.

Most of what I say here comes from three books. The major source is *The Number Sense* (1997). The author, Stanislas Dehaene, is a neuropsychologist with a background in mathematics. He was a researcher at the Institute of Health and Medical Research in Paris at the time this book was written. *A Celebration of Neurons: An Educator's Guide to the Human Brain* (1995) was written by Robert Sylwester. Bob, a former colleague at the University of Oregon, writes and lectures on what's known about the brain and the implications for educators. *The Psychology of Invention in the Mathematical Field* (1944), by Jacques Hadamard, is the third source. Hadamard, a French mathematician, is best known for his part in proving the so-called prime number theorem. He fled France for the United States during the Nazi occupation, returning in 1944. His book is the outgrowth of a lengthy questionnaire sent to mathematicians asking them to describe their thinking. Originally published by Princeton University Press, his book is now available as a Dover reprint. Hadamard died in 1963 at the age of 97.

Everybody Counts, and Adds, and Subtracts

Show a five-month-old baby slides of two objects and measure how long they look at it (Dehaene). The time remains constant or diminishes as habituation sets in. Suddenly change the number of objects to three. The time the baby watches shows a marked increase. The increase depends on the number of objects, not on the type or location of the objects. The increase in attention appears to be dependent only on numerosity, sometimes referred to as cardinality. In a similar vein, show a baby an object behind a screen, and then another object behind a second screen. Remove the screen and measure the time the baby fixates on the collection of objects. The time increases if an object has either been taken away or added behind the screen before the screens are removed. The baby fixates longer if there are one or three objects shown rather than the expected two. Similarly, a baby is shown two objects, a screen is placed over one, and then the other is removed in sight of the baby. If the screen is then removed, the baby will fixate much longer if the removed screen reveals two objects rather than one. Dehaene concludes from these experiments and others that babies are born with an "innate, abstract competence for numbers." There is in the human being (and also in animals) an innate intuition for number. This exhibits itself in babies by their ability to distinguish numerosity between sets of objects and rudimentary knowledge of addition and subtraction.

As language develops, in Dehaene's words, children also show "precocious competence" in counting. By about three and one-half years, children know that in the counting process the names of numerals occur in a particular order, and that in counting a set of objects the order in which you point to them doesn't matter as long as each object is touched exactly once. At first the child apparently doesn't know that counting provides the answer to the question "How many?" If you watch a three-year-old count the number of

Easter eggs they have collected and you ask them, "How many eggs do you have?" they won't necessarily give you the number they reached. However, by four years of age they are aware that counting provides the answer to the question "How many?"

Children also develop their own means of adding and subtracting, mostly by counting on their fingers. They find shortcuts, like commutativity, and hone their strategies, taking into account time and reliability of the process. All this arithmetical activity develops without explicit instruction, apparently based on intuitive understanding of number and the meaning of number calculations.

There's No Black Box

How does our mind operate? What is going on in our brain that manifests itself in this intuitive knowledge of number? Brain imaging techniques reveal that many areas of the brain are involved in arithmetical tasks. To quote Dehaene, "Arithmetic is not…associated with a single calculation center. Each operation recruits an extended cerebral network. Unlike a computer, the brain does not have a specialized arithmetic processor.…Even an act as simple as multiplying two digits requires the collaboration of millions of neurons distributed in many brain areas" (p. 221). Dehaene concludes that in addition to not having a central processing unit—a black box that is a calculation center—there is another significant difference between the neural architecture of our brain and the modern computer. He maintains that the brain is not a digital device but is analogic in the manner in which it perceives quantities—like an analog rather than a digital computer—more like an hour glass than a digital watch. Furthermore, our perception is "fuzzy." We are not bad at approximation and judging differences that are large but our perception is not precise.

Studies show that human beings can recognize one, two, or three dots at a glance. The time required to identify the number of dots grows rapidly after two or three. We can improve on our approximations with practice, but no amount of practice will enable us to say at a glance, with accuracy, "there are 105 dots." For a digital device, finding the exact number would be as easy as finding an approximation.

The farther apart two numbers are, the quicker we are at determining which of the two is larger. For example, we are quicker at determining that 982 is larger than 126 than we are at determining that 272 is larger than 267. To a digital computer the size of the difference between two numbers doesn't matter. Another factor that distinguishes our brain from a computer is that we are always making associations and analogies. If I gave you the task of approximating the number of dots on a page, you couldn't help but notice that the dots were arranged in the shape of a banana. A computer scanning the banana shape counts 107 dots without awareness of the banana shape.

How Are You Feeling?

According to Dehaene and Sylwester, emotions and reason are tightly linked in our cerebral structure and emotions often get the upper hand. That's not a bad thing. When danger was felt, strong survival instincts led our ancestors, and leads us, to flee without cogitating. Better, as Bob Sylwester points out, "to flee unnecessarily many times than to delay once for a more detailed analysis of the threat and so die well informed." But it does lead to impulsive, and as we often say, irrational behavior. You can probably cite your own examples. I remember smelling smoke at the dinner table one evening. Looking for the source, our oldest child went downstairs to check his bedroom. He came running upstairs to announce the basement was full of smoke. I took one peek downstairs, called the fire department and hurried everyone out of the house. Only when we were standing in the driveway waiting for the fire truck did someone remember that Jon was still in the dining room, strapped in his high chair.

It is well accepted that emotional response can impede rational thought. Most of us who teach math are well aware of the effects of mathophobia. Dehaene is convinced that "children of equal initial abilities may become hopeless or excellent at math depending on their love or hatred of the subject." He maintains, "Passion breeds talent." Sylwester suggests one reason emotion is such a powerful force in our behavior is that far more neural fibers project from the limbic system where our emotions are centered, than into the cortex where logical, rational thought is centered. One might say the emotions have more impact on rational thought than rational thought has on emotions. Apparently sensory perception also makes its way to the emotional centers more rapidly than to the rational thought centers. Our emotions get a head start in our reactions to the external world.

We Are Sense-ible, and Reflective, and Creative

We perceive the external world through our senses. Receptors that receive and convert stimuli into neural codes are dense throughout our sense organs—250 in a patch of skin the size of a quarter. The eyes predominate, containing some 70 percent of our body's receptors. About 30 percent of our brain is devoted to visual information. We perceive from the external world around us and we reflect on the input. Not only do we reflect, but we create—we string words together to make sentences. We combine sentences into lectures, perhaps expressing thoughts we have never had before. Where do they come from? Where does the sudden insight come from? The forgotten name that comes to us out of the blue; the solution to the problem we were working on yesterday that hits us while we are taking our morning shower?

Dehaene begs the question with the comment that the "flash of invention is so brief that it can hardly be studied scientifically." Not only is it brief but it doesn't happen on demand. One can attach a bunch of electrodes to my neuronal fields and ask me to compare numbers, solve an equation, or estimate the number of dots on a screen. I can do that and the experimenter can measure what is happening in my brain. However, he can't ask me to have an "aha" or to remember a name I have forgotten or to suddenly see the solution of a problem I have been working on. Someday we may be able to—Dehaene holds the hope that there are physiological traces of neuronal activity below the "threshold of consciousness" that can be measured with brain imaging tools.

Even though for the present we are unable to measure and locate the brain activity that occurs during the process of insight and invention, we can reflect on our thinking and the circumstances of our own creative thought. This is what Hadamard fifty years ago asked his colleagues to do. He sent them a long questionnaire about their habits and work style, about the circumstances leading up to and surrounding moments of insights. In the last question, and almost as an afterthought, he asked them to describe their mechanism of thought.

Sifting over the replies and drawing on his own experiences, Hadamard identified four stages in the process of mathematical invention. He called these preparation, incubation, illumination, and verification. Briefly, preparation is conscious thinking about a problem. Incubation is letting the problem sit without conscious thought. Illumination is the moment when the lights go on, and verification is putting together the rational justification for the insight.

Hadamard based his conclusions on experiences such as the following, reported by Henri Poincarè. Poincarè had been searching for a set of functions that satisfied certain conditions. He continued his search for a fortnight when he interrupted his endeavors to go on an excursion with a group of people. He reported:

> *The incidents of the travel made me forget my mathematical work...we entered an omnibus to go some place or other. At the moment I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define...were identical with those of non-Euclidian geometry. I did not verify the idea; I should not have had time, as, upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return...I verified the result at my leisure.*
>
> *Then I turned my attention to the study of some arithmetical questions without much success.…Disgusted with my failure, I went to spend a few days at the seaside and thought of something else. One morning, walking on the bluff, the idea came to me, with just the same characteristics of brevity, suddenness, and immediate certainty.(pp. 13–14)*

On the matters of thought mechanisms, Hadamard decided that most mathematicians are visual thinkers and their thought process entailed images other than mathematical symbols. The one notable exception he mentions is George Birkhoff who said he was "accustomed to visualizing algebraic symbols and to work with them mentally."

The most celebrated response Hadamard received was from Albert Einstein. Einstein's reply is printed in its entirety in the appendix of Hadamard's book. Einstein describes his thought mechanism this way:

> *The words or the language as they are written or spoken, do not seem to play any role in my mechanism of thought. The physical entities which seem to serve as elements in thought are certain signs*

> *and more or less clear images which can be "voluntarily" reproduced and combined.... The above mentioned elements are, in my case, of visual and some of muscular type. Conventional words or signs have to be sought for laboriously only in a secondary stage. (pp. 142–143)*

The only reference Dehaene cites in his brief discussion of the flash of mathematical invention is Hadamard's book. Apparently not much new has been added in the last fifty years.

Remember When

Human beings are capable of storing vast amounts of information. We have particularly strong visual memories. We can spot a friendly face in a crowd, distinguishing it from hundreds of other faces, even if we haven't seen our friend for ten years. We are not so good at other kinds of memory. What would we do without name badges at conferences? We have all had the experience of seeing a familiar face and not being able to recall how or where we met.

Emotion and memory are closely connected. The limbic system, the part of our brain that processes emotion, also plays an important role in processing memory. We recall emotions along with events and a particular emotion can evoke the memory of an event surrounding that emotion. Events that evoke strong emotion evoke strong memories. I remember a near disastrous event, and the accompanying feelings of terror, as if it occurred yesterday, yet it happened over 35 years ago.

Then there are things we once knew and have forgotten. I suspect all of us can recall mathematical procedures that fall into this category. Tests have shown, however, that it is quicker to refresh memory than learn something for the first time. Associations also bring to mind past memories.

Reunions are good at that, and the conversations may go, "Remember when so and so did such and such?" and, "I haven't thought of that for years!" and on and on it goes.

Dehaene says our memory is associative. That has its good points and its bad points, depending on the task at hand. Associative memory enables us to put together strings of recollections, but it also leads us to focus on bananas instead of dots, and it gets in the way of memorizing arithmetical facts. Experiments have shown that children regress in the time it takes them to recall addition facts once they begin to learn multiplication facts. The theory is that one begins to associate $2 + 3$ with 2×3.

Dehaene offers a lengthy discussion about the difficulty in remembering a sheet of multiplication facts. It's not that we can't do it; if all else fails, drill it into verbatim memory, much as one does a nonsense rhyme. What is sacrificed is meaning. I know things verbatim that I can rattle off without a bit of awareness of the meaning of the words that I am saying. An example is the Pledge of Allegiance that we recited every morning at the elementary school I attended.

Mathematics class can become a matter of rote learning in which one memorizes how to carry out procedures without any sense of what is going on, yet with sufficient skill to pass the course. Carl Jung got good grades in algebra by mimicking what the instructor was doing but understanding nothing. He said he swindled his way through math.

Implications for Teaching and Learning

What are the implications the above observations have for teaching and learning? We can begin by fitting educational practices to how our mind works as opposed to trying to fit how our mind works to educational practices.

Honor Intuition

Children come to us with innate, intuitive knowledge of mathematics. We should build on that strength and be educators in the true meaning of the word. We should educe, that is draw out, and nurture this knowledge. And, by all means, avoid violating it. This means that we make fewer commands, "This is how you do it." We issue more invitations, "How would you do this?" Violating a child's intuition can happen in very subtle and unintentional ways. I once visited a first grade class during arithmetic period. The children were learning how to write the numerals. The teacher had placed a collection of dots on the board and asked for a volunteer to connect the dots to form the figure 5. A little girl volunteered; she started in the upper right corner and connected the dots in a continuous motion without lifting the chalk. What she drew looked like a 5 but the teacher, as nicely as she could, informed the class this wasn't the correct way to write a 5, and, drawing a second set of dots on the board, asked if someone else would like to try. A second child came forward, and starting in the upper left, drew the vertical portion and the bottom curve of the figure. Lifting the chalk, she then drew the top line from left to right. Yes, said the teacher and recited a little poem that described the "proper" way for making a 5, ending by drawing the "cap" on top. "My," I thought, "this is how school math becomes a mysterious and arcane subject." A first-grader does something that makes sense to them, and gives the proper result. Yet without explanation, they are told they're wrong! What impression can this make? Only that school math is an odd subject that has its own set of arbitrary, nonintuitive rules.

Remember that children's number intuition is way ahead of their language skills. This means children will have intuitive knowledge of how to do something and not be able to explain what it is they did. A number of years ago as part of a project to bring college professors into elementary schools, I spent an hour a day teaching math to fifth-graders. Rusty was an alert, quiet child who sat in the back of the class, but he was on top of everything I did. He invariably arrived at correct solutions to problems I posed, often only recording an answer and perhaps a few isolated calculations. When I asked him how he arrived at an answer his usual response was something like, “I just knew.” I, having acquired the common belief that an answer isn’t acceptable unless one can explain how one arrived at it, kept pushing Rusty for explanations, which only frustrated both of us. Finally, it became clear to me that knowing something and explaining it are two different things. I quit asking Rusty how he arrived at his answers. I didn’t want him to think his answers were unacceptable or incorrect because he couldn’t explain how he arrived at them. Above all, I didn’t want to undermine the marvelous intuitive understanding Rusty had of numbers and how they worked. So, rather than ask Rusty to explain his thinking, if I suspected Rusty had a misunderstanding, I would pose a similar question or two, changing the parameters slightly. If Rusty dealt with those correctly, I felt confident he knew what he was doing. Ever since I met Rusty, I’ve attempted to never give the impression that a student’s work is incorrect or unacceptable because they can’t explain how they arrived at their conclusion. As a matter of fact, I have found that sometimes when a student has taken a novel approach to a solution it is I, not the student, who doesn’t understand.

Be Sense-ible

A large part of our brain is devoted to sensory input and its processing—and everything seems to be connected to everything else. Doesn’t it make sense to get as much sensory input as we can into our math instruction? This means handling and exploring things. There are lots of computer simulations available. For example, you can find programs that emulate geoboards or base ten pieces. But my sense is that they are not as good as handling the real thing. Before using one of those programs, I would have students physically move pieces and arrange rubber bands. I would always involve as many senses as possible. Maybe we ought to have scratch and sniff base ten pieces so students also use their sense of smell!

Make Connections

When mathematics is learned by rote, meaning is lost, and conversely, when meaning is absent, mathematics is learned by rote. If that is a concern, math should be taught in a context, a frame of reference that is meaningful to the learner. Meaning may not be a concern if the only goal is passing tests! Dehaene says the child's brain is not a sponge, it is a "structured organ that acquires facts only insofar as they can be integrated into its previous knowledge."

Dehaene talks at length about the difficulty of remembering the times table. One problem is that there is no meaning attached to them. Adults, much less children, have a hard time telling you the meaning of the phrase "eight times seven is fifty-six." Few realize that fifty-six means "5 tens and 6 ones," so the multiplication fact "8 × 7 = 56" is simply a report that if 7 eights are arranged in groups of tens, one gets 5 tens with 6 left over. Once children understand about grouping tens, they can construct their own multiplication tables. (The English language doesn't help. Children in China have much less difficulty with grouping concepts. If counting in English were similar to counting in Chinese, we would, for example, read 56 as "5 tens and 6" and 16 as "1 ten and 6.")

Some think that one creates a frame of reference for a mathematics topic by connecting to some part of the world outside of school. Because something comes from the world outside of school does not mean it creates a frame of reference that is meaningful to the student. For example, knowing how to compute board feet may be crucial if you work in a lumberyard. But, if I walk into the classroom and tell that to my students, give them a formula for computing board feet, and then ask them to solve related problems, I suspect that for most of them I have only promoted rote learning. Furthermore, if I took them to the lumberyard where I worked earning money for college, and expected them to compute board feet in their head while loading an order, I think most of them would be lost. If you want to learn about board feet, go to a lumberyard or bring the lumberyard to the classroom.

Dealing with board feet can be instructive since one encounters lots of fractions, but it is difficult to bring the working lumberyard to the classroom or vice versa. To provide a context for studying a mathematical topic like fractions, it isn't necessary to relate it to an application from the world outside of school. One can create a context for fractions by using egg cartons or manipulatives created for the purposes of studying fractions, such as fraction bars or segment strips. The context provides a frame of reference in which students can become familiar with fractions and devise ways of dealing with them, while developing intuitive understanding. When mathematics becomes disconnected from students' intuitive understanding, the result is innumeracy. We are by nature numerate; numeracy is built into human beings. We don't acquire numeracy, we acquire innumeracy. If we practiced preventive medicine, we woudn't need to search for cures for innumeracy.

Provide Images

To function in an intuitive mode, that is to understand something in other than rote fashion, Dehaene claims the mind needs images, and math education should help children build a rich repertoire of "mental models" of arithmetic. Hadamard mentions the predominance of images in the creative thought of mathematicians. So, manipulatives are not to be used and then discarded to be replaced by abstract thinking. They are to be used to create mental models that we can use to carry information and provide understanding. Our visual memory is very strong. For example, I have a very strong picture of a board foot, but no memory at all of a formula. When I think of a board foot I see a one-foot length of a 1 × 12; or, equivalently, a one-foot length of a 2 × 6. (A 1 × 6 is half of a 1 × 12 so a 10-foot 1 × 6 has 5 board feet; a 2 × 4 is two-thirds of a 2 × 6, so a 10-foot 2 × 4 has two-thirds of 10 or 6 and ⅔ board feet, and so on.) If I wanted a formula, I would have to derive that from my mental picture. That is true for lots of mathematical concepts. The purpose of the Math and the Mind's Eye materials is to build images.

Acknowledge Emotions

We have talked about how emotion easily overwhelms rational thought. The most important thing is to recognize an emotion and let it be. In and of itself, an emotion is neither good nor bad, it simply is. You can deal with emotions in either a constructive or destructive manner. However, it is not constructive to deny them. In other words, you are permitted to hate math. Once a counselor and I team-taught a workshop for secondary teachers on working with math anxious students. One of the things we stressed was to acknowledge the anxiety that existed and not try to make it go away. We role played, one person professing anxiety about a mathematical topic and another responding in an empathic way, giving permission to feel anxious. The exercise was a failure; the teachers could not bring themselves to do that. They kept insisting that things would be okay, or kept trying to find the source of the anxiety. None of them were able to say, "Yes, I hear the anxiety in your voice, and it is okay with me if you are feeling anxious. I still want you to give this activity a go." The teachers were having a difficult time accepting the feelings being expressed and not taking responsibility for them.

Take a Break

I think all of us have experienced those unexpected flashes of cognition such as Poincarè described—when that bit of information or a solution to a problem pops into our conscious mind after we have given up the conscious search. We don't know much about how it works, but it seems to follow a period of conscious effort followed by, in Hadamard's words, a period of incubation during which our attention is diverted elsewhere. Moreover, I believe when we are getting nowhere in working something out, we can facilitate the problem-solving process by deliberately stopping our efforts. I make a conscious effort to do this. It is not easy because once I set my mind on trying to figure something out I don't want to let it go until I am successful. I like word puzzles, double-crostics, and cryptic crosswords. They give me an opportunity to practice letting go. Every morning I do the JUMBLE. It usually involves some rather atrocious pun. If I get stuck trying to figure out the pun, I try to put it aside, do something else and come back to it later. It amazes me how often this works.

When I teach a class, rather than admonish students to work hard, I tell them, "If you are working on something for class and are not making progress, your assignment is to quit before you start to feel frustrated. Say to yourself, 'I will know more about this when I come back to it later.'" Asking them to do this wards off feelings of frustration and messages like, "I'll never get this," or "I must be stupid," or "I hate this." My sense is that feeding these messages to the subconscious doesn't give it permission to keep thinking about the problem. All of this is speculation on my part, but it does create a more relaxed classroom and lots of interesting stories about when a good idea occurred.

Accept Help

As Dehaene points out, there are some things our brains aren't very good at. We can do them, but it takes effort. Computation is one of those things. We don't have a CPU that is devoted to computing, so we call on help from all over our brain. Memorizing isolated information is also difficult. We can do it in verbal memory but then we sacrifice meaning. To compensate for our difficulties we have developed technological solutions; as Sylwester puts it, adding an exterior technological layer to our brain. As far as computation is concerned, we have developed technological devices that aid us in the task of computing: counting boards, Napier's bones, abaci, paper and pencil, trig tables, manual adding machines, electric adding machines, electronic calculators, computers, and so forth. What amazes me is that we don't embrace them and accept the help they offer. What is particularly puzzling to me is how readily we accept a sixteenth-century invention, the lead (actually graphite, which is a form of carbon) pencil, while denying the use of a twentieth-century invention. I am reminded of my school days when I had to use a stick pen rather than a fountain pen (this was prior to ball points!), and wasn't allowed to type papers. Typing was only available to secretarial science students.

I think the rejection of present-day technology is confusion about what is basic to learning mathematics. If I can have a career teaching math, doing research in mathematics, working in industry as a mathematician, and never have used the long division algorithm or anything based on it (as far as I know, there is absolutely no market for long division experts), how can it be a basic skill? Yet we devote countless education resources trying to get our students proficient at long division—at best a school survival skill. Similar things could be said about other algorithms and some of the other things we stress in school. For example the rapid recall of times tables. I think that as math educators we have to be very cautious in our approach to algorithmic learning. Teaching and drilling me on an algorithm for computing board feet before turning me loose in a lumberyard would have been a great disservice to me. What did put me in good stead was a good number sense and the old-timer who showed me what a board foot was, and checked to see that I grasped the idea. In the course of a day I might have used a half dozen different methods for computing board feet, depending on what I was handling. Isn't that what we are striving for? Isn't that the most basic mathematical skill one can possess—a well developed intuitive number sense and arithmetical operations based on innate knowledge that enables one to develop one's own arithmetical procedures as the demand arises?

What They Say About Math and What We Can Learn from It

A number of years ago I ran across the autobiographical excerpts mentioned at the beginning of this article. I have quoted then on a number of occasions, especially to point out to students and teachers that the experiences of Jung and Churchill are strong evidence that mathophobia is not a function of intelligence. I thought if I ever had the time, it would be interesting to find out what other reactions to mathematics might be found in biographical works. My retirement from administrative duties provided that time and occasionally I prowl the shelves of the library looking for such tidbits. What follows is part of what I have found, along with some conclusions I reached about the images of mathematics and mathematicians that have prevailed over the years.

This article is a transcript of a talk given in Chicago on April 14, 2000, at the annual meeting of the National Council of Teachers of Mathematics.

A number of years ago, I read *Memories, Dreams, Reflections,* the autobiography of Carl Jung. It contains a long chapter on his school years and within that chapter a number of pages are devoted to his school mathematics experiences. I found them fascinating. They resonated with much of what I had observed and come to believe about school math. Here are a few passages:

> *I felt a downright fear of the mathematics class. The teacher pretended that algebra was a perfectly natural affair, to be taken for granted, whereas I didn't even know what numbers really were. They were not flowers, not animals, not fossils; they were nothing that could be imagined, mere quantities that resulted from counting. To my*

> *confusion these quantities were now represented by letters, which signified sounds, so that it became possible to hear them, so to speak. Oddly enough, my classmates could handle these things and found them self-evident. No one could tell me what numbers were, and I was unable even to formulate the question. To my horror I found that no one understood my difficulty.…All my life it remained a puzzle to me why it was that I never managed to get my bearings in mathematics when there was no doubt whatever that I could calculate properly.…*
>
> *Equations I could comprehend only by inserting specific numerical values in place of the letters and verifying the meaning of the operation by actual calculation. As we went on in mathematics I was able to get along, more or less, by copying out algebraic formulas whose meaning I did not understand, and by memorizing where a particular combination of letters had stood on the blackboard. I could no longer make headway by substituting numbers, for from time to time the teacher would say, "Here we put the expression so-and-so," and then he would scribble a few letters on the blackboard. I had no idea where he got them and why he did it—the only reason I could see was that it enabled him to bring the procedure to what he felt was a satisfactory conclusion. I was so intimidated by my incomprehension that I did not dare to ask any questions.*
>
> *Mathematics classes became sheer terror and torture to me. Other subjects I found easy; and as, thanks to my good visual memory, I contrived for a long while to swindle my way through mathematics, I usually had good marks.* [1]

Jung's experiences were a clear indication to me that math anxiety is no respecter of intelligence, and I have found recounting Jung's experience to a math-anxious person helps them understand that just because they have been terrorized by a math class doesn't mean they're stupid. Also Jung's experiences are a graphic illustration of how divorced school math can be from one's natural knowledge of number and number operations. Finally, Jung's story gave me a way of describing what I find to be a common phenomenon: swindling one's way through math, that is, getting good marks and not having the slightest notion of what's going on.

Shortly after reading Jung's biography, I was describing his math experiences to an acquaintance who mentioned that they just encountered a description of school math experiences in a biography of Winston Churchill. Churchill too, I discovered, struggled with mathematics. It took him three tries to pass the Civil Service Commissioners' exam that qualified him for entrance to Sandhurst, the Royal Military Academy. Here's Churchill's description of his nightmarish journey into mathematics as he prepared for the exam:

> *Of course what I call Mathematics is only what Civil Service Commissioners expected you to know to pass a very rudimentary examination. I suppose that to those who enjoy this peculiar gift, Senior Wranglers [those who obtain first-class honors in mathematics at Cambridge] and the like, the waters in which I swam must seem only a duck-puddle compared to the Atlantic Ocean. Nevertheless, when I plunged in, I was soon out of my depth. When I look back upon those care-laden months, their prominent features rise from the abyss of memory. Of course I had progressed far beyond Vulgar Fractions and the Decimal System. We were arrived in an 'Alice-in-Wonderland' world, at the portals of which stood 'A Quadratic Equation.' This with a strange grimace pointed the way to the Theory of Indices, which again handed on the intruder to the full rigours of the Binomial Theorem. Further dim chambers lighted by sullen, sulphurous fires were reputed to contain*

> *a dragon called the 'Differential Calculus.' But this monster was beyond the bounds appointed by the Civil Service Commissioners who regulated this stage of Pilgrim's heavy journey. We turned aside, not indeed to the uplands of the Delectable Mountains, but into a strange corridor of things like anagrams and acrostics called Sines, Cosines and Tangents. Apparently they were very important, especially when multiplied by each other, or themselves! They had also had this merit—you could learn many of their evolutions off by heart. There was a question in my third and last Examination about these Cosines and Tangents in a highly square-rooted condition which must have been decisive upon the whole of my after life. It was a problem. But luckily I had seen its ugly face only a few days before and recognised it at first sight.* [2]

Churchill provides another example that math anxiety is not a function of intelligence. And also another example of swindling: his fortuitous circumstance of being asked a math question on his Civil Service exam that he could recall from memory. But there was one thing in Churchill's encounter with mathematics that was missing from Jung's; a teacher who opened up new vistas. Churchill credits his achievement in passing the dreaded exam, not only to his own resolution, "but to the very kindly interest taken in my case by a much respected Harrow master, Mr. C. H. P. Mayo. He convinced me that Mathematics was not a hopeless bog of nonsense, and that there were meanings and rhythms behind the comical hieroglyphics; and that I was not incapable of catching glimpses of some of these." [3]

I found myself relating Churchill's experience to distraught students who had given up all hope of understanding mathematics, along with my belief that they too, along with all other normal human beings, were capable of making sense of mathematics. And it became a personal challenge to help students see "there were meanings and rhythms behind those comical hieroglyphics."

Having found these two biographical excerpts to be particularly revealing and a stimulant for reflecting on the teaching and learning of mathematics, both for me and those to whom I related these stories, I wondered what mathematical tidbits might be lurking in other biographies. Consequently, whenever I came across a biography, I scanned the pages describing educational experiences, picking up a quote here and there, while saying to myself that someday I'm going to make a systematic effort to find out what sort of picture biographies paint of mathematics and mathematics education. Someday actually arrived. Several months ago I began systematically searching biographies to see what I could learn.

So far, I've collected about 150 references to mathematics in biographies and autobiographies. References to mathematics exceed those to any other subject and evoke far more comment. As you might expect, reactions to mathematics run the gamut of human thought and emotion. What's more, these reactions are spontaneous and unguarded, without the inhibitions or biases that occur in the response to and construction of surveys, questionnaires, or other externally imposed assessments.

Here is a list of some of the words and phrases used to describe mathematics that I have encountered:

- Closed to Ordinary Mortals
- Mystically Charming/Intoxicating
- Pinnacle of Intellectual Pecking Order
- Peculiarly Engaging and Delightful
- Abstruse
- Beauty Bare
- Peculiarly Difficult
- Beautiful Road
- Knotty Subject
- Intellectual Adventure
- Intellectual Discipline
- Neat
- Mental Gymnastics
- Fascinating Pastime
- Earnest and Rigorous
- Amusing Brain Stunt
- Useful and Substantial
- Boring and Obstructive

- Rule and Rote
- Pure Wretchedness
- Grab Bag of Diverting Riddles
- Sheer Terror and Torture
- Bog of Nonsense
- Nightmare
- Worse than Useless
- Curse of My Life

As you can see, the views of mathematics range from the sublime to the ridiculous; the heartwarming to the heartrending. For some, math is an arcane and mysterious subject; for others, it has a magical attraction.

The words and phrases used to describe mathematics may be more indicative of the mindset of the biographer than that of the subject of the biography. For example, the first phrase comes from the opening paragraphs of John Kennedy Winkler's biography of the financier J.P. Morgan. All I know about the author is that he also wrote a biography of William Randolph Hearst. He appears to be in awe of mathematics, while reinforcing the popular notion that anything beyond the mathematics of the everyday world is beyond the grasp of common people. Here is the beginning of his biography of Morgan:

> *Perhaps once in a hundred years is born a mind capable of entering a sphere of higher mathematics closed to ordinary mortals. A direct and synthetic mind that cuts across lots and flies straight to conclusions, intuitively and by process unknown to self.*
>
> *Such a mind we call genius.*
>
> *Such was the mind of John Pierpont Morgan.*
>
> *By sheer mental magic, Morgan solved the most complicated problems. He was a mathematical marvel. This quality in itself destined the direction of great affairs.* [4]

As for J.P. himself, what we know is that he was a very good student of mathematics at Goettingen University in Germany; good enough that his professor told him he was making a mistake by going into business and that he should stay at Goettingen, "perhaps becom[ing] the professor's assistant, and even possibly—if he worked diligently and fortune favored him—succeed to the professor's own august chair." [5]

Biographers' biases frequently show in the choice of adjectives they use when referring to a mathematical subject. Thus Longfellow's biographer reports that Longfellow, while a student at Bowdoin, "mastered the peculiarly difficult principles of geometry." [6] It may be that Longfellow found geometry neither peculiar nor difficult. In a similar vein, Huey Long's biographer, when describing the courses Long took in high school, refers to trigonometry and plane and solid geometry as "knotty subjects." [7]

I have come across the word "abstruse," i.e., difficult to comprehend, several times in reference to mathematics. One occurrence is in a biography of the economist John Maynard Keynes. Keynes won every possible math prize while a schoolboy at Eton and went on to study math at King's College, finally giving it up for economics. The biographer, pointing out that although Keynes did well in his mathematical studies, "he did not seek out those abstruse regions which are a joy to the heart of the professional mathematician." [8] Keynes ultimately left math for economics, finding math too narrow. He found a fellow student at King's "only a mathematician, a bore and a precise example of what not to be." [9]

Keynes' biography also provides an example of a myth that's afloat in the biographical literature, namely, that those with a mathematical bent lack humanistic qualities. One of Keynes' classics teachers at Eton expressed the hope that "the more accurate sciences will not dry the readiness of his sympathy and insight for the more inspiring and humane subjects: his little essay on *Antigone* was not like the work of one made for mathematics. He has a well furnished and delightful mind." [10] Thomas Paine's biographer, on reporting that he was an excellent student in both mathematics and poetry finds "this combination unusual." [11] The biographer of James Blaine—U.S. senator and one-time secretary of state—writing of Blaine's "aptitude in mathematical study" found it to "be wondered at and admired; for the mathematical faculty does not usually co-exist, even in great minds, with the excursive and imaginative faculty which Blaine possessed in so high a measure." [12]

The converse myth also occurs, that is, that those interested in humanities are, per se, averse to mathematics. The playwright Tennessee Williams' biographer remarks as "one might expect from a budding writer" his marks in English were high while "to no one's surprise" those in algebra were very low, [13] as if that is to be expected of a literary person.

Actually, a number of writers and poets have done well in math. The poet Sydney Lanier "mastered mathematics beyond any man of his class" at Oglethorpe College. [14] The novelist Upton Sinclair, generally a mediocre student at City College of New York, won a prize in differential calculus. [15] The poet Robert Penn Warren did well in math "and inclined…towards a career in science." [16] The novelist Richard Wright had no trouble with the subject. He said he worked out all his mathematics problems in advance and spent his time in class, when not called on to recite, reading "tattered, secondhand copies of *Flynn's Detective Weekly* and *Argosy All-Story Magazine*, or dream[ing]." [17]

On the other hand, there are those authors who wanted nothing to do with the subject. Lew Wallace "developed a prompt and lasting aversion" to the subject. [18] Gene Stratton-Porter "failed it consistently." [19] F. Scott Fitzgerald found it "boring and obstructive." [20] College mathematics gave George Ade "night terrors." [21] Ellen Glasgow, because of her low standing in arithmetic was put at the foot of the class and, while sitting there, says she "felt a chill crawling up my spine, like a beetle." [22]

The greatest accolade to mathematics I have come across occurs in a biography of Charles Proteus Steinmetz. Steinmetz was born in Europe in 1865 and had nearly finished a Ph.D. degree in mathematics at the University of Breslau when he hurriedly left Germany to avoid being arrested for his socialist activities. He came to the States and ultimately ended up at General Electric where he became an expert on the theory and utilization of alternating current. He continued his interests in pure mathematics until his work at GE left him little time to pursue his interests in synthetic geometry.

To explain Steinmetz' fascination with mathematics, this particular biographer, Jonathan Norton Leonard, included a lengthy section in his biography entitled "My Lady Mathematics." Again, the sentiments expressed are those of the biographer, Leonard, and not those of his subject. (In addition to the biography of Steinmetz, which he wrote when in his twenties, Leonard also wrote on a diversity of other topics including American cooking, Gainsborough, ancient Japan, Atlantic beaches, and the enjoyment of science.)

Here is a portion of Leonard's passage on "My Lady Mathematics":

> *There's a certain almost mystical charm about pure mathematics, a charm which pervades and tinctures the whole soul of the student. It's so totally abstract. You begin with the numbers, 1, 2, 3, etc. You learn that they can be added together, multiplied and manipulated in simple ways to serve the purpose of tradesman and housekeeper. Then you begin to see their more hidden secret qualities. There are negative numbers, for instance. These are interesting things. You play with them for a while and presently you realize that if you multiply one negative number by another negative number you will get a positive number not only larger than either but of an entirely different order of largeness. It is mysterious. You want to know more.*
>
> *Finally, when you've juggled with these simple quantities, turned them upside down, turned them inside out, you begin to see short visions of fascinating qualities hitherto undreamed of. Some numbers are imaginary; they don't exist and can't exist. But nevertheless*

they can be manipulated just like real ones. The answer to a problem done with these unreal ghosts of numbers is just as correct as one done with your own ten fingers. This thrilling revelation is only one of many. Innumerable rules and principles swarm at the gates of the mind and when one of these has become established and naturalized it breeds a host of new ones which in turn present themselves for naturalization. Soon there's a dense population all yelling for attention. Mathematical intoxication is a common disease among students. [23]

Whether or not Steinmetz would agree with everything in this passage is difficult to assess. The evidence suggests that Steinmetz found mathematics intoxicating, but I suspect that he didn't have such a mystical view of numbers. Steinmetz used imaginary numbers in his analyses of electrical current and likely they were just as real to him as any other kind of number, as indeed they are, differing from other numbers, such as the counting numbers or the negative numbers, in their mathematical purpose.

Thomas Jefferson is another person who found mathematics intoxicating. Jefferson said that mathematics was the passion of his life when he was young. Later in life, he observed that "mathematics and natural philosophy [i.e., natural science] are so useful in the most familiar occurrences of life, and are so peculiarly engaging and delightful as would induce every person to wish an acquaintance with them." [24]

One person it didn't induce was William Lyon Phelps, who is responsible for the two phrases at the bottom of our list. Phelps was a professor of literature at Yale where he taught for 41 years and was voted most inspiring professor a number of times. Mathematics takes a real beating in his autobiography:

Mathematics always helped to keep me back; they were the curse of my life at school and college, and had more to do with my un-

happiness than any other thing and I bitterly regret the hours, days, weeks, months, and years that I was forced to spend on this wholly unprofitable study. I shall return to this later with more venom. [25]

And 50 pages later is the additional "venom":

For those who have no gift and no inclination, mathematics are worse than useless—they are injurious. They cast a blight on my childhood, youth, and adolescence. I was as incompetent to deal with them as a child to lift a safe. I studied mathematics because I was forced to do so, faithfully and conscientiously from the age of three to the age of twenty-one, through my Junior year in college. After "long division" nearly every hour spent on the subject was worse than wasted. The time would have been more profitably spent in manual labor, athletics, or in sleep. These studies were a brake on my intellectual advances; a continuous discouragement and obstacle, the harder I worked, the less result I obtained. I bitterly regret the hours and days and weeks and months and years which might have been profitably employed on studies that would have stimulated my mind instead of stupefying it! [26]

I found it interesting that Phelps had a colleague at Yale who had an entirely different experience in math. Wilbur Cross was a professor of English at Yale and later dean of the graduate school. Following his retirement from Yale, he served as governor of Connecticut for 8 years. Although he had a "great dislike for intricate problems concerning the time it would take for A to do a piece of work with the aid of B and often with the further aid of C," [27] geometry was another matter:

Euclid…fascinated me, not because it added anything new to my knowledge of geometry, but because of the art displayed by the old Greek mathematician in proving by a strict deductive method the *truth of propositions which anyone might see were true at a glance. It was like traveling over a beautiful road to the foreseen end of one's journey.* [28]

Cross reminds us that the organization of a subject into a deductive system properly comes after one has a thorough knowledge of the subject matter. A precept that's too often unheeded, especially in introductory geometry courses.

Cross isn't alone in his adulation of Euclidean geometry. Poetess Edna St. Vincent Millay who struggled with math when a student nonetheless wrote a sonnet, the first line of which is "Euclid alone has looked on beauty bare." [29] According to one biographer, "There is a legend that she grew so enamored of her Freshman course in mathematics [at Vassar] that she spent the night before her final examination writing a sonnet about it instead of cramming, and consequently failed to pass." [30] The young Einstein also saw beauty in math, he said that "everything in calculus and geometry is beautifully planned like a Beethoven sonata." [31]

In the biographies and autobiographies in which I have found references to mathematics, geometry is mentioned more frequently in a positive light than algebra. Dwight Eisenhower "despised" high-school algebra; he said he "could see no profit in substituting complex expressions for routine terms and the job of simplifying long, difficult equations bored me." [32] Geometry was another matter.

"The introduction of plane geometry was an intellectual adventure, one that entranced me. After a few months, my teachers conducted an unusual experiment. The principal and my mathematics teacher called me to the office and told me they were going to take away my textbook. Thereafter, I was to work out the geometric problems without the benefit of a book. In other words, the problems would be,

for me, originals. This was a fascinating challenge and particularly delightful because it meant that no advance study was required."[33]

Perhaps if his algebra teacher had set him free in his algebra class he would have found it fascinating also.

The less favorable response to algebra may be that, in contrast to geometry, it is more likely be taught by "rule and rote," a description of mathematics used by the biographer of William Randolph Hearst. "Mathematics he ignored," his biographer writes. "It was ever to be thus, the formal education of rule and rote anathema."[34] While "rule and rote" is helpful in passing math tests, as Churchill pointed out, it's shortcomings are noted. The tutor engaged by the parents of Henry Cabot Lodge to help Henry and his brother overcome their difficulties in math found the task harder than anticipated because their training in arithmetic had been exceedingly poor—mainly consisting, he said, of unreasoning memory work.[35] Richard Feynman relates the following anecdote from his student days at MIT:

One day, in mechanical drawing class, some joker picked up a French curve (a piece of plastic for drawing smooth curves—a curly, funny-looking thing) and said, "I wonder if the curves on this thing have some special formula?"

I thought for a moment and said, "Sure they do. The curves are very special curves. Lemme show ya," and I picked up my French curve and began to turn it slowly. "The French curve is made so that at the lowest point on each curve, no matter how you turn it, the tangent is horizontal."

All the guys in the class were holding their French curves up at different angles, holding their pencil up to it at the lowest point and laying it along, and discovering that, sure enough, the tangent is horizontal. They were all excited by this "discovery"—even though

> *they had already gone through a certain amount of calculus and had already "learned" that the derivative (tangent) of the minimum (lowest point) of* any *curve is zero (horizontal). They didn't put two and two together. They didn't even know what they "knew."*
>
> *I don't know what's the matter with people: they don't learn by understanding; they learn by some other way—by rote, or something. Their knowledge is so fragile!* [36]

Speaking of calculus, Eisenhower relates an incident where his instructor, but not he, relied on rote:

> *About midway in our West Point course we began the study of integral calculus. The subject was interesting but the problems could be intricate. One morning after recitations the instructor said that on the following day the problem would be one of the most difficult of all. Because of this he was giving us, on the orders of the head of the Mathematics Department, an explanation of the approach to the problem and the answer.*
>
> *The explanation was long and involved. It was clear that he was doing his task completely by rote and without any real understanding of what he was talking about. Because I was a lazy student, with considerable faith in my luck, I decided there was little use in trying to understand the solution. After all, with twelve students in the section, only one of us would get this problem to solve, the odds were eleven to one that I would not be tapped.*
>
> *The following morning I was chosen. Going to the board, on which I was required to produce the solution, and then explain it to the instructor, I had not the foggiest notion of how to begin. I did remember the answer given by the instructor and wrote it in the corner of the board.*

I set to work. I had to make at least a good start on the problem, show something *or receive a zero which would do nothing for me in a course where my grades were far from high. Moreover, I could be reported to the disciplinary department for neglect of duty in that I had deliberately ignored the long explanation. With this in the back of my mind I sought in every possible way to jog my memory. I had forty-five or fifty minutes to solve the problem and I really concentrated.*

After trying several solutions that seemed to relate, at least remotely, to the one I dimly remembered from the morning before, I encountered nothing but failure. Finally, with only minutes remaining, I worked out one approach that looked fairly reasonable. No one could have been more amazed than I when this line of action agreed exactly with the answer already written on the board. I carefully went over the work, sat down, and awaited my turn to recite. I was the last man in the section to be called upon.

With some trepidation I started in. It took me a short time to explain my simple solution—indeed it had to be simple or I never would have stumbled upon it. At the end, the instructor turned on me angrily and said, "Mr. Eisenhower, it is obvious that you know nothing whatsoever about this problem. You memorized the answer, put down a lot of figures and steps that have no meaning whatsoever, and then wrote out the answer in the hope of fooling the instructor."

I hadn't been well prepared but this was tantamount to calling me a cheat, something that no cadet could be expected to take calmly. I reacted heatedly and started to protest. Just then I heard Major Bell, the Associate Professor of Mathematics (whom we called "Poopy," a name that was always applied to anyone at West Point who was above *average in academic attainments) who had entered the room for one of his occasional inspections, interrupting. "Just a minute, Captain."*

Of course, I recognized the voice of authority and shut up, although according to my classmates' description that night I was not only red-necked and angry but ready to fight the entire academic department. I would have been kicked out on a charge of insubordination if I had not been stopped.

Major Bell spoke to the instructor, "Captain, please have Mr. Eisenhower go through that solution again."

I did so but in such an emotional state that it is a wonder that I could track it through. The long search for a solution and its eventual simplicity stood me in good stead.

Major Bell heard it out and then said, "Captain, Mr. Eisenhower's solution is more logical and easier than the one we've been using, I'm surprised that none of us, supposedly good mathematicians, has stumbled on it. It will be incorporated in our procedures from now on."

This was a blessing. A moment before, I had an excellent chance of being expelled in disgrace from the Academy. Now, at least with one officer, I was sitting on top of the world. [37]

One's performance in math meant a lot at West Point. Douglas MacArthur attended West Point a decade before Eisenhower and had an outstanding record in math. According to his biographer, "Math counted most of all. It was at the pinnacle of the intellectual pecking order.…More time was devoted to math than any other academic subject. The surest way of getting on course to be one of the Five [the five students selected as the most outstanding in their class] was to do well in math." [38]

West Point began in 1802 as a school for military engineers. In its early years, the beginning West Point student studied two subjects: mathematics and, surprisingly, French. The goal "was to make them, if not fluent, at least to become conversant with French military and engineering treatises." [39] The emphasis on math continued for years, even after the academy broadened its mission to the preparation of army officers in general. Writing in 1928, William E. Woodward, the biographer of Ulysses Grant, questioned the heavy dose of math required of West Point cadets. (Grant was a better math student than French student; in his freshman year—sometime around 1840— he was 16th in math and 49th in French out of 60; the following year he was 10th in math and 44th in French out of 53.)

Woodward was a graduate of the Citadel—at the time, the South Carolina Military Academy—where he lost interest in schooling and graduated third from the bottom in his class. He went into newspaper work, ultimately becoming publicity director of a Wall Street firm and left publicity work to become an executive vice-president and director of 42 banks in which his Wall Street firm had an interest. He grew so bored of banking and finance that he hated the sight of his office. So he quit to become a writer. Despite his low academic ranking at the Citadel, they awarded him an honorary doctor of laws degree. Here's what he wrote about math at West Point:

> *I have never been able to discover any sensible reason why a military education should be so thoroughly saturated with mathematics. In actual warfare there is nothing in mathematical science beyond arithmetic that is of the least value, except to engineering officers, and these are so few in number that a special education in mathematics might be provided for them without forcing every infantry officer to flounder through Descartes and Newton. It is true, indeed, that mathematics is the foundation of the science of ballistics; but,*

> *even so, artillery officers in the field are spared the torture of having to solve differential equations under a heavy fire, as printed tables of ranges and distances are thoughtfully provided by the War Department for their use. It is as simple as looking up a number in a telephone directory.*
>
> *It seems better, from the standpoint of common sense, to do away with everything in mathematics higher than arithmetic in an officer's education, and devote the time thus saved to such important subjects as the relative nutritive value of different kinds of food, the structure of the human body, and the principles of sanitation and medicine.* [40]

Surprisingly little attention is given to the utilitarian value of math in the biographies I've consulted. Mention is made of the value of math to engineers and scientists, as in the above comment, but, as above, not by those who actually use it. I'm reminded of those teachers who tell students how useful mathematics is while never using it themselves outside the classroom. Churchill writes he is "assured that [mathematics] are most helpful in engineering, astronomy and things like that." [41] Phelps says "the truth is that for every occupation except one for which higher mathematics are a prerequisite, like civil engineering, Greek and Latin are more useful." [42] George Ade shuddered when he saw engineering students use textbooks that applied math to engineering problems. He said "it was enough to worry through a mathematics textbook without having to think of using such lessons afterwards." [43]

Mention is made of the general value of mathematics as intellectual discipline. Often, again in reference to others, Phelps, who abhorred mathematics, had "no doubt that for those who had a natural aptitude, mathematics are valuable as an intellectual discipline and training." [44] James Blaine's biographer avers that "without doubt, the possession of mathematical ability is of high value to a public man, particularly if he be destined to deal with economic questions." [45] Charles Frances Adams, an economist and descendant of the presidential Adamses, believed he should "have compelled myself to take some of the more elementary mathematical courses, simply for the mental discipline they afford." He said he needed "the regular mental gymnastics—the daily practice of following a line of sustained thought out to exact results." [46]

There are those who, from their own experience, attest to the value of mathematics as an intellectual discipline. Omar Bradley, a West Point graduate who served a four-year assignment as a math instructor at West Point, said he "benefited from [a] prolonged immersion in math" and that "the study of mathematics, basically a study of logic, stimulates one's thinking and greatly improves one's power of reasoning. In later years, when I was faced with infinitely complex problems, often requiring immediate life-or-death decisions, I am certain that this immersion mathematics helped me think more clearly and logically." [47] Thomas Jefferson maintained that "the faculties of the mind, like the members of the body, are strengthened and improved by exercise" and this is accomplished by "mathematical reasonings and deductions." [48]

Several mentions are made of the value of mental arithmetic, a skill that suffers when heavy emphasis is placed on paper-and-pencil algorithms which, in general, are ill-suited for mental calculations. Wilbur Cross said he owed "a lasting debt" to a teacher for the practice he gave him in mental arithmetic, which, he said, was of very great help to him when dealing with budgets while governor of Connecticut. [49] Chief Justice Charles Evans Hughes' biographer writes his mother's "exercises in 'mental arithmetic' gave Charles the most useful training he ever had. She would have him toe a mark on the floor and, without changing his position, 'do in his head' the various sums she gave him. He was urged to think quickly and accurately without recourse to paper or pencil—a faculty that would add greatly to his prowess as investigator, advocate, and public speaker." [50] Henry Ford had a teacher who "noted that he was naturally fast at figures and made him do sums in his head instead of on the blackboard. Thanks to him, Mr. Ford in later years seldom had to put pencil to paper when working out a problem." [51]

By and large, those who were attracted to mathematics were done so because of its intrinsic appeal and not because of its utilitarian value, something that may be worth remembering the next time one tries to sell math because of its usefulness in other areas. In addition to those we have already mentioned who found math intellectually stimulating and aesthetically pleasing, there are those who enjoyed mathematical puzzles, those who were fascinated by numbers and statistics, and those who simply found it fun. Benjamin Banneker, astronomer and almanac publisher, loved mathematical puzzles and collected them "at every opportunity." [52] Weldon Johnson, an educator and one of the founders of the NAACP, found early in his career that "arithmetic is not only an interesting study, it is also a most fascinating pastime." He "tried to discover and prove the principles that underlay the 'rules of arithmetic'" and, for him, "getting at simpler and more understandable methods of solution became an absorbing game." [53] Noah Webster "took delight in figures and statistics. It is said that the collecting of data interested him, even when there was no apparent purpose to which it could be put....he also counted the houses, examined town lists of votes, consulted records of births and deaths, and noted weather conditions." [54] Helen Keller said she could do "long, complicated quadratic equations in my head quite easily, and it is great fun!" [55] She was also "somewhat elated" upon completing a set of geometry problems "although," she added, "I cannot see why it is so very important to know that the lines drawn from the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal! The knowledge doesn't make life any sweeter or happier, does it?" [56]

Once relieved of any of the demands of school mathematics, many paid little heed to the subject again and seemed to get along just fine. Churchill said he had "never met any of these [mathematical] creatures since. With my third and successful examination they passed away like the phantasmagoria of a fevered dream." [57] Clarence Darrow thought the aim of all learning was to make life easier, which he found mathematics, beyond simple arithmetic, ill-suited to achieve. [58] Emerson, who viewed math as "pure wretchedness," said it was not "necessary to understand Mathematics & Greek thoroughly to be a good, useful, or even *great* man." [59] Hjalmar Schacht, president of the national bank of Germany in the twenties and thirties, who once "distinguished himself by arriving at a different [incorrect] result from all his fellow candidates" in an arithmetic test to pass out of sixth form commented, "In spite of my low marks in arithmetic I have not been entirely unsuccessful in my career as banker and president of the Reichsbank. A bank inspector or manager is not a bookkeeper. His work entails expert knowledge of quite different subjects; for example, psychology, economics, common sense, ability to make decisions, but above all, insight into the intricacies and the nature of credit." [60]

One wonders what role teachers and others had in determining attitudes towards mathematics. Often an individual is referred to who was helpful, but the nature of the help isn't described. Churchill mentions the "respected Harrow master, Mr. C. H. P. Mayo who convinced him that mathematics was not a hopeless bog of nonsense, and that there were meanings and rhythms behind the comical hieroglyphics," and perhaps, most important of all, "that [he] was not incapable of catching glimpses of some of these." [61] But nothing is said of how Mayo accomplished this.

We do get some glimpses from other biographies of teachers' traits and methods that were valued. Helen Keller, who said that, as a young child, arithmetic was the only subject she did not like, credits Merton Keith, her private math tutor, for changing her outlook. She said it was "much easier and pleasanter to be taught by myself than to receive instruction in class. There was no hurry, no confusion. My tutor had plenty of time to explain what I did not understand.…Even mathematics Mr. Keith made interesting; he succeeded in whittling problems small enough to get through my brain. He kept my mind alert and eager, and trained it to reason clearly, and to seek conclusions calmly and logically, instead of jumping wildly into space and arriving nowhere. He was always gentle and forbearing, no matter how dull I might be." [62]

Thomas Jefferson credits his successes to Dr. William Small, a professor of mathematics at the College of William and Mary. Jefferson describes Small as having "a happy talent of communication, correct gentlemanly manners, and an enlarged and liberal mind." [63] The artist John Trumbull mentions a schoolmaster who "had the wisdom to vary my studies, as to render them rather a pleasure than a task"—he mentions how he was given an arithmetic problem that he had difficulty solving, but the master would not help him solve it and forbade others from helping him. For three months, he said, the problem was unsolved when "at length the solution seemed to flash upon my mind at once, and I went forward without further let or hindrance." [64] Eli Lilly recalls an elementary school teacher, Jane Graydon who "had an unusual element of freshness, and electricity of the spirit" and who "inspired me out of my arithmetic slump to make a perfect grade on my final test." [65]

Nikola Tesla mentions a calculus professor at the polytechnic school in Graz, Austria, who "was the most brilliant lecturer to whom I ever listened" and "would frequently remain for an hour or two in the lecture room, giving me problems to solve, in which I delighted." [66] Earlier, we mentioned how delighted Eisenhower was when he was left to work on his own. Richard Feynman, the Nobel Prize winner in physics had a similar experience, albeit in a high school physics class. One day the instructor asked him to stay after class and told him he talked too much, not necessarily about the matter at hand, and he believed it was because he was bored. So he gave Feynman a book and told him from now on he was to sit in the back of the room and study that book and when he had finished that he could talk again. It was an advanced calculus book—Feynman had already worked his way through a beginning calculus text. Feynman comments on how much he learned on his own from that book and how useful it turned out to be in later work he did. [67]

Of a different stripe, are those teachers who simply accommodated their students. Poet Vachel Lindsay entered Hiram College with the eventual goal of studying medicine. He was supposed to take physics, but was told he needed trig, a subject "with which his mind could scarcely grapple at all, even though the instructor, to help him out, worked all the examples himself." [68] The historian William Hickling Prescott was "noted for his horsemanship, his charm, his wit, but never his studiousness" when a student at Harvard. Prescott, as a sophomore, took a required course in geometry from a Professor John Farrar:

> *For a time he laboriously memorized propositions and processes in geometry and reproduced them in class exactly as they appeared in the textbook. Wearying, however, of the drudgery which stamped him acceptable to his teacher but grossly ignorant of the subject, Prescott confided his secret to Farrar. Convincing the professor of the impossibility of his mastering the subject, he was told that regular attendance, without recitation, would suffice.* [69]

Sometimes it was friends or relatives who saved the day or sparked interest. Jack London's friend Bess Maddern's "skillful coaching eliminated many of the mazes and pitfalls in the mathematics," enabling London to pass the math in the Berkeley entrance exam. [70] Albert Einstein learned mathematics from his Uncle Jake and Max Talmey, a university medical student who came to dinner once a week. Einstein said Max was better at explaining things than anyone at the gymnasium. [71] Later, reflecting on his gymnasium experiences, where he found the teachers severe and the lessons boring, Einstein wrote, "It is, in fact, nothing short of a miracle that the modern methods of instruction have not yet entirely strangled the holy curiosity of inquiry: for this delicate little plant, aside from stimulation, stands mainly in need of freedom; without this it goes to wreck and ruin without fail. It is a very grave mistake to think that the enjoyment of seeing and searching can be promoted by means of coercion and a sense of duty." [72]

The actress Myrna Loy struggled with grammar-school math. She reports that she went for help to an uncle to prepare for a "big test." When she passed, the teacher accused her of cheating. She walked out of class, reported the teacher to the principal and went home; refusing to return to class until the teacher apologized. [73] Being falsely accused of cheating seems particularly devastating. Eisenhower says the calculus instructor who accused him of cheating "was the only man at West Point for whom I ever developed any lasting resentment." [74]

In contrast to those who communicated well, with civility and forbearance, there were those teachers who didn't communicate at all or, if they did, were sarcastic or severe. William Lloyd Phelps had a teacher tell him, "In mathematics, you are slow, but not sure." [75] Robert Kennedy, who attended Milton Academy during the second World War at the time the German general Rommel was being defeated in Africa, wrote in a letter home that "on our last day of school...the math teacher made a small speech to the class in which he said that two great things had happened to him; one that Rommel was surrounded in Egypt and 2nd that Kennedy had passed a math test." [76]

In the 1770s, when Alexander Hamilton was a student at King's College, the forerunner of Columbia University, mathematics was taught by a “testy” professor, Robert Harpur. “More exacting than his colleagues,” Hamilton’s biographer writes, “the students frequently met his discipline with individual defiance or collective jeers.” [77] King’s College records reveal that one Edward Thomas, a student, “was ordered before Governors ‘for abusing, along with many others, Mr. Harpur, the Evening before.’ ...Thomas proved his innocence, but soon seven more…were compelled to ask public pardon ‘for ill-using Mr. Harpur, by *Calling Names* in the Dark.’” Later a student was suspended “for using Mr. Harpur in the most scandalous manner.” [78]

A century later, in the 1860s at Harvard, Oliver Wendell Holmes took math from a professor who, in the classroom, “was brief and impatient. Stupid students were terrified of him, the brilliant greeted him with joy.” [79] An example of a phenomenon known to most of us, the teacher who is able to teach only those who don’t need teaching.

And a century later, in the 1940s, Lee Iacocca tells another familiar story. Iacocca tells how he almost flunked freshman physics at Lehigh University: “We had a professor named Bergmann,” he writes, “a Viennese immigrant whose accent was so thick that I could hardly understand him. He was a great scholar, but he lacked the patience to teach freshmen.” [80]

The oft-encountered caricature of the math teacher as a social misfit, living in their own little world, arises. William Woodward, the biographer of Ulysses Grant and author of the comments about math at West Point mentioned earlier, reports that the young Ulysses Grant, who enjoyed math and once inquired if there were any math teaching positions at West Point, “built a daydream of himself as a teacher. Woodward writes, “He saw himself standing throughout the years by the stream of life, a half-recluse, sprinkling algebra and calculus generously upon the heads of the passing generations.” [81]

George Stigler was an economist who spent his professional life in academia. In his autobiography he describes how universities are willing to put up with the idiosyncrasies of experts in their fields. He chose to cast his example as a mathematician: "Universities cater to more highly specialized human beings than most other callings in life. If X is a great mathematician, he will be more or less silently endured even though he dresses like a hobo, has the table manners of a chimpanzee, and also achieves new depths of incomprehensibility in teaching. His great strength is highly prized; his many faults are tolerated." [82]

Steinmetz, the GE engineer we encountered earlier, while not as boorish as Stigler's example, is one person who fed the image of the eccentric mathematics professor. Steinmetz, who had all but finished his Ph.D. degree in math before fleeing to the U.S., wished to continue his academic involvement. To satisfy this, a lectureship was arranged for him at Union College in Schenectady where GE was headquartered. His biographer Jonathan Leonard describes his classroom:

> *He would write nervously on the blackboard, talking all the time, and then without missing a word whirl round in a tempest of questions. After the first fifteen minutes the minds of the students became rather numb. No one ever followed him in all of his calculations. He'd plunge into a flood of figures like a diver into a whirlpool; he'd struggle furiously with weird symbols which meant nothing at all to anyone but himself; he'd cover the board with writing too small to be seen beyond the first row, and finally he would emerge with a conclusion which should have been on Page 347, two chapters ahead.* [83]

Despite his students not learning anything, Leonard claims his classes weren't a total failure:

> *The students got very little mathematical information out of his lectures but they did get a great deal of inspiration. And mathematics in its higher forms is very inspirational. The sight of the little man on the platform there, bursting with enthusiasm and performing chalk*

> *miracles before their eyes, was enough to put energy into any ambitious young engineer. There aren't many lecturers like Steinmetz. If there were, no one would learn anything definite. But one Steinmetz in the intellectual adolescence of every man would make that man higher minded and less apt to become a mere stodgy technician.* [84]

According to Leonard, there was one other thing that set Steinmetz apart from ordinary people. Commenting on the fact that Steinmetz never married, Leonard tells us that "mathematics occupied completely that central part of his mind which if he had been a normal man would have been dominated by sex." [85]

Benjamin Banneker's biographer also maintains that mathematics got in the way of romance. He writes that Banneker's "consuming interest in reading and mathematical studies, and his jealous preservation of the little leisure he had for pursuing them, disinclined him to seek a wife." [86]

Lest one begins to believe that mathematics is a deterrent to romance, I end with the story of the courtship of Barnes Wallis. Wallis was a pioneer in the British aircraft industry—an aeronautical engineer before the term existed—and a very good mathematics student.

He fell in love with Molly Bloxam, a woman 15 years his younger. Molly was quite taken by Barnes but Molly's father was not in favor of Molly marrying an older man. However, as Barnes discovered, Molly was terrified of taking mathematics exams that were required in her degree program, so he began tutoring her, an activity to which her father did not object. When separated, while Molly continued her education and Barnes pursued his career, he continued his mathematics instruction by letter. Here is the beginning of his correspondence course on calculus:

> *Now here begins lecture one, from me, Barnes, to you, Molly, on the very delightful subject of the Calculus.... The calculus is a very beautiful and simple means of performing calculations which either*

> *cannot be done at all in any other way, or else can only be performed by very clumsy, roundabout and approximate methods.* [87]

"As some men carry forward their courting with imperfect poetry," the biographer writes, "so Wallis conducted his most comfortably with the perfection of sine and cosine." [88]

Molly insisted to her father that passing her exams was only possible if Barnes continued his correspondence course in mathematics. Her "persistence—and the mathematics coaching—began first to circumvent and then to erode her father's obduracy." [89] Molly and Barnes got married and lived happily ever after.

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About the Author

Eugene Maier is past president and cofounder of The Math Learning Center, and professor emeritus of mathematical sciences at Portland State University. Earlier in his career, he was chair of the Department of Mathematics at Pacific Lutheran University and, later, professor of mathematics at the University of Oregon. He has a particular interest in visual thinking as it relates to the teaching and learning of mathematics. He is coauthor of the *Math and the Mind's Eye* series and has developed many of the mathematical models and manipulatives that appear in Math Learning Center curriculum materials. He has directed fourteen projects in mathematics education supported by the National Science Foundation and other agencies, has made numerous conference and inservice presentations, and has conducted inservice workshops and courses for mathematics teachers throughout the United States and in Tanzania. Born in Tillamook, Oregon, he is a lifelong resident of the Pacific Northwest.

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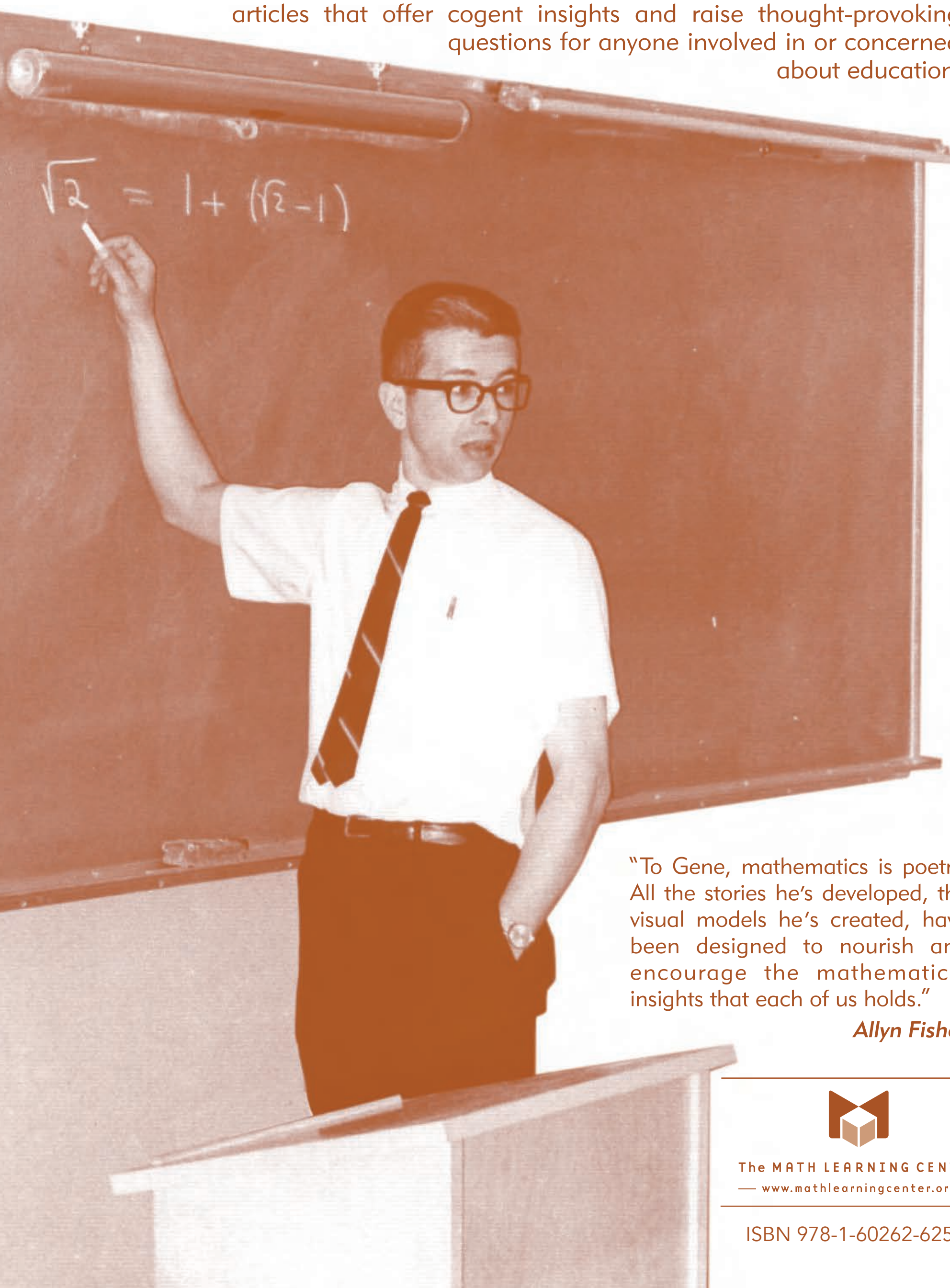

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ISBN 978-1-60262-625-6